Levi factors of linear algebraic groups

AMS Western Sectional Meeting UCRiverside

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2024-10-27

Levi factors

- k: field
- G: linear algebraic group $/k$.
  (i.e. *smooth affine group scheme over* k)
- we make the following assumption (R):
  The unipotent radical U of G is defined and split over k.
- (R) is always valid if k is *perfect* but can fail in general.
- A Levi factor of G is a k-subgroup M of G such that $\pi_{|M} : M \to G/U$ is an isomorphism, where $\pi : G \to G/U$ is the quotient map.
- If k has char. 0, G always has a Levi factor. (*Mostow*)

non-existence of Levi factors

- Suppose k has char. $p > 0$
- let M be reductive $/k$, let V a linear rep of M and let $\alpha \in H^2(M, V)$.
- α determines a SES

$$0 \to V \to G_\alpha \to M \to 1$$

  which is not split whenever $\alpha \neq 0$
  in particular, if $\alpha \neq 0$ then G_α has no Levi factor.
- since k has char. $p > 0$, there are many representations with non-vanishing H^2

Descent of Levi factors

- Let ℓ be a finite separable extension of k
- Question: If G_ℓ has a Levi factor ("over ℓ"), does G has a Levi factor ("over k")?
- partial answer:

Theorem (McNinch 2013)

Suppose ℓ is Galois over k with $\gcd(p, [\ell : k]) = 1$. If G_ℓ has a Levi factor, then G has a Levi factor.

Linear actions

- Let the vector group U be an M-group. The action of M on U is linear provided that there is an equivariant isomorphism of algebraic groups $U \simeq \mathrm{Lie}(U)$.

Proposition

If the unipotent radical U of G is a vector group with linear action of G, then G_ℓ has a Levi factor $\implies$ G has a Levi factor.

Failure of descent of Levi factors

- suppose $p \neq 2$
- Let H be the extension

$$0 \to \mathbf{G}_a \to H \to \mathbf{G}_a \times \mathbf{G}_a \to 0$$

  defined by the cocycle $(v, w) \mapsto \beta(v, w)^p - \beta(v, w)$ where $\beta : \mathbf{G}_a \times \mathbf{G}_a \to \mathbf{G}_a$ is a non-degenerate alternating form.
- for $t \in k$ let

$$V_t = \langle (t, 0), (0, 1) \rangle \subset (\mathbf{G}_a \times \mathbf{G}_a)(k)$$

  so that $V_t \simeq \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$.
- consider the extension

$$0 \to \mathbf{G}_a \to \mu_t \to V_t \to 0$$

  determined by the alternating form β.

Failure of descent, continued

- setting $E_t = \mu_t \times_{\mathbf{G}_a} H$ we find an extension

$$1 \to H \to E_t \to V_t \to 0 \qquad (0.1)$$

- the extension Equation (0.1) is split $\iff$ $F(X) = X^p - X - t \in k[X]$ is reducible over k.
- In particular if $F(X) = X^p - X - t$ is irreducible and α is a root, set $\ell = k(\alpha)$. Then E_t has no Levi factor, but $E_{t,\ell}$ has a Levi factor.
- see [1] for more details, and see also [3] for a similar construction where H is replaced by a commutative connected unipotent group of exponent p^2.
- On the other hand, I am not currently aware of any connected group G satisfying (R) such that G has no Levi factor but G_ℓ has a Levi factor.

Non-abelian cohomology

- if U is an M-group a *1-cocycle* on M with values in U is a morphism $f : M \to U$ satisfying

$$f(xy) = f(x) \cdot {}^{x}f(y)$$

- The 1-coycles f, g are *cohomologous* – written $f \sim g$ – if there is $u \in U(k)$ such that

$$f(x) = u^{-1} \cdot g(x) \cdot {}^{x}u$$

- Write $H^1(M, U) =$ 1-cocycles$/ \sim$ for the resulting *first cohomology set.*

Non-abelian cohomology and semidirect products

- Let
$$1 \to U \to G \xrightarrow{\pi} M \to 1$$
be an extension.
- Write $\mathrm{Sect}(G \xrightarrow{\pi} M)$ for the $U(k)$-orbits of homoms $M \to G$ which are sections to π. i.e. two such homomorphism are equiv if
$$s \sim s' \iff \exists u \in U(k) \quad \text{s.t.} \quad s = us'u^{-1}$$

Non-abelian cohomology and semidirect products

Proposition

If there is a homomorphism which is a section $s_0 : M \to G$ to π, there is a bijection

$$H^1(G, M) \xrightarrow{\sim} \mathrm{Sect}(G \xrightarrow{\pi} M).$$

- of course, existence of the section s_0 as in the previous Proposition means that $G = M \ltimes U$ is a semidirect product.

Main result

- Suppose that G satisfies condition (R).

Theorem (McNinch 2024)

Let ℓ a finite separable extension of k, and assume the following:

(a) G_ℓ *has a Levi factor*

(b) $U_\ell^{M_\ell} = 1$.

(c) $H^1(M_\ell, U_\ell) = 1$.

Then G has a Levi factor.

Remarks on the proof

- The proof of the Theorem uses both the non-abelian cohomology set $H^1(M_\ell, U_\ell) = 1$ and the Galois cohomology set $H^1(k, U)$.
- Since U is a split unipotent group (by assumption (R)), $H^1(k, U) = 1$.
- in giving the proof, may suppose ℓ is Galois over k; write $\Gamma = \mathrm{Gal}(\ell/k)$.
- Let $s_0 : M_\ell \to G_\ell$ is a fixed section and $\gamma \in \Gamma$. Since $H^1(M_\ell, U_\ell) = 1$ we know

$$^{\gamma}s_0 = u_\gamma^{-1} \cdot s_0 \cdot u_\gamma$$

for some $u_\gamma \in U(\ell)$

Remarks on the proof

- Now argue using hypothesis (b) that $\gamma \mapsto u_\gamma$ is a Galois 1-cocycle. Since $H^1(k, U)$ is trivial, there is $u \in U(k)$ such that
$$^{\gamma}u = u \cdot u_\gamma.$$
- Now $s = u \cdot s_0 \cdot u^{-1}$ is a section with $^{\gamma}s = s$ for each $\gamma \in \Gamma$. Thus s is define over k.

Linear filtrations

Definition

A filtration

$$1 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_{m-1} \subset U_m = U$$

by G-invariant closed subgroups U_i of U is a *central linear filtration* for the action of G if

(a) U_{i+1}/U_i is a vector group with linear G action for each i, and

(b) U_{i+1}/U_i is central in U/U_i for each i.

Theorem (McNinch 2014)

Assume R holds for G. If G is connected, there is a central linear filtration of U for the action of G.

Linear actions (continued)

Corollary (McNinch 2024)

Assume that U has a central linear filtration for the action of G and suppose the following:

- *(a)* G_ℓ *has a Levi decomposition (over* ℓ*),*
- *(bb)* *the group scheme* $(U_{i+1}/U_i)^M$ *is trivial for* $i = 0, \cdots, m-1$, *and*
- *(cc)* $H^1(M, U_{i+1}/U_i) = 0$ *for* $i = 0, \cdots, m-1$.

Then G has a Levi decomposition.

Why the interest in descent of Levi factors?

Origin of interest: I wanted info on Levi factors of special fibers of parahoric group schemes. Here's an example:

- Let $\mathscr{A}$ be a complete DVR with fractions K and residue field $k = \mathscr{A}/\pi\mathscr{A}$, and let L be a ramified separable quadratic extension of K. Assume that the residue char. p is $\neq 2$.
- Let V be a $2n$-dimensional L-vector space equipped with a quasi-split hermitian form h over K, and let $G = \mathrm{SU}(V, h)$ be the corresponding unitary group. G is a linear algebraic group over K and $G_L \simeq \mathrm{SL}_{2n,L}$.
- Write $\mathscr{B}$ for the integral closure of $\mathscr{A}$ in L. Notice that $\mathscr{B} \otimes_{\mathscr{A}} k \simeq k[\epsilon] = k[E]/\langle E^2 \rangle$.

Why the interest … ? (continued)

- a suitable choice of $\mathscr{B}$-lattice $\mathscr{L}$ in V determines an $\mathscr{A}$-group scheme $\mathcal{P}$ with the following properties:
- $\mathcal{P}_K$ identifies with G.
- $\mathcal{P}_k$ is an extension

  $$0 \to U \to \mathcal{P}_k \to \mathrm{Sp}(M) \to 1$$

  where U is a vector group with linear action of $\mathrm{Sp}(M)$
- in fact U is the unique $\mathrm{Sp}(M)$-invariant subspace of $\bigwedge^2 M$ of codimension 1.
- $\mathcal{P}_k$ has a Levi decomposition.
- Assume $\dim M = \dim V \equiv 0 \pmod{p}$. Then $H^1(\mathrm{Sp}(M), U) \neq 0$, so that $\mathcal{P}_k$ has non-conjugate Levi factors.

Bibliography

[1] George McNinch. "Levi decompositions of linear algebraic groups and non-abelian cohomology". In: *Pacific J. Math (special issue in memory of Gary Seitz)* (2024). to appear.

[2] George McNinch. "Linearity for Actions on Vector Groups". In: *Journal of Algebra* 397 (2014), pp. 666–688. DOI: 10.1016/j.jalgebra.2013.08.030.

[3] George McNinch. "On the Descent of Levi Factors". In: *Archiv der Mathematik* 100.1 (2013), pp. 7–24. DOI: 10.1007/s00013-012-0467-y.