Linear actions and Levi factors

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Notation and assumptions

- k: arbitrary field, G: linear algebraic group over k
- we insist on the following assumption:

  (♣): the unipotent radical $U = R_u(G)$ of G is defined and split over k.
- thus G determines a strictly exact sequence of linear algebraic k-groups

$$(\heartsuit) : 1 \to U \to G \xrightarrow{\pi} M \to 1$$

  where $M = G/U$ is reductive.
- notice that G acts on U by conjugation, while e.g. M acts on the center $Z(U)$ but in general M does not act on U.

Levi factors

- A Levi factor of G is a k-subgroup L such that $\pi_{|L} : L \to M$ is an isomorphism.
- If k has char. 0, G always has a Levi factor.

  This is a result of Mostow.

  For the rest of the talk, suppose that k has char. $p > 0$.
- If M is reductive and $P \subset M$ a parabolic subgroup, P has a Levi factor.
- When k is imperfect, we are avoiding some substantial issues:
  - $R_u(G)$ may fail to be defined over k (this is related to the existence of pseudo-reductive groups).
  - even when defined over k, $R_u(G)$ could in general fail to be *k-split*.

The special fiber of parahoric group scheme

- Suppose K is a local field - the field of fractions of a complete DVR $\mathcal{A}$.
- suppose that G is a reductive algebraic group over K.
- The parahoric group schemes attached to G are certain smooth affine group scheme $\mathcal{P}$ over $\mathcal{A}$ – the ring of integers of K – for which $\mathcal{P}_K = G$.
- If $k = \mathcal{A}/\pi\mathcal{A}$ is the residue field of $\mathcal{A}$, the k-group $\mathcal{P}_k$ is in general not reductive.
- If G splits over a tamely ramified extension of K, then $\mathcal{P}_k$ has a Levi factor.

see (McNinch 2014a) and (McNinch 2020).

Examples of non-existence of Levi factors

- Let V be a linear representation of the reductive k-group V and let

  $$\alpha \in H^2(M, V).$$

  Then α determines a SES

  $$0 \to V \to G_\alpha \to M \to 1$$

  which is split if and only if $\alpha = 0$.

  In particular, G_α has no Levi factor if $\alpha \neq 0$.

- since k has positive characteristic, there are many representations with non-vanishing H^2. So there are many linear algebraic groups having no Levi factor.

The problem

- Let ℓ be a finite separable field extension of k. And let G linear algebraic group over k as before (so we continue to assume U is defined and split over k.)
- **Question:** if G_ℓ has a Levi factor ("over ℓ"), does G have a Levi factor ("over k")?
- partial answer:

Theorem (McNinch 2013): Suppose that ℓ is Galois over k with

$$\gcd(p, [\ell : k]) = 1.$$

If G_ℓ has a Levi factor, then G has a Levi factor.

Failure of Descent

- suppose $p \neq 2$
- Let H be the extension

  $$0 \to \mathbb{G}_a \to H \to \mathbb{G}_a \times \mathbb{G}_a \to 0$$

  defined by the cocycle $(v, w) \mapsto \beta(v, w)^p - \beta(v, w)$ where $\beta : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ is a non-degenerate alternating form.
- for $t \in k$ let

  $$V_t = \langle (t, 0), (0, 1) \rangle \subseteq (\mathbb{G}_a \times \mathbb{G}_a)(k)$$

  so that

  $$V_t \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}.$$

Failure of Descent

- consider the extension

$$0 \to \mathbb{G}_a \to \mu_t \to V_t \to 0$$

  determined by the alternating form β.

- setting $E_t = \mu_t \times_{\mathbb{G}_a} H$ we find an extension

$$1 \to H \to E_t \to V_t \to 0$$

- this extension is split if and only if the polynomial

$$F(T) = T^p - T - t \in k[T]$$

  is reducible over k.

Failure of Descent

- See (McNinch 2025) for details.
- And see (McNinch 2013) for a similar construction with H replaced by a commutative connected unipotent group of dimension 2 and exponent p^2.
- On the other hand, with notations as before, I'm unaware of an example of a *connected* linear algebraic group H over a field such that H_ℓ has a Levi factor but H itself does not.

Non-abelian cohomology

- if U is an M-group a *1-cocycle* on M with values in U is a morphism $f : M \to U$ satisfying

$$f(xy) = f(x) \cdot {}^x f(y).$$

- The 1-coycles f, g are *cohomologous* – written $f \sim g$ – if there is $u \in U(k)$ such that

$$f(x) = u^{-1} \cdot g(x) \cdot {}^x u$$

- Write

$$H^1(M, U) = \text{1-cocycles} \ / \sim$$

  for the resulting *first cohomology set.*

Semidirect products

- Let $1 \to U \to G \xrightarrow{\pi} M \to 1$ be an extension.
- Write $\operatorname{Sect}\left(G \xrightarrow{\pi} M\right)$ for the $U(k)$-orbits of homoms $M \to G$ which are sections to π. Thus $\operatorname{Sect}\left(G \xrightarrow{\pi} M\right)$ is the set of equiv classes for the relation

$$s \sim s' \Leftrightarrow \exists u \in U(k) \text{ s.t. } s = us'u^{-1}$$

Proposition: If there is a homomorphism which is a section $s_0 : M \to G$ to π, there is a bijection

$$H^1(G, M) \simeq \operatorname{Sect}\left(G \xrightarrow{\pi} m\right).$$

- of course, if $\operatorname{Sect}\left(G \xrightarrow{\pi} M\right)$ is non-empty, then G is the semidirect product of M and U.

Result

Theorem ((McNinch 2025)): Let ℓ a finite separable extension of k, and assume the following:

a. G_ℓ has a Levi factor
b. $U_\ell^{M_\ell} = 1$.
c. $H^1(M_\ell, U_\ell) = 1$.

Then G has a Levi factor.

- The proof of the Theorem uses both the non-abelian cohomology set $H^1(M_\ell, U_\ell) = 1$ and the Galois cohomology set $H^1(k, U)$.

  since U is connected and split unipotent, $H^1(k, U)$ is trivial.

sketch of proof

- in giving the proof, may suppose ℓ is Galois over k; write $\Gamma = \mathrm{Gal}(\ell/k)$.
- Let $s_0 : M_\ell \to G_\ell$ a fixed section and $\gamma \in \Gamma$.

  Since $H^1(M_\ell, U_\ell) = 1$, we know that

  $$^\gamma s_0 = u_\gamma^{-1} \cdot s_0 \cdot u_\gamma$$

  for some $u_\gamma \in U(\ell)$.
- Now argue using hypothesis b that $\gamma \mapsto u_\gamma$ is a Galois 1-cocycle.

  Since $H^1(k, U)$ is trivial, there is $u \in U(k)$ such that $^\gamma u = u \cdot u_\gamma$.
- Then $s = u \cdot s_0 \cdot u^{-1}$ is a section with $^\gamma s = s$ for each $\gamma \in \Gamma$. Thus s is defined over k as requiried.

Definitions

Definition: Let U be a vector group on which G acts. We say that the action of G on U is linear provided that there is a G-equivariant isom $U \xrightarrow{\sim} \mathrm{Lie}(U)$ for the adjoint action of G on $\mathrm{Lie}(U)$.

Definition: A filtration

$$1 = U_0 \subset U_1 \subset \dots \subset U_{n-1} \subset U_n = U$$

by G-invariant closed subgroups U_i of U is a central linear filtration provided that for each i, U_{i+1}/U_i is a vector group with linear action of G which is central in U/U_i.

Application

- **Theorem** ((McNinch 2025)): Assume that U has a central linear filtration for the action of G and suppose the following:

  a. G_ℓ has a Levi decomposition (over ℓ),
  b. the group scheme $\left(U_{i+1}/U_i\right)^M$ is trivial for $i = 0, ..., m-1$, and
  c. $H^1(M, U_{i+1}/U_i) = 0$ for $i = 0, ..., m-1$. Then G has a Levi decomposition.

Existence of linear filtrations

- **Theorem** ((McNinch 2014b)): If G is connected and $U = R_{u(G)}$ is defined and split over k, there is a central linear filtration of U for the action of G.

- There are examples of vector groups U which are M groups for which the action is not linear.
- For these examples, $\mathrm{Lie}(U)$ is not a simple G-module.
- Let $\mathcal{A}(U) = \mathrm{Hom}(U, \mathbb{G}_a)$. In these examples, $\mathcal{A}(U)$ is not completely reducible as M-representation.

Existence of linear filtrations

- **Theorem** ((McNinch 2014b)): Assume G is connected and that U is a G-group. If $\mathrm{Lie}(U)$ is a simple representation for G^0 then $\mathcal{A}(U)$ is completely reducible and the action of G on U is linear.

Bibliography

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