Centralizers of nilpotent elements

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Outline

Introduction

- This talk will describe some applications of “comparison results” for centralizers of nilpotent elements in the Lie algebra of a linear algebraic group.
- Part of the results described appear in the joint paper McNinch and Testerman (2016) in Proc. AMS with Donna Testerman (EPFL).
- The second part describes an improved version of a result from McNinch (2008); it will appear in McNinch (2016a).

Standard reductive groups

We want to define a notion of *standard* reductive groups over a field $\mathcal{F}$:

- Semisimple groups in "very good" characteristic are standard, and tori are standard.
- If G is standard and H is *separably isogenous* to G, then H is also standard.
- If G_1 and G_2 are standard, so is $G_1 \times G_2$.
- If $D \subset G$ is a diagonalizable subgroup scheme and if G is standard, then also $C_G^o(D)$ is standard.
- In particular: GL_n is standard for all $n \geq 1$.
- If G is standard and if L is a Levi factor of a parabolic of G, then L is standard.
- Not standard: symplectic or orthogonal groups in char. 2.

Standard reductive groups: properties

Suppose that G is a standard reductive group over the field $\mathcal{F}$.

Theorem

(a) *The center Z of G (as a group scheme) is smooth over $\mathcal{F}$.*

(b) *The centralizers $C_G(X)$ and $C_G(x)$ are smooth over $\mathcal{F}$ for every $X \in \mathrm{Lie}(G)$ and every $x \in G(\mathcal{F})$.*

(c) *There is a G-invariant nondegenerate bilinear form on $\mathrm{Lie}(G)$.*

(d) *There is a G-equivariant isomorphism – a Springer isomorphism – $\varphi : \mathcal{U} \to \mathcal{N}$ where $\mathcal{U} \subset G$ is the unipotent variety and $\mathcal{N} \subset G$ is the nilpotent variety.*

Theorem (McNinch and Testerman (2009))

For $X \in \mathrm{Lie}(G)$ and $x \in G(\mathcal{F})$, $Z(C_G(X))$ and $Z(C_G(x))$ are smooth over $\mathcal{F}$.

Nilpotent elements for a standard reductive group over a field

- Let G a "standard" reductive alg gp over the field $\mathcal{F}$.
- Let $X \in \text{Lie}(G)$ nilpotent. A cocharacter $\phi : \mathbf{G}_m \to G$ is *associated to* X if $X \in \text{Lie}(G)(\phi; 2)$ and if ϕ takes values in (M, M) where $M = C_G(S)$ for a maximal torus $S \subset C_G(X)$.

Theorem

(a) *There are cocharacters associated to X ("defined over $\mathcal{F}$").*

(b) *Any two cocharacters associated to X are conjugate by an element of $U(\mathcal{F})$ where $U = R_u C_G(X)$.*

(c) *Each cocharacter ϕ associated to X determines the same parabolic subgroup $P = P(\phi)$. In fact,*

$$\text{Lie}(P) = \sum_{i \geq 0} \text{Lie}(G)(\phi; i).$$

Nilpotent elements: associated cocharacters

Let X nilpotent and let ϕ be a cocharacter associated to X.

- If $\mathcal{F}$ has characteristic 0, let (Y, H, X) be an $\mathfrak{sl}_2$-triple containing X. Then up to conjugacy by $U(\mathcal{F})$, $\mathrm{Lie}(G)(\phi; i)$ is the i-eigenspace of $\mathrm{ad}(H)$.
- For general $\mathcal{F}$, we have the following result:

Theorem (McNinch (2005))

If $X^{[p]} = 0$ there is a unique $\mathcal{F}$-homomorphism $\psi : \mathrm{SL}_{2,\mathcal{F}} \to G$ such that $d\psi(E) = X$ and $\psi_S = \phi$, where $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and where $S \simeq \mathbf{G}_m$ is the diagonal torus of SL_2.

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Even nilpotent elements

G is a standard reductive group over $\mathcal{F}$ and $X \in \mathrm{Lie}(G)$ nilpotent.

- Let ϕ be a cocharacter associated to X.
- X is *even* if $\mathrm{Lie}(G)(\phi; i) \neq 0 \implies i \in 2\mathbf{Z}$.
- If X is *even*, then $\dim C_G(X) = \dim M$ where $M = C_G(\phi)$ is a Levi factor of $P = P(\phi)$.

Main result

Theorem (McNinch and Testerman (2016))

If X is even, $\dim Z(C_G(X)) \geq \dim Z(M)$. [Where $Z(-)$ means "the center of -"].

- In fact, Lawther-Testerman already proved that equality holds (for G semisimple). Their methods were "case-by-case".
- The argument I'll describe here is more direct.
- Reason for interest: let the unipotent u correspond to X via a Springer isomorphism. In char. $p > 0$, one has in general no well-behaved exponential map, but one might still hope to embed u in a "nice" abelian connected subgroup. $Z(C_G(X))^0 = Z(C_G(u))^0$ is a starting point.

Reductions

- One knows that

$$\mathrm{Lie}(Z(C_G(X))) = \mathfrak{z}(\mathrm{Lie}(C_G(X))^{\mathrm{Ad}(B)} = \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(X)) \cap \mathfrak{g}^{\mathrm{Ad}(B)}$$

  where $B = C_{C_G(X)}(\phi)$.
- In particular, to prove the main result, it is enough to argue that $\dim \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(X)) \cap \mathfrak{g}^{\mathrm{Ad}(B)} \geq \dim \mathfrak{z}(\mathrm{Lie}(M))$.
- (This reduction requires to know: the center of the standard reductive group M is smooth!)
- Let $A = k[T] \subset K = k(T)$. For simplicity of exposition, we note here if the char. of k is 0, a proof of the Theorem can be given by studying the center of the centralizer of $X + TH$ in $\mathrm{Lie}(G) \otimes_k A$. We now sketch some of this argument.

Modules over a Dedekind domain

- Let A be a *Dedekind domain* – e.g. a *principal ideal domain*.
- For a maximal ideal $\mathfrak{m} \subset A$ and an A-module N, write $k(\mathfrak{m}) = A/\mathfrak{m}$, and $N(\mathfrak{m}) = N/\mathfrak{m}N = N \otimes_A k(\mathfrak{m})$,
- let K be the field of fractions of A and write $N_K = N \otimes_A K$.
- Let M be a fin. gen A-module. Then $M = M_0 \oplus M_{\text{tor}}$ where M_{tor} is torsion and M_0 is projective.

Homomorphisms (notation)

- Let $\phi : M \to N$ be an A-module homom where M and N are f.g. projective A-modules.
- let $P = \ker \phi$ and $Q = \operatorname{coker} \phi$.
- write $Q = Q_0 \oplus Q_{\text{tor}}$ as before.
- M/P is torsion free and thus projective, so for any max'l ideal $\mathfrak{m}$, we may view $P(\mathfrak{m})$ as a subspace of $M(\mathfrak{m})$.
- Write $\phi(\mathfrak{m}) : M(\mathfrak{m}) \to N(\mathfrak{m})$ for $\phi \otimes 1_{k(\mathfrak{m})}$.

Fibers of a kernel

Recall $\phi : M \to N$, $P = \ker \phi$, and $Q = \operatorname{coker} \phi$.

Theorem

(a) $P(\mathfrak{m}) \subset \ker \phi(\mathfrak{m})$, *with equality* $\iff$ $Q_{\mathrm{tor}} \otimes k(\mathfrak{m}) = 0$.

(b) $P(\mathfrak{m}) = \ker \phi(\mathfrak{m})$ *for all but finitely many* $\mathfrak{m}$.

- Pf of (a) uses the following fact: for a finitely generated A-module M

$$(\clubsuit) \quad \mathrm{Tor}^1_A(M, k(\mathfrak{m})) \simeq M_{\mathrm{tor}} \otimes k(\mathfrak{m})$$

.

- For (b), one just notes that Q_{tor} has *finite length*.
- If one knows that $\dim_{k(\mathfrak{m})} \ker \phi(\mathfrak{m})$ is equal to a constant d for all $\mathfrak{m}$ in some infinite set Γ of prime ideals, then $d = \dim_K \ker \phi(K)$.

Fibers of the center of an A-Lie algebra

- Let L be a Lie algebra over A which is f.g. projective as A-module.
- Let $Z = \{X \in L \mid [X, L] = 0\}$ be the center of L.

Theorem

(a) *L/Z is torsion free.*

(b) $\dim_{k(\mathfrak{m})} Z(\mathfrak{m})$ *is constant.*

(c) *For each maximal $\mathfrak{m} \subset A$, $Z(\mathfrak{m}) \subset \mathfrak{z}(L(\mathfrak{m}))$, and equality holds for all but finitely many $\mathfrak{m}$.*

- Here $\mathfrak{z}(L(\mathfrak{m}))$ means the center of the $k(\mathfrak{m})$-Lie algebra $L(\mathfrak{m})$.
- The result essentially follows from the result for kernels.

Center example

- Let $A = k[T]$ for alg. closed k, and identify maximal ideals of A with elements in k.
- let $L = Ae + Af$, with e and f an A-basis where $[e, f] = T \cdot f$.
- Now $Z(L) = 0$, and $\mathfrak{z}(L(t)) = 0$ for $t \neq 0$.
- But $L(0)$ is abelian, i.e $\mathfrak{z}(L(0)) = L(0)$.

Center of the centralizer

Return to the setting of even nilpotent $X \in \mathfrak{g}$.

- Write $D = \mathfrak{c}_{\mathfrak{g}_A}(X + T \cdot H)$.
- Write Z for the center of the A-Lie algebra D.
- And write $H = \mathfrak{g}^B \otimes A \subset L$.
- Ultimately, must argue that

$$(Z \cap H)(1) \subset \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(X)) \cap \mathfrak{g}^B$$

  while for almost all $t \neq 1$,

$$(Z \cap H)(t) = Z(t) = \mathfrak{c}_{\mathfrak{g}}(X + tH).$$

- This implies the “main result”.

Outline

Reductive group schemes

- Let $\mathcal{A}$ be a complete discrete valuation ring with field of fractions K and residue field k.
- Let $\mathcal{G}$ be a reductive $\mathcal{A}$-group scheme with connected fibers $\mathcal{G}_{\mathrm{K}}$ and $\mathcal{G}_{\mathrm{k}}$.
- The fibers $\mathcal{G}_{\mathrm{K}}$ and $\mathcal{G}_{\mathrm{k}}$ are reductive linear algebraic groups. The group scheme $\mathcal{G}$ is affine, smooth, and of finite type over $\mathcal{G}$.
- Since $\mathcal{G}$ is smooth over $\mathcal{A}$, Lie($\mathcal{G}$) is a projective (hence free) $\mathcal{A}$-module of finite rank.
- If $X \in$ Lie($\mathcal{G}$) and if X_{K} is nilpotent in Lie($\mathcal{G}_{\mathrm{K}}$), then also X_{k} is nilpotent, and we say that X is a *nilpotent section*.

Balanced sections

- Consider a $\mathcal{G}$-module $\mathcal{L}$ which is free of finite rank as $\mathcal{A}$-module.
- Given $X \in \mathcal{L}$, one can form the *scheme theoretic stabilizer* $C = \mathrm{Stab}_{\mathcal{G}}(X)$. Then C is a group scheme over $\mathcal{A}$, and we have

$$C_{\mathrm{K}} = \mathrm{Stab}_{\mathcal{G}_{\mathrm{K}}}(X_{\mathrm{K}}) \quad \text{and} \quad C_{\mathrm{k}} = \mathrm{Stab}_{\mathcal{G}_{\mathrm{k}}}(X_{\mathrm{k}}).$$

- We say that X is *balanced* for the action of $\mathcal{G}$ if C_{K} is smooth over K, if C_{k} is smooth over k, and if $\dim C_{\mathrm{K}} = \dim C_{\mathrm{k}}$.

Recognizing balanced sections

Proposition (McNinch (2016a))

Let $X \in \mathcal{L}$. Write $\mathfrak{g} = \mathrm{Lie}(\mathcal{G})$, and assume the following:

(a) *the $\mathcal{G}_{\mathrm{K}}$ orbit of X_{K} is smooth – i.e.* $\dim \mathrm{Stab}_{\mathcal{G}_{\mathrm{K}}}(X_{\mathrm{K}}) = \dim_{\mathrm{K}} \mathfrak{c}_{\mathfrak{g}_{\mathrm{K}}}(X_{\mathrm{K}})$, *and*

(b) $\dim_{\mathrm{K}} \mathfrak{c}_{\mathfrak{g}_{\mathrm{K}}}(X_{\mathrm{K}}) = \dim_{\mathrm{k}} \mathfrak{c}_{\mathfrak{g}_{\mathrm{k}}}(X_{\mathrm{k}})$.

Then X is balanced for the action of $\mathcal{G}$.

- The main points are: (i) $\dim C_{\mathrm{K}} \geq \dim C_{\mathrm{k}}$ by Chevalley's upper semicontinuity theorem, and (ii) smoothness on the generic fiber implies that $\dim C_{\mathrm{K}}$ coincides with the dimension of the stabilizer of x_{K} in $\mathfrak{g}_{\mathrm{K}}$.

Balanced nilpotent sections

- Now suppose that the fibers $\mathcal{G}_{\mathrm{K}}$ and $\mathcal{G}_{\mathrm{k}}$ are *standard reductive groups*, that $\mathcal{L} = \mathrm{Lie}(\mathcal{G})$ is the adjoint $\mathcal{G}$-module, and let $X \in \mathrm{Lie}(\mathcal{G})$.
- Then the centralizer in $\mathcal{G}_{\mathrm{K}}$ of X_{K} and the centralizer in $\mathcal{G}_{\mathrm{k}}$ of X_{k} are automatically smooth, so X is balanced if and only if the Lie algebraic centralizers on the fibers have the same dimension.

Existence and conjugacy of balanced nilpotent sections.

Theorem (McNinch (2016a))

Let $X_0 \in \mathrm{Lie}(\mathcal{G}_{\mathrm{k}})$ *nilpotent.*

(a) *There is a balanced, nilpotent section* $X \in \mathrm{Lie}(\mathcal{G})$ *s.t. that* $X_{\mathrm{k}} \in \mathrm{Lie}(\mathcal{G}_{\mathrm{k}})$ *coincides with* X_0.

(b) *There is an* $\mathcal{A}$*-homom* $\phi : \mathbf{G}_m \to \mathcal{G}$ *s.t.* $X \in \mathrm{Lie}(\mathcal{G})(\phi; 2)$, ϕ_{k} *is a cochar assoc with* X_{k} *and* ϕ_{K} *is a cochar assoc with* X_{K}.

(c) *Let* $X, X' \in \mathrm{Lie}(\mathcal{G})$ *be balanced nilpotent sections with* $X_{\mathrm{k}} = X'_{\mathrm{k}} = X_0$. *Then there is an element* $g \in \mathcal{G}(\mathcal{A})$ *such that* $X' = \mathrm{Ad}(g)X$.

- The "Bala-Carter data" of X_{K} and X_{k} are "the same".
- Using results of McNinch (2016b), the result is extended in McNinch (2016a) to so-called parahoric group schemes (under some further assumptions).

SL_2 over $\mathcal{A}$

Theorem

Let $X \in \mathrm{Lie}(\mathcal{G})$ be a balanced nilpotent section and let $\phi : \mathbf{G}_m \to \mathcal{G}$ be an $\mathcal{A}$-homomorphism such that $\phi_{\mathcal{F}}$ is a cocharacter associated to $X_{\mathcal{F}}$ for $\mathcal{F} \in \{\mathrm{k}, \mathrm{K}\}$. If $(X_{\mathrm{k}})^{[p]} = 0$, there is a unique $\mathcal{A}$-homomorphism

$$\Phi : \mathrm{SL}_{2/\mathcal{A}} \to \mathcal{M}$$

such that $d\Phi(E) = X$, and $\Phi_{|S} = \phi$, where $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{Lie}(\mathrm{SL}_{2,\mathcal{A}})$ and $S \simeq \mathbf{G}_{m,\mathcal{A}}$ is the diag torus of SL_2.

Smoothness

Theorem (Brian Conrad)

Let $\mathcal{H}$ be a group scheme of finite type over $\mathcal{A}$ for which the fibers $\mathcal{H}_{\mathrm{K}}$ and $\mathcal{H}_{\mathrm{k}}$ are each smooth of the same dimension. Then there is a locally closed subgroup scheme $\mathcal{M} \subset \mathcal{H}$ such that:

(a) *$\mathcal{M}$ is smooth, affine, and of finite type over $\mathcal{A}$,*

(b) *$\mathcal{M}_{\mathrm{K}} = (\mathcal{H}_{\mathrm{K}})^0$ and $\mathcal{M}_{\mathrm{k}} = (\mathcal{H}_{\mathrm{k}})^0$.*

Corollary

If $X \in \mathrm{Lie}(\mathcal{G})$ is balanced section, there is a locally closed subgroup scheme $\mathcal{M} \subset C = C_{\mathcal{G}}(X)$ such that:

- *$\mathcal{M}$ is smooth, affine and of finite type over $\mathcal{A}$, and*
- *$\mathcal{M}_{\mathrm{K}} = C^0_{\mathcal{G}_{\mathrm{K}}}(X_{\mathrm{K}})$ and $\mathcal{M}_{\mathrm{k}} = C^0_{\mathcal{G}_{\mathrm{k}}}(X_{\mathrm{k}})$*

Smoothness, continued

- In McNinch (2008), it was claimed that $C = C_{\mathcal{G}}(X)$ is smooth when X is balanced, but the argument is incorrect (it fails to justify why C is *flat* over $\mathcal{A}$).
- Results on the previous slide essentially fix the problem for the identity component C^0.
- However, with knowing the smoothness of the "full centralizer group scheme" C, the given arguments for McNinch (2008, Theorem B) are incorrect. That theorem concerns a comparison of the component groups $C_{\mathrm{K}}/C_{\mathrm{K}}^0$ and $C_{\mathrm{k}}/C_{\mathrm{k}}^0$. I don't know whether the conclusion of the Theorem is correct.

The reductive quotient of a nilpotent centralizer

Theorem (Theorem A of McNinch (2008))

Let $X \in \mathrm{Lie}(\mathcal{G})$ a balanced nilpotent section. The geom root datum of the reduc quotient of the conn centralizer $C^0_{\mathcal{G}_{\mathrm{K}}}(X_{\mathrm{K}})$ is the same as the geom root datum of the reduc quotient of $C^0_{\mathcal{G}_{\mathrm{k}}}(X_{\mathrm{k}})$.

- This proof can be found in McNinch (2016a).
- In fact, let $\phi : \mathbf{G}_m \to \mathcal{G}$ be an $\mathcal{A}$-homom s.t. ϕ_{K} is a cochar assoc to X_{K} and ϕ_{k} is a cochar assoc to X_{k}.
- And let $\mathcal{M} \subset C$ be the smooth locally closed subgp scheme of the Corollary above.
- Then the centralizer $L = C_{\mathcal{M}}(\phi)$ is a reductive subgroup scheme of $\mathcal{M}$ for which L_{K} is a Levi factor of $C^0_{\mathcal{G}_{\mathrm{K}}}(X_{\mathrm{K}})$ and L_{k} is a Levi factor of $C^0_{\mathcal{G}_{\mathrm{k}}}(X_{\mathrm{k}})$.
- Now use: $\mathcal{M}$ splits over some unramified extension of $\mathcal{A}$.

Outline

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