SEMISIMPLICITY IN POSITIVE CHARACTERISTIC

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1. Introduction

The purpose of this note is to provide a description of some recent results conerning semisimple modules in the representation theory of groups over fields of positive characteristic p. These results share some common features. Each gives conditions for representations to be semisimple, or completely reducible. In each case, the conditions are somehow related to the vector space dimension of the representation.

1.1. REPRESENTATIONS OF GROUPS AND SEMISIMPLICITY

The study of finite dimensional representations V for a group G, say over the field k, has been of considerable interest in mathematics. Of course, the simple representations play an important role in this study; one obtains an invariant of an arbitrary representation by taking the list of simple modules making up its composition factors.

Semisimple representations of G are precisely those which are direct sums of their composition factors. In an important sense, they have the most transparent structure. Understanding the semisimple representations is thus a logical first step in the study of representation theory.

There are some well-known cases in which certain categories of representations are semisimple; we recall a few here.

Theorem 1.1 *(Mashke) If char$(k) = 0$ and G is finite, the algebra kG is semisimple.*

Theorem 1.2 *If G is a compact connected Lie group, the category of finite dimensional continuous representations is semisimple. If G is a connected, reductive algebraic group over the algebraically closed field k of characteristic 0, the category of rational representations is semisimple.*

In contrast, if k is a field of characteristic $p > 0$, and p divides the order of the finite group G, then kG is *never* semisimple. If in addition k is algebraically closed, and G is a reductive algebraic group over k with positive semisimple rank, the category of rational representations is *never* semisimple.

1.2. ALGEBRAIC GROUPS IN POSITIVE CHARACTERISTIC

Denote by k a field of characteristic p, and by G a reductive algebraic k group. We describe some of the essential features of the representation theory of G. We give as a reference [5], especially II.1-6.

Fix $T \subset B$ a "borus" (i.e. a maximal torus contained in a Borel subgroup of G). Let $X = \mathrm{Hom}(T, k^{\times})$ denote the character group of T. Any T-module M is the direct sum of *weight spaces* M_λ for $\lambda \in X$; T acts through the character λ on M_λ. The non-zero weights of the adjoint module $\mathbf{L} = \mathrm{Lie}(\mathbf{G})$ form the root system $\Phi \subset X$.

The weight lattice X is isomorphic to $\mathbf{Z}^\ell$ where $\ell = \dim(T)$. The choice of Borel determines a system of positive roots in Φ and a basis $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of simple roots. Let $\varpi_1, \ldots, \varpi_\ell$ be the corresponding fundamental dominant weights.

Let X_+ denote the dominant region of X. For each weight $\lambda \in X_+$, there are several important indecomposable highest weight representations of G:

$$\begin{array}{ll} L(\lambda) & \text{the simple module} \\ \nabla(\lambda) & \text{the standard module, with socle } L(\lambda) \\ \Delta(\lambda) & \text{the Weyl module, with head } L(\lambda) \end{array} \tag{1}$$

The modules $L(\lambda)$ exhaust all of the simple rational G modules. When $p = 0$, all three of these high weight modules coincide; assume for now on that $p > 0$.

In this paper, our interest is in identifying semisimple modules. To proceed with this study, one needs some understanding of the possible extensions between simple G modules. Recall that (equivalence classes of) extensions between L and L' are parameterized by the group $\mathrm{Ext}^1_G(L, L')$.

In the current setting, take $L = L(\lambda)$, and $L' = L(\lambda')$. Assume that $\lambda \not\leq \lambda'$. Then

$$\begin{array}{rcl} \mathrm{Ext}^1_G(L, L') & \simeq & \mathrm{Hom}_G(\mathrm{rad}\Delta(\lambda), L') \\ & \simeq & \mathrm{Hom}_G(L', \nabla(\lambda)/\mathrm{soc}\nabla(\lambda)) \end{array} \tag{2}$$

Let $\mathbf{C}$ denote the *lowest dominant alcove* for the action of the affine Weyl group $\mathcal{W}_p$ on X. The *linkage principle* (see [5], II.) implies that whenever

$\lambda \in \mathbf{C}$, $\Delta(\lambda)$ is simple. In particular, (2) implies:

$$\mathrm{Ext}^1_G(L(\lambda), L(\lambda')) = 0 \text{ for } \lambda, \lambda' \in \mathbf{C} \tag{3}$$

Let $\rho \in X_+$ denote the half sum of the positive roots, and let α_0 denote the short root in Φ of maximal height. For a dominant weight $\lambda \in X_+$, one has

$$\lambda \in \mathbf{C} \text{ just when } \langle \lambda + \rho, \alpha_0\check{}\,\rangle < p \tag{4}$$

Fix $m \geq 1$, and let $X_m = \{\lambda \in X_+ \mid \langle \lambda, \alpha_i \rangle < p^m \text{ for } 1 \leq i \leq \ell\}$. Of particular importance is the set X_1, the so-called *restricted* weights. An arbitrary dominant weight $\lambda \in X_+$ has a p-adic expansion $\lambda = \lambda_0 + \lambda_1 p + \cdots + \lambda_n p^n$ with all $\lambda_i \in X_1$. When λ is expressed in this way, Steinberg's tensor product theorem describes the simple module $L(\lambda)$:

$$L(\lambda) \simeq L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \cdots \otimes L(\lambda_n)^{[n]} \tag{5}$$

where the exponent $[i]$ indicates the i-fold Frobenius twist of the representation.

1.3. FINITE GROUPS OF LIE TYPE

Let k and G be as in 1.2; we keep the assumption that $p > 0$. Assume that G is defined over the finite field $\mathbf{F}_p$, and furthermore assume that G is split over $\mathbf{F}_p$, i.e. G contains a maximal torus T defined over $\mathbf{F}_p$. Let F be the Frobenius endomorphism of G. Consider $q = p^r$ for $r \geq 1$, and fix a possibly trivial diagram automorphism σ of G (determined by a corresponding automorphism of the root system Φ). The corresponding finite group of Lie type $G(\mathbf{F}_q)$ is defined as the fixed points in G of the map $F^r \circ \sigma$.

The rational representation theory of G is important in the study of representations of $G(\mathbf{F}_q)$ over k. Indeed, the simple $kG(\mathbf{F}_q)$ modules are precisely the restrictions of the simple rational G modules $L(\lambda)$ for $\lambda \in X_r$. However, the extension theory of these simple modules over the finite group $G(\mathbf{F}_q)$ is not in general determined by the extension theory over G; this topic is discussed a bit more below in 3.5.

2. Dimensional Results

2.1. SERRE'S TENSOR PRODUCT RESULT

In this section, G is any group. Given two finite dimensional representations V and W over k of the group G, one can form the tensor product representation $V \otimes_k W$. If k has characteristic zero, the tensor product is well behaved with respect to semisimplicity: namely, if V and W are semisimple then $V \otimes W$ is semisimple ([3], p. 88).

If the characteristic of k is $p > 0$, the tensor product is not well behaved in the above sense. Indeed, for $G = SL(V)$, a semisimple group of type A_ℓ, every Weyl module $\Delta(\lambda)$ ($\lambda \in X_+$) appears as a section of a tensor power $V^{\otimes r}$ of the natural representation V, but in general Weyl modules aren't semisimple.

However, J.-P. Serre ([11], Théorème 1) has established the following condition for semisimplicity:

Theorem 2.1 *(Serre) Assume that k has characteristic $p > 0$, and that V and W are semisimple. If $dim_k V + dim_k W < p + 2$, then $V \otimes W$ is again semisimple.*

Remark 1 There are examples showing that the bound in Theorem 2.1 is sharp; see [11], 1.3.

Remark 2 Serre, in [11], Thèoréme 2, has established a similar result establishing the semisimplicity of $\wedge^2(V)$ for suitably small semisimple V. Furthermore, in [12], partial converses to these results are established.

Remark 3 Roughly speaking, the result is established in two stages. Serre first proves the result when G is a connected almost simple algebraic group and V and W are restricted simple rational modules for G. The extension of this result to connected algebraic groups, and even algebraic groups with $(G : G^o)$ prime to p, is then relatively straightforward.

The extension to arbitrary G involves the process of *saturation*; the reader is referred to [11], §4.

2.2. JANTZEN'S RESULT

In this section, G denotes a reductive algebraic k group defined over $\mathbf{F}_p$, with $G(\mathbf{F}_q)$ the group of $\mathbf{F}_q$ rational points (where $q = p^r$) for the possibly twisted Frobenius map as in 1.3.

The following theorem was established by Jantzen in [6].

Theorem 2.2 *(Jantzen)*

1. *Let V be a rational G-module with $dim_k V \leq p$. Then V is semisimple.*
2. *Let V be a module for $kG(\mathbf{F}_q)$ with $dim_k V \leq p$. If the root system of G has no component A_1, then V is completely reducible.*

Remark 4 Part (a) of Theorem 2.2 was conjectured by M. Larsen; in [7], Larsen established a similar result but with a weaker bound.

2.3. RESULTS INVOLVING THE RANK

In this section, we take for G an almost simple algebraic k group. In this context, we can improve on Theorem 2.2. We first develop some preliminaries.

For almost simple G, the root system Φ is irreducible, say of rank ℓ. So Φ is of type A, B, $\dots, G$. Let $\tilde{\alpha}, \alpha_0 \in \Phi_+$ be the long, respectively short, root of maximal height, and put

$$\mathcal{C} = \mathcal{C}(\Phi) = \max\left\{\frac{|\mathcal{W}\tilde{\alpha}|}{2}, \frac{|\mathcal{W}\alpha_0|}{2}\right\}, \tag{6}$$

where $\mathcal{W}$ is the Weyl group of Φ.

Computation of the quantity $\mathcal{C}$ is straightforward; the resulting values are listed in Table 1.

We are concerned with modules of small dimension which fail to be semisimple. The following propostion records some interesting cases (see [9], Proposition 5.1.1).

Proposition 2.1 *For each pair $\{\xi, \zeta\}$ in Table 2, there is an indecomposable module $E = E(\xi, \zeta)$ of length two with composition factors $L(\xi)$ and $L(\zeta)$ whose dimension is given in the table. In each case, $Ext^1_G(L(\xi), L(\zeta))$ is 1 dimensional. Furthermore, for each pair there exist ℓ and p so that $dim_k E \leq \mathcal{C}p$.*

We remark that the modules $E(\xi, \zeta)$ are explicitly constructed. For example, let $\Phi = B_\ell$, and for simplicity assume $p \neq 2, 3$. The simply connected group G has as a quotient the special orthogonal group of a space V. The symmetric power S^3V contains a submodule isomorphic to V. When $2\ell + 3 \equiv 0 \pmod p$, the quotient S^3V/V is the indecomposable length two module $E(\varpi_1, 3\varpi_1)$. Most of the other modules $E(\xi, \zeta)$ have similar constructions; see [9], section 4.

The following results are due to the author; see [9], Theorem 1 and Corollary 1.1.1.

Theorem 2.3 *Let G be an almost simple algebraic group, and let V be a rational G module with $dim_k V \leq \mathcal{C} \cdot p$. If V is not semisimple, then V has a sub-quotient isomorphic to a Frobenius twist of one of the indecomposable modules described in Proposition 2.1.*

Inspection of the indecomposalbe modules $E(\xi, \zeta)$ shows that all have dimension at least ℓp. We obtain thus the following:

Corollary 2.1 *Suppose that G is an almost simple group and that V is a rational G module. If $dim_k V \leq \ell p$, then V is semisimple.*

We can also give a "rank based" improvement on Theorem 2.2, (2). The next result is obtained in [8].

Theorem 2.4 *Let $q = p^r$, where p is a prime number. Let $G(\mathbf{F}_q)$ be a finite group of Lie type as above. Let V be a $kG(\mathbf{F}_q)$ module with $dim_k V \leq \ell p$. If (Φ, p, r) does not appear on Table 3, then V is semisimple.*

Remark 5 In 3.5, we discuss the necessity of restricting the triples (Φ, p, r) in Table 3.

Table 1. Definition of $\mathcal{C}$

Type	$\mathcal{C}$
A_ℓ	$\binom{\ell+1}{2}$
B_ℓ, C_ℓ, D_ℓ	$2\binom{\ell}{2} = \ell(\ell-1)$
E_6	36
E_7	63
E_8	120
F_4	12
G_2	3

Table 2. Small Indecomposable Modules

Φ	$\{\xi, \zeta\}$	Condition	$\dim_k E$
A_ℓ	$\{\varpi_1 + \varpi_\ell, 0\}$	$\ell + 1 \equiv 0 \pmod p$	$\ell(\ell+2)$
A_ℓ	$\{2\varpi_1, \varpi_2\}$ or $\{2\varpi_\ell, \varpi_{\ell-1}\}$	$p = 2$	$\binom{\ell+2}{2}$
B_ℓ	$\{2\varpi_1, 0\}$	$2\ell + 1 \equiv 0 \pmod p$	$\binom{2\ell+2}{2} - 1$
B_ℓ	$\{3\varpi_1, \varpi_1\}$	$2\ell + 3 \equiv 0 \pmod p, p \neq 3$	$\binom{2\ell+3}{3} - 2\ell - 1$
B_ℓ	$\{\varpi_1, 0\}$	$p = 2$	$2\ell + 1$
C_ℓ	$\{\varpi_2, 0\}$	$\ell \equiv 0 \pmod p$	$\binom{2\ell}{2} - 1$
C_ℓ	$\{2\varpi_1, 0\}$	$p = 2$	$2\ell + 1$
D_ℓ	$\{2\varpi_1, 0\}$	$\ell \equiv 0 \pmod p, p \neq 2$	$\binom{2\ell+1}{2} - 1$
E_6	$\{\varpi_2, 0\}$	$p = 3$	78
F_4	$\{\varpi_4, 0\}$	$p = 3$	26

Table 3. Restrictions on q in the finite case.

Φ		p	r
A_1		any	any
A_2		2	1
A_ℓ	$\ell \geq 2$	3	$1, 2$
C_ℓ	$\ell \geq 2$	5	1
B_2		5	1

3. Techniques and Observations

In this section, we describe some of the techniques involved in obtaining the results of section 2. Our goal is not to describe fully the proofs of these results, but rather to describe those techniques which are interesting, which illustrate the method of proof, and which may have further application.

We first point out that, roughly speaking, results like Theorems 2.1, 2.2, 2.3 are derived from corresponding results about vanishing of Ext; one wants to show (for the latter two results, at least) that if L and L' are simple modules with $\dim_k L + \dim_k L'$ suitably bounded, then $\mathrm{Ext}^1_G(L, L') = 0$.

3.1. LOW ALCOVE MEMBERSHIP

We have seen in (2) that high weight membership in $\mathbf{C}$ is one way of forcing vanishing of Ext between simple modules. We pointed out in Remark 3 that Serre first proves Theorem 2.1 in case V and W are simple, restricted modules for G an almost simple algebraic group. What he actually proves in this case is that, for V and W simple, restricted, and satisfying the dimensional hypothesis, all of the composition factors of $V \otimes W$ have highest weights in $\mathbf{C}$; see [11], Proposition 7.

Similarly, the proof of Theorem 2.2 relies in part on the observation that simple restricted modules L with $\dim_k L \leq p$ have highest weights in $\overline{\mathbf{C}}$, the closure of the lowest alcove; see [6], Lemma 1.4.

Of course, we recall that (4) determines when a weight lies in the lowest alcove. Since (4) involves the quantity

$$l(\lambda) = \langle \lambda + \rho, \alpha_0\check{\ } \rangle, \tag{7}$$

one needs a method of relating the dimension of $L(\lambda)$ to $l(\lambda)$.

The arguments in [6] and [11] achieve an estimate of the dimension of $L(\lambda)$ by considering the $G_\alpha = SL_2$ sub-modules determined by the various α strings through λ; see [6], Lemma 1.2.

In order to get the rank-related result, Theorem 2.3, it was necessary to find stronger dimension estimates for $L(\lambda)$. We describe how this was done in the next sections.

3.2. PREMET'S RESULT

Premet's paper [10] establishes dimension estimates for simple modules with restricted highest weight. Let G be a reductive algebraic k group, and let λ denote a restricted dominant weight. Premet's theorem gives information on the weight spaces of $L(\mu)$ provided the prime p is not "special", where special means the following: If G has a component of type B_ℓ, C_ℓ, or F_4, the prime $p = 2$ is special. If G has a component of type G_2, the primes $p = 2, 3$ is special. No other prime is special.

Theorem 3.1 *Assume that the prime p is not special, and let λ as above. If $\mu \in X$ is such that $\Delta(\lambda)_\mu \neq 0$, then $L(\lambda)_\mu \neq 0$.*

In particular, let $\Pi(\lambda)$ denote the set of weights μ with $\Delta(\lambda)_\mu \neq 0$. Then for restricted λ, we have

$$|\Pi(\lambda)| \leq \dim_k L(\lambda). \tag{8}$$

The important observation about (8) is the fact that this inequality is independent of p (so long as p is not special). Indeed, the weight space dimensions of $\dim_k \Delta(\lambda)$ are independent of p; they are given by the Weyl character formula.

3.3. AN INEQUALITY

One of the key results of [9] is that

$$l(\lambda) \cdot \mathcal{C} < |\Pi(\lambda)| \tag{9}$$

for all but finitely many dominant λ. The exceptions are explicity computed; the reader is referred to [9], Table 3.1.1 for the list of exceptions, and §3 for details of the argument. This inequality is again independent of p.

Assume that $L = L(\lambda)$ is a simple, restricted G-module with $\dim_k L \leq \mathcal{C}p$. If p is not special, and λ is not an exception to (9), then one has

$$l(\lambda) < \mathcal{C}^{-1} \cdot |\Pi(\lambda)| \leq \mathcal{C}^{-1} \dim_k L \leq p, \tag{10}$$

i.e. one has $\lambda \in \mathbf{C}$. Of course, p has reentered the picture; the condition that λ be restricted depends on p.

Together with a careful analysis of those λ which *are* exceptions to (9), (10) leads immediately to a proof of Theorem 2.3 in the restricted case, i.e. when all composition factors are restricted; see [9], Proposition 5.2.1.

3.4. INFINITESIMAL RESULTS

In order to prove results like Theorems 2.2 and 2.3, one must have Ext vanishing results which apply to simple modules which are not necessarily restricted.

Such results are achieved by studying cohomology groups $H^1(G_1, L(\mu))$ for dominant, restricted weights μ, where G_1 is the first Frobenius kernel of G. The representation theory of G_1 is "the same" as that of the p-Lie algebra $\mathbf{L} = \mathrm{Lie}(G)$. We refer the reader to the proof of [6] Lemma 1.7 and [9], Proposition 5.3.4, and the results in section 5.4 to see the application of these cohomology groups; but we point out here some cohomology vanishing results.

Jantzen uses the following (see [6], proof of Lemma 1.7):

Proposition 3.1 *Let $\lambda \in \mathbf{C}$ be a dominant restricted weight. Then*

$$H^1(G_1, L(\lambda)) = 0.$$

In [9], we have the following vanishing result:

Proposition 3.2 *Assume that p is not special. Let λ be a restricted dominant weight so that $dim_k L(\lambda) \leq \mathcal{C}p$. Then $H^1(G_1, L(\lambda)) = 0$, unless λ is listed in Table 4.*

Table 4. Non-vanishing G_1 Cohomology

Φ	p	λ	$H^1(G_1, L(\lambda))^{[-1]}$
A_ℓ	2	ϖ_2 or $\varpi_{\ell-1}$	$L(\varpi_1)$ or $L(\varpi_\ell)$ $(\ell \geq 4)$
$A_\ell,\ \ell > 2$	$p \mid \ell + 1 \equiv 0 \pmod{p}$	$\varpi_1 + \varpi_\ell$	k
A_2	$p = 3$	$\varpi_1 + \varpi_2$	$L(0) \oplus L(\varpi_1) \oplus L(\varpi_2)$
A_3	$p = 3$	$\varpi_1 + \varpi_2$	$L(\varpi_1)$
A_3	$p = 3$	$\varpi_2 + \varpi_3$	$L(\varpi_2)$
B_ℓ	$p \mid 2\ell + 1 \equiv 0 \pmod{p}$	$2\varpi_1$	k
C_ℓ	$p \mid \ell \equiv 0 \pmod{p}$	ϖ_2	k
D_ℓ	$p \mid \ell \equiv 0 \pmod{p}$	$2\varpi_1$	k

3.5. TECHNIQUES IN THE FINITE CASE

To handle the finite groups of Lie type $G(\mathbf{F}_q)$, one hopes to exploit the results already obtained in the algebraic case. In particular, one needs to relate the extension theory for the simple modules of $G(\mathbf{F}_q)$ to that of G. Fundamental work of Cline, Parshall, Scott and van der Kallen, [4], develops this relationship. Their results gives conditions for the natural map

$$\mathrm{Ext}^1_G(L, L') \to \mathrm{Ext}^1_{G(\mathbf{F}_q)}(L, L') \tag{11}$$

to be an isomorphism; see [4], conditions 5.2 – 5.5 and the "twisted" analogues of these conditions given by Avrunin in [2]. Theorems 2.2 (b) and 2.4 rely on these conditions in an essential way.

Let $\lambda, \lambda' \in X_r$ be the highest weights of L, L'. In [6], Corollary 2.3, Jantzen proves: if $\Phi \neq G_2$ and $l(\lambda + \lambda') \leq p^r - 2p^{r-1} - 2$, then (11) is an isomorphism. Jantzen's condition relies on those given in [4], but is not equivalent to them. He gives also a similary condition for G_2. This condition leads to a proof of Theorem 2.2 (b).

One now obtains a rank related analogue to Jantzen's result as follows: one shows that if no p-adic term of λ or λ' is an exception to (9), and $\dim_k L + \dim_k L' \leq \mathcal{C}p$, then (11) is an isomorphism.

Theorem 2.4 is then proved by some ad hoc considerations, and one is forced to make the exceptions listed in Table 3.

Remark 6 In some cases, the exceptions in Table 3 are necessary. Consider the following examples:

Let p by any prime and consider the group $G = \mathrm{SL}_2(\mathbf{F}_p)$. It was pointed out in [6] that G possesses indecomposable modules of length two with dimension $p-1$; see [6] Remark 2.3 and [1], p. 49.

Let $p = 2$, and consider the group $G = \mathrm{SL}_3(\mathbf{F}_2)$. Then P, the projective cover of the trivial representation in characteristic 2, has length 4. P possesses 4 distinct isomorphism classes of indecomposable length 2 subquotients; each of these subquotients has dimension 4.

Remark 7 On the other hand, some of the exceptions in Table 3 are not known to be necessary; these arise when the conditions in [4] are not sufficiently sharp to detect whether (11) is an isomorphism. An example of this situation is: $\Phi = C_\ell$, $q = p = 5$, $\lambda = \lambda' = \varpi_1$. See [8], Remark 6.10 for more examples and discussion of this deficiency.

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