An overview of representations of reductive algebraic groups

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Linear algebraic groups and linear representations

Let k be a field, let G be a linear algebraic group over k, and consider a linear representation of G on a k-vector space V

- usually call V – or (ρ, V) – a G-module or G-representation.
- If V is fin. dim. the notation (ρ, V) implies a morphism $\rho : G \to \mathrm{GL}(V)$ of alg. groups. Thus V is a "rational repr".
- V is a co-module for the Hopf alg $k[G]$; i.e. $\exists$ a "co-module map" $\Delta_V : V \to k[G] \otimes_k V$ encoding action of G.
- The co-module point-of-view allows to speak of G-modules which are infinite dim'l; e.g. the left regular repr $(\rho_\ell, k[G])$.

Cohomology and extensions

Proposition

(a) *If W is a G-module, there is an injective resolution of G-modules:* $0 \to W \to I^0 \to I^1 \to I^2 \to \cdots$.

(b) *The "fixed-point functor"* $(V \mapsto V^G) : G\text{-mod} \to k\text{-Vect}$ *is left-exact*

- thus, take the cohomology groups $H^i(G, V)$ for $i \geq 0$ to be the derived functors of $V \mapsto V^G$
- The reg repr $k[G]$ is injective, and $\exists$ injective resolution of k with $I^j = \bigoplus^j k[G]$ a sum of j copies of $k[G]$.
- So, can describe cohomology using "regular functions" $G \times \cdots \times G \to V$ which satisfy usual "cocycle condition" of the cohomology of (abstract) groups.

Cohomology and extensions

- For a G-module V, $\mathrm{Ext}^i_G(V,-)$ are the derived functors of the left-exact functor $\mathrm{Hom}_G(V,-)$.
- For fin dim'l V, $\mathrm{Ext}^i_G(V,W) = H^i(G, \mathrm{Hom}_k(V,W))$.

Proposition

Applying $\mathrm{Hom}_k(C,-)$ *to the SES* $(\clubsuit) \quad 0 \to A \to B \to C \to 0$ *of* G*-modules yields a connecting map*

$$\mathrm{End}_G(C) \xrightarrow{\partial} \mathrm{Ext}^1_G(C,A)$$

and the extension $(\clubsuit)$ *is classified (up to isom. of extension) by the class* $\alpha = \partial(1_C)$*.*

Reduction to reductive groups

- Assume that the unip radical of $G_{\bar{k}}$ is defined over k – this is always the case if k is *perfect*; call this radical R. Thus G/R is a reductive k-group. Since R acts trivially on any simple G module, we have:

Theorem

The Grothendieck group of $G\text{-}\mathrm{mod}$ *coincides with that of* $(G/R)\text{-}\mathrm{mod}$.

Standing assumptions

- For the remainder of this talk, we will consider a (connected) split reductive group G over a field k of characteristic $p \geq 0$.
- In particular, we fix a Borel subgroup $B \subset G$ with unipotent radical U, and a split maximal torus $T \subset B$.

Outline

The "big cell" and triangular decomposition

- Write $\mathcal{B} = G/B$ for flag variety – "variety of Borel subgps of G" – and let $\pi : G \to \mathcal{B}$ be the orbit map $g \mapsto gBg^{-1}$.
- Let $T \subset B^+$ Borel subgp "opposite" to B.
- Roots of T in $\mathrm{Lie}(B)$ are *neg*, those in $\mathrm{Lie}(B^+)$ *pos*.
- Recall $W = N_G(T)/T$ is "the" Weyl group of G. So $B^+ = \tilde{w}_0^{-1} B \tilde{w}_0$ for $\tilde{w}_0 \in N_G(T)$ repr the "long word" $w_0 \in W$.

Proposition

$\tilde{w}_0^{-1} B \tilde{w}_0 B = U^+ B$ is an open subset of G, and the product mapping yields an isom of varieties

$$U^+ \times T \times U \to U^+ B.$$

Triangular decomp of $U(\mathrm{Lie}(G))$

Since U^+B is an open subset of G containing 1, we obtain:

$$\mathfrak{g} = \mathfrak{u}^+ \oplus \mathfrak{t} \oplus \mathfrak{u},$$

where $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{u} = \mathrm{Lie}(U)$ etc.
This implies that

$$\mathbf{U}(\mathfrak{g}) = \mathbf{U}(\mathfrak{u}^+) \otimes \mathbf{U}(\mathfrak{t}) \otimes \mathbf{U}(\mathfrak{u})$$

where $\mathbf{U}(?)$ denotes the enveloping alg, and if $p > 0$ it implies that

$$\mathbf{U}^{[p]}(\mathfrak{g}) = \mathbf{U}^{[p]}(\mathfrak{u}^+) \otimes \mathbf{U}^{[p]}(\mathfrak{t}) \otimes \mathbf{U}^{[p]}(\mathfrak{u})$$

where $\mathbf{U}^{[p]}(?)$ denotes restricted env alg. ; i.e. the quotient of the enveloping algebra by ideal generated by all $X^{[p]} - X^p$.

The algebra of distributions

For a linear algebraic group H, the algebra $\mathrm{Dist}(H)$ consists of all linear mapping $\lambda : k[H] \to k$ such that $\lambda_{|\mathfrak{m}^e} = 0$ for some $e \geq 1$, where $\mathfrak{m}$ is the kernel of the augmentation homomorphism – i.e. the maximal ideal corresponding to the identity element of H.

- The group structure of H gives $\mathrm{Dist}(H)$ the structure of an *algebra*.
- If the char. of k is 0, $\mathrm{Dist}(H)$ may be identified with $\mathbf{U}(\mathfrak{h})$.
- If the char. of k is $p > 0$, there is an embedding $\mathbf{U}^{[p]}(\mathfrak{h}) \to \mathrm{Dist}(H)$.
- For the reductive group G, again have triangular decomp

$$\mathrm{Dist}(G) = \mathrm{Dist}(U^+) \otimes \mathrm{Dist}(T) \otimes \mathrm{Dist}(U).$$

Distributions on $\mathbf{G}_a$ and $\mathbf{G}_m$

- The additive gp $\mathbf{G}_a$ and the mult gp $\mathbf{G}_m$ arise by base change from smooth gp schemes $\mathbf{G}_{a,\mathbf{Z}}$ and $\mathbf{G}_{m,\mathbf{Z}}$ over $\mathbf{Z}$.
- Fix basis vectors

$$H \in \mathrm{Lie}(\mathbf{G}_{m,\mathbf{Q}}) \quad \text{and} \quad X \in \mathrm{Lie}(\mathbf{G}_{a,\mathbf{Q}}),$$

  which we view as elts of $\mathrm{Dist}(\mathbf{G}_{m,\mathbf{Q}})$ resp. $\mathrm{Dist}(\mathbf{G}_{a,\mathbf{Q}})$. These choices determine $\mathbf{Z}$-forms of $\mathbf{G}_{a,\mathbf{Q}}$ and $G_{m,\mathbf{Q}}$ with:

$$\mathrm{Dist}(\mathbf{G}_{m,\mathbf{Z}}) = \sum_{r \geq 0} \mathbf{Z}\binom{H}{r} \subset \mathrm{Dist}(\mathbf{G}_{m,\mathbf{Q}}), \quad \text{and}$$

$$\mathrm{Dist}(\mathbf{G}_{a,\mathbf{Z}}) = \sum_{r \geq 0} \mathbf{Z}X^{(r)} \subset \mathrm{Dist}(\mathbf{G}_{a,\mathbf{Q}}) \quad \text{where} \quad X^{(r)} = \frac{X^r}{r!}$$

- $\mathrm{Dist}(\mathbf{G}_m) = \mathrm{Dist}(\mathbf{G}_{m,\mathbf{Z}}) \otimes k$ and $\mathrm{Dist}(\mathbf{G}_a) = \mathrm{Dist}(\mathbf{G}_{a,\mathbf{Z}}) \otimes k$

More on distributions

- Let Φ be the roots of G (non-zero weights of T in $\mathrm{Lie}(G)$; see later). Then $U = \prod_{\alpha<0} U_\alpha$ for root subgroups U_α, and

$$\mathrm{Dist}(U) = \sum_{\vec{n} \geq \vec{0}} k \prod_{\alpha<0} X_\alpha^{(n_\alpha)}.$$

- If G is semisimple, $\mathrm{Dist}(G)$ may be identified with $\mathbf{U}_\mathbf{Z} \otimes k$ where $\mathbf{U}_\mathbf{Z}$ is the Kostant $\mathbf{Z}$-form of $\mathbf{U}(\mathfrak{g}_\mathbf{Q})$ for a split semisimple Lie algebra $\mathfrak{g}_\mathbf{Q}$ over $\mathbf{Q}$ with root sys of G.

Theorem

Let M, N be G-modules, and let $M' \subset M$ a k-subspace.

(a) M' *is a* G*-submodule* $\iff$ M' *is a* $\mathrm{Dist}(G)$*-submodule.*

(b) $\mathrm{Hom}_G(M, N) = \mathrm{Hom}_{\mathrm{Dist}\, G}(M, N)$.

Outline

Flag variety of a split reductive group

Proposition

$\mathcal{B} = G/B$ is a projective algebraic variety over k, called the flag variety of G.

Write $X = X^*(T) = \mathrm{Hom}(T, \mathbf{G}_m) \simeq \mathbf{Z}^{\dim T}$ for the lattice of *characters* of T.

Proposition

There is a bijection $\lambda \mapsto \mathcal{L}(\lambda)$ between X and the collection of G-linearized invertible sheaves on $\mathcal{B}$.

- Explicitly, λ determines a B-module k_λ, and $\mathcal{L}(\lambda)$ is the sheaf of sections of the line bundle $G \times^B k_\lambda \to G/B$.
- This bijection depends on the choice of B containing T.

Flag variety and standard modules

Induction and global sections

Can form $\mathrm{ind}_B^G(-) = \Gamma(\mathcal{B}, \mathcal{L}(-))$, a left-exact functor with right derived functors $R^i \mathrm{ind}_B^G(?) = H^i(?)$. We write $H^0(\lambda) = \mathrm{ind}_B^G(\lambda)$.

Frobenius recip in this context implies:

Proposition

For $\lambda \in X$, $H^0(\lambda) = H^0(\mathcal{B}, \mathcal{L}(\lambda))$ has the universal mapping property: for a G-module M, there is a natural isomorphism $\mathrm{Hom}_B(M_{|B}, k_\lambda) \simeq \mathrm{Hom}_G(M, H^0(\lambda))$.

Terminology

The $H^0(\lambda)$ are known as *standard modules*.

Flag variety and standard modules

Proposition

If $H^0(\lambda) \neq 0$*, then* $H^0(\lambda)^{U^+} = H^0(\lambda)_\lambda$ *is 1 dimensional.*

Sketch

An elt of $H^0(\lambda)^{U^+}$ *corresponds to* $f \in k[G]$ *for which*

$$f(u_1 t u_2) = \lambda(t)^{-1} f(1)$$

for $t \in T$*,* $u_1 \in U^+$*,* $u_2 \in U$*.*
Now the result follows since U^+B *is dense in* G*.*

Dominant weights

Proposition

$H^0(\lambda) \neq 0 \iff \lambda$ is dominant; i.e. $\langle \lambda, \alpha^\vee \rangle \geq 0$ for $\alpha \in \Phi^+$.

Sketch

$\Rightarrow$: The main point is to argue that any weight μ of $H^0(\lambda)$ satisfies $w_0\lambda \leq \mu \leq \lambda$. Dominance follows by considering the weight $s_\alpha\lambda$ of $H^0(\lambda)$ for simple roots α.
$\Leftarrow$: Consider the regular function f_λ on U^+B defined by $u_1 t u_2 \mapsto \lambda(t)^{-1}$. Must argue that λ dominant $\implies f_\lambda$ can be extended to a regular function on G. Enough to argue can extend f_λ over the open subvarieties $\dot{s}_\alpha U^+ \dot{B}$ for simple α, since the complement of $U^+B \cup \bigcup_{\alpha\in\Delta} \dot{s}_\alpha U^+ B$ has codimension ≥ 2 and G is a normal variety.

Simple G-modules

Proposition

(a) *$L(\lambda) = \operatorname{soc} H^0(\lambda)$ is a simple G-module $\forall$ dominant λ.*

(b) *The simple G-modules are precisely the $L(\lambda)$.*

Sketch

(a) $L_1, L_2 \subset H^0(\lambda)$ distinct irr submods $\implies \dim H^0(\lambda)^{U^+} \geq 2$.
(b) L irr $\implies L^{U^+}$ non-0 and T-inv so UMP gives $L \subset H^0(\lambda)\ \exists\lambda$

Relation to char. 0

Let λ a dominant weight.

Proposition

If char. k is 0, $H^0(\lambda) = L(\lambda)$ is simple as G- and $\mathfrak{g}$-module

Proposition

For any k, the formal character $\chi(\lambda) = \mathrm{ch}(H^0(\lambda))$ - an elt of $\mathbf{Z}[X]^W$ - of is given by Weyl's character formula.

Remark

For $\mu \in X$, we set $\chi(\mu) = \sum_{i \geq 0} (-1)^i \,\mathrm{ch}\, H^i(\mu)$. Then $\chi(w \bullet \mu) = (-1)^{\ell(w)} \chi(\mu)$ for the "dot-action" of $w \in W$. The Borel-Bott-Weil Thm implies $\chi(\lambda) = \mathrm{ch}\, H^0(\lambda)$ for dom λ.

Properties of standard and Weyl modules

- if $\lambda \not< \mu$, $\mathrm{Ext}^1_G(L(\mu), L(\lambda)) \simeq \mathrm{Hom}_G(L(\mu), H^0(\lambda)/L(\lambda))$
- can dualize: set $V(\lambda) = H^0(-w_0\lambda)^\vee$
- the $V(\lambda)$ are known as "Weyl modules"
- any length 2 indecomposable is either a quotient of a Weyl module, or a submodule of a standard module.

Proposition

For dominant λ, μ,

$$\mathrm{Ext}^i_G(V(\lambda), H^0(\mu)) = \begin{cases} 0 & \text{if } i \neq 0 \text{ or } \lambda \neq \mu \\ k & \text{if } i = 0 \text{ and } \lambda = \mu. \end{cases}$$

Outline

Integral forms

Let $\mathcal{A}$ be a DVR with valuation ν, residues $\mathcal{A}/\pi\mathcal{A} = k$ and fractions K of char. 0. e.g. $\mathcal{A} = \mathbf{Z}_{(p)}$, $\pi = p$, $k = \mathbf{F}_p$ and $K = \mathbf{Q}$.

Proposition

There is a split reductive group scheme $\mathcal{G} = G_{\mathcal{A}}$ over $\mathcal{A}$ with $G = \mathcal{G}_k$ for which $G_K = \mathcal{G}_K$ is a split reductive group over K having "the same root datum" as G.

Sketch

Take G_K split reduc with specified root datum, and use choice of a Chev. basis in $\mathfrak{g}_K$ to construct $\mathcal{A}$-forms of $T_K \simeq \prod \mathbf{G}_m$ and each $U_{\alpha,K} \simeq \mathbf{G}_a$. These $\mathcal{A}$-forms determine

$$\mathrm{Dist}(\mathcal{G}) = \mathrm{Dist}(U^+_{\mathcal{A}}) \otimes_{\mathcal{A}} \mathrm{Dist}(T_{\mathcal{A}}) \otimes_{\mathcal{A}} \mathrm{Dist}(U_{\mathcal{A}}).$$

And now $\mathcal{A}[\mathcal{G}] = \{f \in K[G_K] \mid \mu(f) \in \mathcal{A}\ \forall \mu \in \mathrm{Dist}(\mathcal{G})\}$.

Some $\mathcal{G}$-modules

Proposition

If V is a fin dim G_K-module and $M \subset V$ an $\mathcal{A}$-lattice, then M is $\mathcal{G}$-stable $\iff \mathrm{Dist}(\mathcal{G})M = M$.

Constructions

- Let $H_K^0(\lambda)$ simple G_K-module, fix $0 \neq v_0 \in H_K^0(\lambda)_\lambda$, and consider $V_\mathcal{A}(\lambda) = \mathrm{Dist}(\mathcal{G})v_0$.
- $\lambda \in X$ determines $B_\mathcal{A}$-mod $\mathcal{A}_\lambda$; now form $\mathcal{G}$-module

$$H^i_\mathcal{A}(\lambda) = R^i \operatorname{ind}_{B_\mathcal{A}}^{\mathcal{G}} \mathcal{A}_\lambda = H^i(\mathcal{G}/B_\mathcal{A}, \mathcal{L}_\mathcal{A}(\lambda)), \quad i \geq 0.$$

Induced modules over $\mathcal{A}$

Proposition

(a) $H^0_{\mathcal{A}}(\lambda)$ *is a* $\mathcal{G}$*-stable lattice in* $H^0_K(\lambda)$

(b) $H^0(\lambda) = H^0_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} k$ *as* $G = \mathcal{G}_k$*-module.*

(c) *Let* $n = |\Phi^+| = \ell(w_0)$*. Then*

$$H^n_{\mathcal{A}}(w_0 \bullet \lambda) \simeq V_{\mathcal{A}}(\lambda) \simeq H^0_{\mathcal{A}}(-w_0\lambda)^\vee.$$

In particular, $V(\lambda) = V_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} k$ *as* G*-modules.*

Sketch

Part (c) depends in part on "Serre duality": namely,

$$H^i(\lambda) \simeq H^{n-i}(-(\lambda + 2\rho))^\vee.$$

Interlude on lattices

Let M, M' fin. gen $\mathcal{A}$-modules and $\phi : M \to M'$ $\mathcal{A}$-linear s.t. $\phi_K : M \otimes_{\mathcal{A}} K \to M' \otimes_{\mathcal{A}} K$ bijective.

- Put $\nu(\phi) = \ell(\operatorname{coker} \phi)$ where $\ell(?)$ denotes length of $\mathcal{A}$-module.
- If ϕ is morphism of $T_{\mathcal{A}}$-modules, put $\nu^c(\phi) = \sum_\mu \nu(\phi_{|M_\mu} : M_\mu \to M'_\mu) e(\mu) \in \mathbf{Z}[X]$.

Proposition

Suppose M,M' are torsion free, put $M^i = \{x \in M \mid \phi(x) \in \pi^i M'\}$, and let $\overline{M^i}$ image of M^i in $\overline{M} = M/\pi M = M \otimes_{\mathcal{A}} k$. Then:

(a) $\sum_{i>0} \dim_k \overline{M^i} = \nu(\phi)$.

(b) *if ϕ is map of $T_{\mathcal{A}}$-modules, then $\sum_{i>0} \operatorname{ch}(\overline{M^i}) = \nu^c(\phi)$.*

The Jantzen sum formula

Theorem

There is a $\mathcal{G}$-morphism $T : V_{\mathcal{A}}(\lambda) = H^n_{\mathcal{A}}(w_0 \bullet \lambda) \to H^0_{\mathcal{A}}(\lambda)$. The resulting filtration $\{V^i(\lambda)\}$ of $V(\lambda) = \overline{V_{\mathcal{A}}(\lambda)}$ satisfies

(a) $V(\lambda)/V^1(\lambda) \simeq L(\lambda)$, *and*

(b) $\sum_{i>0} \mathrm{ch}\, V^i(\lambda) = \nu^c(T) = \sum_{\alpha>0} \sum_{i=1}^{\langle \lambda+\rho, \alpha^\vee \rangle - 1} \nu(i) \cdot \chi(\lambda - i\alpha)$.

Remark

In fact, since $\mathrm{Ext}^1_G(V(\lambda), H^0(\lambda)) = 0$,
$\mathrm{Hom}_{\mathcal{G}}(V_{\mathcal{A}}(\lambda), H^0_{\mathcal{A}}(\lambda)) \simeq \mathcal{A} \cdot T$, so T is ! determ up to $\mathcal{A}^\times$.
But the formula in (b) depends on an explicit description of T obtained by choosing a reduced decomposition of w_0.

Examples

Notation

Let $\mathcal{L}$ be an $\mathcal{A}$-lattice in $V_K = \mathcal{L} \otimes_{\mathcal{A}} K$. Suppose that γ is a non-degenerate bilinear form on V_K.

- $\mathcal{L}^* = \{x \in V_K \mid \gamma(x, \mathcal{L}) \subset \mathcal{A}\}$ is again an $\mathcal{A}$-lattice.
- If $\beta(\mathcal{L}, \mathcal{L}) \subset \mathcal{A}$, then $\mathcal{L} \subset \mathcal{L}^*$.

Examples: some representations

Symplectic group and $\lambda = \varpi_2$

- Let $G = \mathrm{Sp}(V, \beta)$ with $\dim_k V = 2\ell$, and suppose $p \neq 2$.
- Let $V_K = H^0_K(\varpi_1)$ and fix G_K-invt non-deg form on $\bigwedge^2 V_K$.
- Then $\bigwedge^2 V_K = K\beta \oplus \beta^\perp$ as G_K-modules, and $\beta^\perp \simeq H^0_K(\varpi_2)$.
- sum formula gives $\sum_{i>0} \mathrm{ch}\, V^i(\varpi_2) = \nu(\ell)\chi(0)$.

Special Linear group and $\lambda = \varpi_1 + \varpi_\ell$

- Let $G = \mathrm{SL}(V)$ with $\dim_k V = \ell + 1$.
- Let $V_K = H^0_K(\varpi_1)$ and consider trace form on $\mathrm{End}_K(V_K)$.
- Then $\mathrm{End}_K(V_K) = K\,\mathrm{Id} \oplus \mathrm{Id}^\perp$, and $\mathrm{Id}^\perp \simeq H^0_K(\varpi_1 + \varpi_\ell)$.
- sum formula gives $\sum_{i>0} \mathrm{ch}\, V^i(\varpi_1 + \varpi_\ell) = \nu(\ell + 1)\chi(0)$.

Examples: Lattice descriptions

Common features

For both $\mathrm{Sp}(V)$ and $\mathrm{SL}(V)$, have $\mathcal{L} = V_{\mathcal{A}}(\varpi_1) = H^0_{\mathcal{A}}(\varpi_1) \subset V_K$.
May s'pose $\bigwedge^2 \mathcal{L} = (\bigwedge^2 \mathcal{L})^*$, resp. $\mathrm{End}_{\mathcal{A}}(\mathcal{L}) = \mathrm{End}_{\mathcal{A}}(\mathcal{L})^*$.

Proposition

(a) *For* $\mathcal{G} = \mathrm{Sp}(\mathcal{L})$ *and* $\lambda = \varpi_2$,

$$V_{\mathcal{A}}(\lambda) = \bigwedge^2 \mathcal{L} \cap \beta^{\perp} \quad \textit{and} \quad V_{\mathcal{A}}(\lambda) = H^0_{\mathcal{A}}(\lambda)^*.$$

Moreover, $H^0_{\mathcal{A}}(\lambda)/V_{\mathcal{A}}(\lambda) \simeq \mathcal{A}/\ell\mathcal{A}$.

(b) *For* $\mathcal{G} = \mathrm{SL}(\mathcal{L})$ *and* $\lambda = \varpi_1 + \varpi_\ell$,

$$V_{\mathcal{A}}(\lambda) = \mathrm{End}_{\mathcal{A}}(\mathcal{L}) \cap \mathrm{Id}^{\perp} \quad \textit{and} \quad H^0_{\mathcal{A}}(\lambda) = V_{\mathcal{A}}(\lambda)^*.$$

Moreover $H^0_{\mathcal{A}}(\lambda)/V_{\mathcal{A}}(\lambda) \simeq \mathcal{A}/(\ell+1)\mathcal{A}$.

Examples: conclusion

Proposition

(a) *If* $G = \mathrm{Sp}(V)$*,* $V(\varpi_2)$ *is simple unless* $p \mid \ell$*, in which case* $\mathrm{rad}\, V(\varpi_2) = k$ *is 1 dimensional.*

(b) *If* $G = \mathrm{SL}(V)$*,* $V(\varpi_1 + \varpi_\ell)$ *is simple unless* $p \mid \ell + 1$*, in which case* $\mathrm{rad}\, V(\varpi_1 + \varpi_\ell) = k$ *is 1 dimensional.*

Remark

By itself, the sum formula doesn't quite yield the preceding facts e.g. if $\ell = p^3$ resp. $\ell + 1 = p^3$.