Reductive groups over local fields

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Outline

1. Local fields
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Local fields and so forth

- By a *local field*, I mean the field of fractions K of a complete DVR $\mathcal{A}$ with residue field $\mathcal{A}/\pi\mathcal{A} = \mathrm{k}$.
- Of course, that is a mouthful...
- Two interesting origins:
  - number fields
  - algebraic curves
- First recall: $\mathcal{A}$ is a DVR means that $\mathcal{A}$ is a local, commutative ring which is a principal ideal domain

Local fields from number fields

- First, suppose $\mathbf{Q} \subset \mathcal{F}$ is a finite ext; i.e. $\mathcal{F}$ is a *number field*.
- For each prime number p, the field $\mathbf{Q}$ is the field of fractions of the discrete valuation ring $\mathbf{Z}_{(p)}$
- The integral closure of $\mathbf{Z}_{(p)}$ in $\mathcal{F}$ has finitely many non-zero prime ideals; the localization $\mathcal{A}$ of this integral closure at each of these prime ideals yields a discrete valuation subring of $\mathcal{F}$.
- Now the completion

$$\widehat{\mathcal{A}} = \lim_{\leftarrow} \mathcal{A}/\pi^n\mathcal{A}$$

  of $\mathcal{A}$ is a complete DVR, where $\pi\mathcal{A}$ is a chosen a maximal ideal of $\mathcal{A}$.
- And the field of fractions $\mathcal{F}_\pi$ of $\widehat{\mathcal{A}}$ is a local field.

Local fields from number fields

- In particular, $\mathbf{Q}_p$ is the field of fractions of the completion $\mathbf{Z}_p$ of $\mathbf{Z}_{(p)}$.

Proposition

$\mathbf{Q}_p \subset \mathcal{F}_\pi$ is a finite extension, $\mathrm{k} = \mathcal{A}/\pi\mathcal{A} \simeq \widehat{\mathcal{A}}/\pi\widehat{\mathcal{A}}$ and $[\mathrm{k} : \mathbf{F}_p]$ divides $[\mathcal{F}_\pi : \mathbf{Q}_p]$.

"Mixed characteristic" local fields

- The local fields $\mathcal{F}_\pi$ arising from number fields just discussed are of characteristic zero, and their "residue fields" have characteristic $p > 0$ – "mixed characteristic".
- More generally, to a field k of characteristic $p > 0$, one can functorially associate the ring $\mathcal{A} = W(\mathrm{k})$ of Witt vectors; $W(\mathrm{k})$ is a complete DVR with residue field k and field of fractions of characteristic 0.

Local fields from algebraic curves

- Let k be a field and let X be an absolutely irreducible smooth algebraic curve over k.
- The "closed points" P of X are in one-to-one correspondence with the k-valuation subrings $\mathcal{A}_P$ of the field of rational functions $\mathcal{F} = \mathrm{k}(X)$ on X.
- E.g. $k[t]_{(t-a)}$ is the valuation ring of the field of rational functions $k(t)$ on $\mathbf{P}^1$ determined by $a \in k$.
- If $\pi \in \mathcal{A}_P$ is a uniformizer, the completion $\widehat{\mathcal{A}_P}$ of $\mathcal{A}_P$ identifies with $\ell[\![\pi]\!]$ where $\ell = \mathcal{A}_P/\pi\mathcal{A}_P$ is the residue field of $\mathcal{A}_P$, so $\ell \supset \mathrm{k}$ is a finite extension.
- The field of fractions $\mathcal{F}_P$ of $\widehat{\mathcal{A}_p}$ is a local field of **equal characteristic**, and $\mathcal{F}_p \simeq \ell(\!(\pi)\!)$.

Ramification

Let (♡) $\mathcal{F}_1 \subset \mathcal{F}_2$ be a finite extension of local fields, with rings of integers $\mathcal{A}_1 \subset \mathcal{F}_1$ and $\mathcal{A}_2 \subset \mathcal{F}_2$. Write $\mathrm{k}_1, \mathrm{k}_2$ for the respective residue fields.

- $\mathcal{A}_2$ is the integral closure of $\mathcal{A}_1$ in $\mathcal{F}_2$
- The extension (♡) is *unramified* if $\mathrm{k}_1 \subset \mathrm{k}_2$ is separable and if $\pi_1 \mathcal{A}_2 = \pi_2 \mathcal{A}_2$.
- If $\mathrm{k}_1 \subset \mathrm{k}_2$ is separable, we have $[\mathcal{F}_2 : \mathcal{F}_1] = [\mathrm{k}_2 : \mathrm{k}_1] \cdot e$ where $\pi_1 \mathcal{A}_2 = \pi_2^e \mathcal{A}_2$.
- "ramified" means $e > 1$.

Ramification

Example: totally ramified extensions

- $\mathbf{Q}_p \subset \mathbf{Q}_p(\sqrt{p})$ and $k((t^2)) \subset k((t))$ are totally ramified ($e = 2$).

Proposition

For each finite separable extension $\mathrm{k} \subset \ell$, there is a unique unramified extension $\mathcal{F} \subset \mathcal{F}'$ for which $\mathcal{A}'$ has residue field ℓ.

Example: Unramified extensions

- If $\mathrm{k} \subset \ell$ is a separable extension, $\mathrm{k}((t)) \subset \ell((t))$ is the corresponding unique unramfied extension
- If $p \neq 2$ $\mathbf{Q}_p \subset \mathbf{Q}_p(i)$ is unramified. Of course, $[\mathbf{Q}_p(i) : \mathbf{Q}_p] = 2$ if $p \equiv 3 \pmod 4$ while $\mathbf{Q}_p(i) = \mathbf{Q}_p$ if $p \equiv 1 \pmod 4$.

Outline

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2 Reductive groups over Local fields

3 Main result

Tori in reductive groups

- If G is a reductive group over a field $\mathcal{F}$, then G has a subgroup T which is a maximal torus.
- This means: after possibly extending scalars to a finite separable extension of $\mathcal{F}$, T becomes a *split torus* – i.e. $T \simeq \mathbf{G}_m \times \cdots \times \mathbf{G}_m$ where $\mathbf{G}_m = \mathrm{GL}_1$ is the "multiplicative group" of $\mathcal{F}$, so e.g. $\mathbf{G}_m(\mathcal{F}) = \mathcal{F}^\times$
- Maximal tori play an important role governing the structure of G – for example, if $\mathcal{F}$ is separably closed, all $\mathcal{F}$-maximal tori are conjugate under $G(\mathcal{F})$.
- A reductive group is *split* over $\mathcal{F}$ if it has a split maximal torus over $\mathcal{F}$.

Split reductive groups

- split reductive groups are classified by some combinatorial information – "root datum"; roughly, their Dynkin diagram.
- partial list: $\mathrm{GL}(V), \mathrm{Sp}(V), \mathrm{O}(V), G_2, F_4, \ldots$
- Compare with: classification of compact Lie groups, or of semisimple complex Lie algebras
- important fact, (Chevalley): for any root datum, there is a group scheme $\mathcal{G}$ (smooth, affine, and of finite type) over $\mathbf{Z}$ with the property that $\forall$ fields $\mathcal{F}$, the linear algebraic group $\mathcal{G}_{\mathcal{F}}$ is a split reductive group over $\mathcal{F}$ with the given root datum.
- In particular, if K is a local field with integers $\mathcal{A}$, and if G is a split reductive group over K, G has a "reductive model over $\mathcal{A}$: i.e. there is a group scheme $\mathcal{G}$ over $\mathcal{A}$ for which $G = \mathcal{G}_{\mathrm{K}}$ and for which $\mathcal{G}_{\mathrm{k}}$ is a reductive group over k "with the same root datum as G".

Ramification and splitting

Let G be a reductive group over the local field $\mathcal{F}$.

Proposition

If G has a reductive model $\mathcal{G}$ over $\mathcal{A}$, then $G_{\mathcal{F}'}$ is split for some unramified extension $\mathcal{F} \subset \mathcal{F}'$.

- Indeed, the reductive k-group $\mathcal{G}_{\mathrm{k}}$ has a splitting field
- i.e. after a finite separable extension ℓ of k, $\mathcal{G}_{\ell}$ becomes split reductive
- a Hensel-type argument shows that one can "lift" a split max torus to $\mathcal{G}$.
- Then G splits over the corresponding unramified extension of $\mathcal{F}$.

Example: a classical split reductive group

Let V be a $2d$ dimensional $\mathbb{K}$-vector space, and let β be a non-degenerate alternating form on V.

- β gives an involution $X \mapsto X^*$ on $R = \mathrm{End}_{\mathbb{K}}(V)$ by the formula $\beta(Xv, w) = \beta(v, X^* w)$ for $v, w \in V$.
- The gp $G = \mathrm{Sp}(V) = \mathrm{Sp}(V, \beta)$ is given by the functor $(\clubsuit) \quad G(\Lambda) = \{g \in R_\Lambda \mid g \cdot g^* = 1\}$.
- G is a split reduc gp with Dynkin diagram of type C_d.
- maximal tori of G are determined by $*$-stable maximal étale subalgebras $\mathcal{E}$ of R.
- If E and F are max'l isotropic subsp s.t. $V = E \oplus F$, choose a basis $e_1, \dots, e_d$ for E and the dual basis $f_1, \dots, f_d$ for F. The étale subalgebra $\mathcal{E} = \langle E_{ii}, F_{ii} \rangle \subset R$ spanned by the corresponding idempotent matrices dets a maxl split torus.

Example: a classical split reductive group

- $G = \mathrm{Sp}(V)$ is determined by the "algebra with involution" $(R, *)$
- To get a reductive model $\mathcal{G}$, choose a full $\mathcal{A}$-lattice $\mathcal{L} \subset V$ for which the $\mathcal{A}$-subalgebra $\mathcal{R} = \mathrm{End}_{\mathcal{A}}(\mathcal{L}) \subset R$ satisfies $\mathcal{R} = \mathcal{R}^*$.
- The model $\mathcal{G}$ is determined by the functor given by the analogue of (♣): i.e. for all $\mathcal{A}$-algebras Λ

$$(\clubsuit) \quad \mathcal{G}(\Lambda) = \{g \in \mathcal{R}_\Lambda \mid g \cdot g^* = 1\}$$

Models

- important theorem: a real Lie group H has a unique conjugacy class of maximal compact subgroup.
- Consider a split reductive algebraic group G over a local field K with reductive model $\mathcal{G}$ over $\mathcal{A}$.
- If the residue field k is finite, $\mathcal{G}(\mathcal{A})$ is a maximal compact subgroup of the topological group $G(\mathrm{K})$.
- (Well, note at least that $\mathcal{G}(\mathcal{A}) = \varprojlim \mathcal{G}(\mathcal{A}/\pi^n\mathcal{A})$ is *profinite*, hence compact.)

Models

- But there are in general non-conjugate maximal compact subgroups of $G(\mathrm{K})$.
- Work –especially of **F. Bruhat and J. Tits**– shows: the maximal compact subgroups of $G(\mathrm{K})$ arise (essentially) from groups $\mathcal{P}(\mathcal{A})$ for certain $\mathcal{A}$-models $\mathcal{P}$ of G – the parahoric group schemes.
- These parahoric include the reductive models, but in general there are non-reductive parahoric group schemes.

Example: non-reductive models for split reductive groups

Let $G = \mathrm{GL}(V)$ for a finite dimensional K-vector space V.

- Choose a full $\mathcal{A}$-lattice $\mathcal{L}$ in V.
- And choose a second lattice $\mathcal{M}$ with $\pi\mathcal{L} \subset \mathcal{M} \subset \mathcal{L}$ (so $\pi\mathcal{L} \subset \mathcal{M} \subset \mathcal{L}$ is a "lattice flag").
- Identify G with the "diagonal subgroup" $\Delta G \subset G \times G = \mathrm{GL}(V) \times \mathrm{GL}(V)$.
- note: $\mathrm{GL}(\mathcal{L}) \times \mathrm{GL}(\mathcal{M})$ is reduc model for $G \times G$.
- The "schematic closure" $\mathcal{P}$ of $G = \Delta G$ in $\mathrm{GL}(\mathcal{L}) \times \mathrm{GL}(\mathcal{M})$ is a model for G which in some sense is "the stabilizer of the chosen lattice flag".
- The linear algebraic k-group $\mathcal{P}_{\mathrm{k}}$ has reductive quotient $\mathrm{GL}(\mathcal{L}/\mathcal{M}) \times \mathrm{GL}(\mathcal{M}/\pi\mathcal{L})$ and is not reductive.

Example: ramified splitting field

Example: supppose the resid. char p of k is $\neq 2$. Consider quasi-split unitary gp $G = \mathrm{SU}(V)$ with $\dim_{\mathrm{L}} V = 2d$ splitting over a totally ramif quad ext $\mathrm{K} \subset \mathrm{L}$.

- L is a splitting field for G, and G has no reductive model over $\mathcal{A}$.
- suitable $\mathcal{A}_{\mathrm{L}}$-lattice $\mathcal{L}$ in V leads to model $\mathcal{P}$ of G such that:
- $M = \mathcal{L}/\pi_{\mathrm{L}}\mathcal{L}$ is a k-vector space of dim $2d$ with a symplectic form and there is an exact seq

$$0 \to W \to \mathcal{P}_{\mathrm{k}} \to \mathrm{Sp}(M) \to 1$$

  where W is the ! codim 1 $\mathrm{Sp}(M)$-submodule of $\bigwedge^2 M$.
- $\mathcal{P}_{\mathrm{k}}$ does have a Levi factor
- but $H^1(\mathrm{Sp}(M), W) \neq 0$ if $d \equiv 0 \pmod p$. So in general $\mathcal{P}_{\mathrm{k}}$ has non-conjugate Levi factors.

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Structure of parahoric group schemes

Let G reductive over local field K, *s'pose G splits over unramified extension of* K. Let $\mathcal{P}$ a parahoric gp scheme / $\mathcal{A}$ with $\mathcal{P}_{\mathrm{K}} = G$. Even when k is perhaps imperfect, notice:

Remark

The unipotent radical of $\mathcal{P}_{\mathrm{k}}$ is defined and split over k ("Condition **(R)** holds for $\mathcal{P}_{\mathrm{k}}$").

Main result

Theorem (McNinch 2018)

There is a reductive subgroup scheme $\mathcal{M} \subset \mathcal{P}$ such that

(a) $\mathcal{M}_{\mathrm{K}}$ *is a reductive subgroup of G containing a maximal split torus of G, and*

(b) $\mathcal{M}_{\mathrm{k}}$ *is a Levi factor of $\mathcal{P}_{\mathrm{k}}$*

Remark

In fact, the groups $M = \mathcal{M}_{\mathrm{K}}$ that occur are the "groups of type $C(\mu)$" of the next slide.

Remark

I first proved in (McNinch 2010) that $\mathcal{P}_{\mathrm{k}}$ has a Levi factor (at least when k is perfect) – but without the existence of $\mathcal{M}$ – and that any two Levi factors are "geometrically conjugate".

Reductive subgroups of G containing a maximal torus

Let G be a split reduc over $\mathcal{F}$ with split max torus T.

- reductive subgps of G containing T admit a combinatorial description, essentially due to (Borel and Siebenthal 1949).
- The nodes α of the "extended Dynkin diagram" attached to G label certain *root subgroups* $U_\alpha \simeq \mathbf{G}_a$ with respect to T.
- For a proper subset Δ of the nodes of the extended Dynkin diagram, one can form $M = \langle T, U_\alpha \mid \alpha \in \Delta \rangle$.
- The M formed in this way are the reduc subgps of type $C(\mu)$ – M is the connected centralizer in G of the image of a homomorphism $\mu_n \to T$.

Example

If $V = V_1 \perp V_2$ is an "orthogonal sum" of symplectic spaces, then $\mathrm{Sp}(V_1) \times \mathrm{Sp}(V_2)$ is a subgroup of $\mathrm{Sp}(V)$ of type $C(\mu)$.

Ramified unitary group, again

As before, let $G = \mathrm{SU}(V)$ a quasi-split unitary group splitting over a totally ramified quadratic extension $\mathrm{K} \subset \mathrm{L}$.

- We saw that G has a parahoric group scheme $\mathcal{P}$ such that a Levi factor of $\mathcal{P}_{\mathrm{k}}$ is the split symplectic group over k.
- It follows that $\mathcal{P}$ can not have a reductive subgroup scheme $\mathcal{M}$ as described in the preceding theorem. (because G has no reductive subgroup containing a maximal torus which is isomorphic to a symplectic group).

Tame ramification

Suppose that G splits over a tamely ramified extension of K and let $\mathcal{P}$ be a parahoric for G.
Following a suggestion of Gopal Prasad, I proved:

Theorem (McNinch 2014)

If k is perfect and if $\overline{\mathrm{k}}$ denotes an algebraic closure, then $\mathcal{P}_{\overline{\mathrm{k}}}$ has a Levi factor.

Remark

I don't know whether the Levi factor of the Theorem "descends" to a Levi factor of $\mathcal{P}_{\mathrm{k}}$. See (McNinch 2013) for some partial results on descent – e.g. if $\mathcal{P}_{\mathrm{k}}$ acquires a Levi factor over a galois extension $\mathrm{k} \subset \ell$ with $[\ell : \mathrm{k}]$ relatively prime to the characteristic, then $\mathcal{P}_{\mathrm{k}}$ has a Levi factor.

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