SEMISIMPLE MODULES FOR FINITE GROUPS OF LIE TYPE

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1. INTRODUCTION

Let k be an algebraically closed field of characteristic $p > 0$, and let G be a connected, reductive algebraic group over k. In [8] and [11], conditions on the dimension of rational G modules were seen to imply semisimplicity of these modules. In [8], certain of these conditions were extended to cover the finite groups of Lie type. In this paper, we extend some of the results of [11] to cover these finite Lie type groups. The main such extension is the following result:

Theorem 1. *Let* $q = p^r$, *where* p *is a prime number. Let* $G(\mathbb{F}_q)$ *be a finite group of Lie type, arising as the fixed points of an automorphism of an almost simple algebraic group* G *of rank* ℓ *with root system* Φ, *and let* V *be a* $kG(\mathbb{F}_q)$ *module. Assume that* (Φ, p, r) *does not appear in Table 1 below. If* $\dim_k V \leq \ell p$, *then* V *is semisimple.*

Table 1. Restrictions on q.

Φ		p	r
A_1		any	any
A_2		2	1
A_ℓ	$\ell \geq 2$	3	1, 2
C_ℓ	$\ell \geq 2$	5	1

The proof of this theorem proceeds roughly as follows. As in [11], the proof involves studying the group

$$\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L, L')$$

where L, L' are simple modules for $G(\mathbb{F}_q)$ with the property that $\dim_k L + \dim_k L' \leq \ell p$; Theorem 1 follows provided that this Ext group, which measures extensions between L and L', vanishes in all cases.

The main idea of the proof of Theorem 1 is to compare the rational representation theory of the algebraic group G with the representation theory of $kG(\mathbb{F}_q)$. In Section 2, we point out some of the basic features of the representation theory of $kG(\mathbb{F}_q)$. One of the chief aspects of interest is the fact that simple modules for $kG(\mathbb{F}_q)$ "lift" to the algebraic group G. Much of the notation for the remainder of the paper is fixed in this section.

We utilize two main results from the work in [11]. The first is the rational analogue of Theorem 1, namely Theorem 1 of [11]. The second is the "combinatorial" work establishing the list $\mathcal{I}$ of

weights which fail to be *allowable*. We summarize much of this combinatorial work in Section 3.

In Section 4, we collect some of the methods for studying extensions between $G(\mathbb{F}_q)$ representations. Proposition 4.1 states Jantzen's adaptation of the generic cohomology conditions given in [5]. Together with Theorem 1 of [11], these conditions form the basis for proof of the main theorem for "most" of the finite groups of Lie type; they imply that one has an isomorphism

$$\mathrm{Ext}^1_G(L, L') \simeq \mathrm{Ext}^1_{G(\mathbb{F}_q)}(L, L'),$$

for sufficiently large q and for fixed L, L'.

In Section 5.4 we utilize the results of Section 4 together with work in [11] to obtain some key reductions involved in the proof of Theorem 1. In particular, Proposition 5.6 provides a reduction to the case where p is small; this Proposition establishes the vanishing of the appropriate Ext group whenever $p > p_{\max}$, where $p_{\max}$ is specified in Table 5. Furthermore, the assertion of Theorem 1 is verified when L' is the trivial module; see Proposition 5.13. Finally, we obtain the crucial reduction to the situation where the highest weights of L and L' are "tensor indecomposable;" see 5.12. The section concludes with Theorem 2. This theorem implies Theorem 1 for most q, though some additional restrictions on q beyond those in Table 1 are made; see Table 3.

Section 6 describes the generic cohomology conditions given in [5] and [2] (for *first* cohomology groups); these results are then applied in 6.2 to establish the required vanishing of Ext when (Φ, p, r) is among the triples listed in Table 3.

The paper concludes with Section 7; here we provide the proof of Theorem 1. We also describe some small indecomposable modules which necessitate certain of the restrictions from Table 1.

Remark 1.1. For exceptional-type groups, the bound ℓp can be improved somewhat; see Table 4 and the statement of Theorem 2.

Remark 1.2. One should refer to sections 6 and 7.2 for discussion concerning the values of q ruled out by Table 1 and 3. For some of these q, there are indecomposable modules violating the dimension condition of Theorem 1; these modules are described in section 7.2. For the remaining values of q, we provide no answer here. In some cases the existing techniques are inadequate to verify the theorem; remarks concerning this inadequacy are given in section 6.

2. NOTATION

Let $r \geq 1$, and let $q = p^r$. Denote by $\mathbb{F}_q$ the finite field with q elements. We assume that the group G is defined over $\mathbb{F}_p$; furthermore, assume that G is *split* over $\mathbb{F}_p$, i.e. that G contains a maximal torus T defined over $\mathbb{F}_p$. Let F be the corresponding Frobenius endomorphism of G.

Let σ denote a fixed (and possibly trivial) automorphism of the Dynkin diagram. Abusing notation slightly, we denote also by σ the corresponding map $G \to G$ and $X \to X$. Let $G(\mathbb{F}_q)$ denote the group of fixed points of the map $F^r \circ \sigma$.

Let $X = \mathrm{Hom}(T, k^\times)$ denote the weight lattice of G; X is isomorphic to $\mathbb{Z}^\ell$ where ℓ is the *rank* of G. Let $\Phi \subset X$ denote the *root system* of G, and let $\Phi_+ \subset \Phi, X_+ \subset X$ denote a fixed choice of positive roots and corresponding positive region in X. A weight $\lambda \in X_+$ is called dominant; we recall that the dominant weights parametrize the simple rational G modules $L(\lambda)$. Let $\alpha_1, \alpha_2, \ldots, \alpha_\ell \in \Phi_+$ denote the simple roots, and let $\varpi_1, \varpi_2, \ldots, \varpi_\ell \in X_+$ denote the corresponding fundamental dominant weights.

For $t \geq 1$, let $X_t = \{\lambda \in X \mid 0 \leq \langle \lambda, \alpha_i \rangle < p^t$ for $i = 1, 2, \ldots, \ell\}$. It is known that the restrictions $L(\lambda)|_{G(\mathbb{F}_q)}$ for $\lambda \in X_r$ form a complete set of simple $kG(\mathbb{F}_q)$ modules; see for example Theorem 43, [13].

Any weight $\lambda \in X_r$ may be uniquely expressed as

$$\lambda = \lambda_0 + p\lambda_1 + p^2\lambda_2 + \cdots + p^{r-1}\lambda_{r-1} \tag{2.a}$$

where $\lambda_i \in X_1$ is a *restricted* weight. The structure of $L(\lambda)$ can be understood in part from such a decomposition; Steinberg's tensor product theorem asserts that

$$L \simeq L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \cdots \otimes L(\lambda_{r-1})^{[r-1]}. \tag{2.b}$$

Here, $L^{[d]}$ denotes the dth Frobenius twist of the module L; namely the representation resulting from twisting the representation L by the dth power of the Frobenius map F. Note in particular that

$$\dim_k L(\lambda_i) \leq \dim_k L(\lambda) \quad (0 \leq i \leq r). \tag{2.c}$$

Remark 2.1. Let $\lambda \in X_r$ be the highest weight of the simple $G(\mathbb{F}_q)$ module L. Then $L^{[d]}$ has highest weight $p^{d'}\lambda$ where d' is determined by the conditions $0 \leq d' \leq r-1$ and $d \equiv d' \pmod{r}$.

For a dominant weight λ, recall that $\Pi(\lambda)$ denotes the saturated set of weights with highest weight λ. Suppose that λ is *restricted* and that p is not a *special* prime (see [11] Definition (2.2.2)). Premet's theorem, [12] (see also [11] (2.2.3) for a statement consistent with the notation used here) gives the following dimension estimate:

$$|\Pi(\lambda)| \leq \dim_k L. \tag{2.d}$$

Remark 2.2. In this paper, the only twisted groups of Lie type that we consider are those arising from automorphisms of the Dynkin diagram. The twisted groups which do not arise in this way are as follows:

$$^2F_4(2^r), \quad {}^2G_2(3^r), \quad \text{and } {}^2B_2(2^r).$$

See [4], 1.19, for a discussion of the classification of finite Lie type groups, as well as for the values of r for which the above notations have meaning. For these groups, the parameters ℓ and p are fixed (and are small). One observes that ℓp exceeds the minimal non-trivial module dimension in each case; thus Theorem 1 is vacuous for these groups.

Remark 2.3. The semisimple algebraic groups are classified by their root systems Φ. In this paper, we shall refer to root systems of type A_ℓ, B_ℓ, C_ℓ, and D_ℓ as being of *classical type*, and those of type $E_\ell, 6 \leq \ell \leq 8$, F_4, and G_2 as being of *exceptional type*. Similarly, we may refer to a semisimple algebraic group as being either classical or exceptional.

3. ALLOWABLE WEIGHTS

Throughout this section, Φ denotes an *irreducible* root system of rank ℓ. Let $\tilde{\alpha}$ (respectively α_0) $\in \Phi_+$ be the long (respectively short) root of maximal height, and let

$$\mathcal{C} = \mathcal{C}(\Phi) = \max\left\{\frac{|\mathcal{W}\tilde{\alpha}|}{2}, \frac{|\mathcal{W}\alpha_0|}{2}\right\}.$$

Computation of the quantity $\mathcal{C}$ is straightforward. One uses the tables in [3] to determine the long and short root of maximal height; they are simply the long and short root which are dominant weights. It is then a simple matter to apply the definition of $\mathcal{C}$ to obtain the following

$$
\begin{array}{llll}
\text{For type } A_\ell, & \mathcal{C} = \dbinom{\ell+1}{2}. & \text{For type } C_\ell, & \mathcal{C} = 2\dbinom{\ell}{2} = \ell(\ell-1). \\
\text{For type } B_\ell, & \mathcal{C} = \ell(\ell-1). & \text{For type } D_\ell, & \mathcal{C} = \ell(\ell-1). \\
\text{For type } E_6, & \mathcal{C} = 36. & \text{For type } E_7, & \mathcal{C} = 63. \\
\text{For type } E_8, & \mathcal{C} = 120. & \text{For type } F_4, & \mathcal{C} = 12. \\
\text{For type } G_2, & \mathcal{C} = 3.
\end{array}
\tag{3.a}
$$

Definition 3.1. A weight $\lambda \in X_+$ will be called *allowable* provided that

$$|\Pi(\lambda)| > \mathcal{C} \cdot \langle \lambda + \rho, \alpha_0^{\vee} \rangle.$$

The notion of an allowable weight is important because, combined with (2.d), it permits us to relate the size of $\langle \lambda, \alpha_0^{\vee} \rangle$ with $\dim_k L(\lambda)$. In Table 2 below, we specify a set $\mathcal{I}$ of weights for each irreducible root system Φ. The following result was an important tool used in the results of [11]; its proof can be found in [11] 3.2.

Proposition 3.2. *Let Φ be an irreducible root system of rank $\ell \geq 2$, and let $\mathcal{I}$ be the set specified in table 2. Suppose that λ is a non-0 weight. If $\lambda \in X_+ \setminus \mathcal{I}$, then λ is allowable.*

Table 2. The set $\mathcal{I}$.

Exceptional Types

Type E_6. $\mathcal{I} = \{\varpi_2,\ \varpi_1 + \varpi_6,\ \varpi_1,\ \varpi_6,\ \varpi_5,\ \varpi_3,\ 2\varpi_1,\ 2\varpi_6\}$.
Type E_7. $\mathcal{I} = \{\varpi_1,\ \varpi_6,\ 2\varpi_7,\ \varpi_7,\ \varpi_2\}$.
Type E_8. $\mathcal{I} = \{\varpi_1,\ \varpi_8\}$.
Type F_4. $\mathcal{I} = \{\varpi_1,\ \varpi_3,\ \varpi_4,\ 2\varpi_4\}$.
Type G_2. $\mathcal{I} = \{\varpi_1,\ \varpi_2,\ 2\varpi_2,\ 3\varpi_2\}$.

For the classical types of root systems, $\mathcal{I}$ consists of the diagram automorphism conjugates of the following weights.

The classical types.			
Type $A_\ell, \ell \geq 2$	Type $B_\ell, \ell \geq 3$	Type $C_\ell, \ell \geq 2$	Type $D_\ell, \ell \geq 4$
$\varpi_i, \ell > 2i$ $(i = 1, 2, 3)$	$\varpi_i, \ell > i$ $(i = 1, 2, 3)$	$\varpi_i, \ell > i$ $(i = 1, 2, 3)$	$\varpi_i, \ell > i+1$ $(i = 1, 2, 3)$
$\varpi_4, 7 \leq \ell \leq 15$	$\varpi_\ell, 3 \leq \ell \leq 11$	$\varpi_4, 4 \leq \ell \leq 6$	$\varpi_4, \ell = 6$
$\varpi_5, 9 \leq \ell \leq 11$		$\varpi_5, \ell = 5, 6$	$\varpi_\ell, 4 \leq \ell \leq 12$
		$\varpi_6, \ell = 6$	
$r\varpi_1$ $(r = 2, 3)$	$r\varpi_1$ $(r = 2, 3)$	$r\varpi_1$ $(r = 2, 3)$	$2\varpi_1$
$4\varpi_1, \ell = 4, 5$			
$\varpi_1 + \varpi_2, \ell \geq 3$	$\varpi_1 + \varpi_2$	$\varpi_1 + \varpi_2$	$\varpi_1 + \varpi_2, \ell \geq 5$
$2\varpi_1 + \varpi_\ell, \ell \geq 3$	$\varpi_2 + \varpi_\ell, \ell = 3, 4$	$2\varpi_2, \ell = 2, 3$	$2\varpi_4, \ell = 5$
$\varpi_i + \varpi_\ell, \ell > i$ $(i = 1, 2, 3)$	$2\varpi_\ell, \ell = 3, 4$	$\varpi_1 + \varpi_3, \ell = 3$	$\varpi_1 + \varpi_\ell, 4 \leq \ell \leq 7$
$2\varpi_2, 3 \leq \ell \leq 6$	$\varpi_1 + \varpi_\ell, 3 \leq \ell \leq 5$		$\varpi_4 + \varpi_5, \ell = 5$
$2\varpi_1 + \varpi_2, 2 \leq \ell \leq 5$			
$3\varpi_1 + \varpi_2, \ell = 2$			
$\varpi_4 + \varpi_6, \ell = 6$			
$\varpi_2 + \varpi_3, \ell = 4$			

We make the following additional technical observations which will be exploited later.

Proposition 3.3. *Let $\Phi = A_\ell$.*

(a) *Let $\lambda \in \mathcal{I}$, and assume that $p \leq 7$, and that $\dim_k L(\lambda) < \ell p$. Then*

$$\langle \lambda, \alpha_0^{\vee} \rangle = 1 \tag{3.b}$$

unless λ is one of the following:

ℓ	p	λ
$2, 3$	7	$3\varpi_1$
≤ 11	7	$2\varpi_1$
≤ 7	5	$2\varpi_1$
3	7	$\varpi_1 + \varpi_2$
$2, 3, 4$	7	$\varpi_1 + \varpi_\ell$
$2, 3$	5	$\varpi_1 + \varpi_\ell$
3	7	$\varpi_2 + \varpi_3$
3	7	$2\varpi_2$

(b) *Let $p = 2$, let $\lambda \in \mathcal{I}$ be a restricted weight, assume that $\dim_k L(\lambda) < \ell p = 2\ell$. Then $\lambda = \varpi_1$ or ϖ_ℓ.*

Proposition 3.4. *Let $\Phi = B_\ell$ with $\ell \geq 3$.*

(a) *Assume that $p \leq 13$, $\lambda \in \mathcal{I}$, and $\dim_k L(\lambda) < \ell p$. Then $\langle \lambda, \alpha_0^{\vee} \rangle \leq 2$ unless $\lambda = 2\varpi_1$, $\ell = 3, 4, 5$, and $p > 2\ell + 3$.*

(b) *Assume that $p = 3$, $\lambda \in \mathcal{I}$, $\dim_k L(\lambda) < 3\ell$. Then either $\lambda = \varpi_1$ or $\ell = 3$ and $\lambda = \varpi_3$.*

Proposition 3.5. *Let $\Phi = C_\ell$, $\ell \geq 2$.*

(a) *Suppose that $\lambda \in \mathcal{I}$ is a restricted weight. If $\ell \geq 4$, suppose that $p \leq 7$, while if $\ell = 2, 3$ suppose that $p \leq 11$. If $\dim_k L(\lambda) < \ell p$ then $\langle \lambda, \alpha_0^\vee \rangle \leq 2$.*
(b) *Let $p = 3$. If $\lambda \in \mathcal{I}$ is restricted and $\dim_k L(\lambda) < 3\ell$, then $\lambda = \varpi_1$.*

Proposition 3.6. *Let $\Phi = D_\ell$, $\ell \geq 4$.*

(a) *Suppose that $\lambda \in \mathcal{I}$ is a restricted weight, and assume that $p \leq 7$. If $\dim_k L(\lambda) < \ell p$ then $\langle \lambda, \alpha_0^\vee \rangle \leq 2$.*
(b) *Let $p = 2, 3$. If $\lambda \in \mathcal{I}$ is restricted and $\dim_k L(\lambda) < p\ell$, then $p = 3$ and, up to diagram automorphism, $\lambda = \varpi_1$.*

Sketch of proof for Propositions 3.3, 3.4, 3.5, and 3.6. The proof of each of these four results is similar. For (a) of each result, the dimensions of the modules $L(\lambda)$ for $\lambda \in \mathcal{I}$ were determined or estimated in [11]; see in particular Table 4.5.2, Proposition 4.2.2. For type A_ℓ, one also needs Proposition 4.6.8 of [11], and for types B_ℓ, C_ℓ, and D_ℓ, one should refer to Proposition 4.9.2, Remark 4.9.3, Proposition 4.7.4, and Lemma 4.8.2 of [11].

It is straightforward to check that when p satisfies the indicated condition and $\dim_k L(\lambda) > \ell p$, then $\langle \lambda, \alpha_0^\vee \rangle$ exceeds the indicated bound.

As to (b) when $\Phi = A_\ell$, one can apply [11] Lemma 5.4.4 to obtain restrictions on the possible support of λ; the result then follows by consideration of the dimensions of the simple modules corresponding to the weights in $\mathcal{I}$ having the indicated support.

For (b), when $\Phi \neq A_\ell$ is one of the remaining possibilities, the argument proceeds much as for A_ℓ. Since $|\mathcal{W}\lambda| \leq \dim_k L(\lambda) \leq \ell 3 \leq 2\mathcal{C}$, Lemma 5.4.4 applies and one obtains restrictions on the support of λ. It is then straightforward to verify that the listed weights are the only possibilities. $\qquad\square$

4. GENERIC COHOMOLOGY CONDITIONS

In this section, we describe certain conditions proved in [8] which guarantee that the natural map

$$H^1(G, V) \to H^1(G(\mathbb{F}_q), V) \tag{4.a}$$

is an isomorphism. We point out that the arguments in [8] are proved using the techniques of [5]; for more discussion of the latter techniques, see section 6. Let $r \geq 1$, and let

$$f(r, p) = \begin{cases} p^r - 3p^{r-1} - 3 & \text{if } \Phi = G_2 \\ p^r - 2p^{r-1} - 2 & \text{otherwise} \end{cases} \tag{4.b}$$

Proposition 4.1. *Fix $q = p^r$.*

(a) *Let V be a finite dimensional rational G-module such that each weight λ of V satisfies $\langle \lambda, \alpha_0^\vee \rangle \leq f(r, p)$. Then the map in (4.a) is an isomorphism.*
(b) *Let λ and μ be dominant weights satisfying*

$$\langle \lambda + \mu, \alpha_0^\vee \rangle \leq f(r, p). \tag{4.c}$$

Then the natural map

$$\mathrm{Ext}^1_G(L(\mu), L(\lambda)) \to \mathrm{Ext}^1_{G(\mathbb{F}_q)}(L(\mu), L(\lambda)) \tag{4.d}$$

is an isomorphism.

Proof. This result is 2.2 of [8]. □

Proposition 4.2. *Let $\mathcal{G}$ be a finite group, and let ϕ be an automorphism of $\mathcal{G}$. For any $k\mathcal{G}$ module M, denote by M^ϕ the $k\mathcal{G}$ module which is M as a vector space, but with $\mathcal{G}$ action twisted by ϕ. Then $H^i(\mathcal{G}, M^\phi) \simeq H^i(\mathcal{G}, M)$.*

Proof. The functor $M \mapsto M^\phi$ is an automorphism of the category; the result follows at once. □

Corollary 4.3. *Let $\mathcal{G} = G(\mathbb{F}_q)$ be a finite group of Lie type, and let M be a $k\mathcal{G}$ module. Then*

(a) *$H^1(\mathcal{G}, M^{[d]}) \simeq H^1(\mathcal{G}, M)$ for all $d \in \mathbb{Z}$, where $M^{[d]}$ denotes the d-th Frobenius twist of the module M.*
(b) *If M is simple, $H^1(\mathcal{G}, M^*) \simeq H^1(\mathcal{G}, M)$, where M^* denotes the dual or contragredient module.*

Proof. This result follows from Proposition 4.2 where for (a) we take $\phi = F^d$, the d-th power of the Frobenius automorphism of $\mathcal{G}$, and for (b) we take ϕ to be the "diagram automorphism" of $\mathcal{G}$ given by the action of $-w_0$ where w_0 is the longest word in the Weyl group of Φ. □

5. THE PROOF FOR ALMOST ALL CASES

We shall first provide a proof of Theorem 1 when (Φ, p, r) is not among the triples listed in Tables 1, 3.

Table 3. Temporary Restrictions on $q = p^r$.

Φ		p	r
A_ℓ	$\ell \geq 2$	5	1
B_ℓ	$\ell \geq 3$	5	1
D_ℓ	$\ell \geq 4$	5	1
G_2		7	1

We shall also require a constant, c, defined as follows: If Φ is of classical type, take $c = \ell$. If Φ is of exceptional type, let the number c be given by Table 4. Note that in all cases $c \geq \ell$.

Table 4. Value of c for Exceptional Groups

Φ	E_6	E_7	E_8	F_4	G_2
c	$\dfrac{53}{7}$	$\dfrac{31}{2}$	$\dfrac{495}{7}$	$\dfrac{51}{11}$	2

Actually, we first prove the following theorem:

Theorem 2. *Let Φ be an irreducible root system and let σ be a fixed automorphism of Φ. Let $q = p^r$, and let $G(\mathbb{F}_q)$ denote the corresponding finite group of Lie type. Assume that (Φ, p, r) does not appear in Table 1 or 3. If $\lambda, \lambda' \in X_r$ and*

$$\dim_k L + \dim_k L' \leq cp$$

then $\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L, L') = 0$, where $L = L(\lambda), L' = L(\lambda')$.

Theorem 2 implies Theorem 1 for the triples (Φ, p, r) which are not listed in either Table 1 or 3. We concentrate first on proving Theorem 2. However, unless otherwise stated, the results we prove hold for the triples in Table 3. The only results which exploit the exclusions of the triples in Table 3 are Proposition 5.13 and the proof of Theorem 2.

Fix L, L' non-0 simple modules for $kG(\mathbb{F}_q)$. Let

$$\lambda = \sum_{i=0}^{r-1} p^i \lambda_i, \quad \lambda' = \sum_{i=0}^{r-1} p^i \lambda'_i, \quad \lambda_i, \lambda'_i \in X_1 \tag{5.a}$$

denote the highest weights of these simple modules, written in their p-adic expansions.

We shall be interested in representations L, L' for $kG(\mathbb{F}_q)$ satisfying the following:

Condition 5.1. The inequality

$$\dim_k L + \dim_k L' \leq cp$$

holds, where $c = \ell$ when Φ is classical, and c is given in Table 4 when Φ is exceptional.

Proposition 5.2. *Let L, L' be simple rational modules for the algebraic group G. If L, L' satisfy Condition 5.1, then $\mathrm{Ext}^1_G(L, L') = 0$.*

Proof. For Φ of classical type, this follows from [11], Corollary 1. When Φ is exceptional, one uses [11] Theorem 1 together with the observation that every indecomposable module described in [11] Proposition 5.1.1 for exceptional groups has dimension exceeding cp. $\square$

Proposition 5.3. *If $L = L' = L(0)$, then $\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L, L') = 0$.*

Proof. The standing rank assumption $\ell > 1$ guarantees that the group $G(\mathbb{F}_q)$ contains a subgroup H which is a central extension of a simple group; furthermore, $[G(\mathbb{F}_q) : H]$ is prime to p. Thus, restriction induces an injection $0 \to H^1(G(\mathbb{F}_q), k) \to H^1(H, k)$. One knows as well that $H = H'$ is its own derived group. Now, a 1-cocycle on H with coefficients in k is nothing more than a group homomorphism from H to the additive group of k; since $H = H'$, all such are trivial. $\square$

Remark 5.4. Of course, the assumption on the rank of G is much stronger than necessary for Proposition 5.3; the result holds for rank 1 provided that $q \geq 4$.

Proposition 5.5. *Let p be a* special *prime. If Condition 5.1 holds, then*

$$\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L, L') = 0.$$

Proof. By proposition 5.3, it suffices to show that $L = L' = L(0)$ whenever Condition 5.1 holds. Let m be the minimal dimension for a non-trivial $G(\mathbb{F}_q)$-module. Evidently, one need only observe that $m \geq cp$; the following data immediately yield this inequality.

Φ	p	c	m
B_ℓ	2	ℓ	2ℓ
C_ℓ	2	ℓ	2ℓ
G_2	3	2	6
F_4	2	$\dfrac{51}{11}$	26

$\square$

We assume from now on that p is not special.

Proposition 5.6. *Let $\lambda, \lambda' \in X_r$, and consider their corresponding p-adic expansions as in 2.a.*

(a) *Suppose for every $0 \leq i \leq r$ that the following condition holds:*

$$\langle \lambda_i + \lambda'_i, \alpha_0^{\vee} \rangle \leq \begin{cases} p - 3 & \text{if } \Phi = A_2 \\ p - 6 & \text{if } \Phi = G_2 \\ p - 4 & \text{for all other root systems} \end{cases} . \tag{5.b}$$

Then $\langle \lambda + \lambda', \alpha_0^{\vee} \rangle \leq f(r, p)$.

(b) *Assume that $dim_k L(\lambda) + \dim_k L(\lambda') \leq cp$. Suppose that each p-adic term of λ and λ' is an allowable weight. Then*

$$\langle \lambda + \lambda', \alpha_0^{\vee} \rangle \leq f(r, p).$$

(c) *Assume that $dim_k L(\lambda) + \dim_k L(\lambda') \leq cp$. Suppose that $\langle \lambda + \lambda', \alpha_0^{\vee} \rangle > f(r, p)$. Then $p \leq p_{max}$ where p_{max} is given in Table 5.*

Table 5. Possibilities for p.

Φ	A_ℓ	B_ℓ	C_ℓ	C_ℓ	D_ℓ	E_6	E_7	E_8	F_4	G_2
ℓ	$\ell \geq 2$	$\ell \geq 3$	$\ell \geq 4$	$\ell = 2,3$	$\ell \geq 4$					
p_{max}	7	13	7	11	7	7	7	7	11	17

Proof. (a) is proved in [8], 2.1(3). (b) will follow from (a) after we verify that (5.b) holds whenever λ_i, λ'_i are allowable. We first observe that since p is not special and λ_i is restricted, (2.d) together with (2.c) yield

$$|\Pi(\lambda_i)| \leq \dim_k L.$$

Since λ_i is allowable, we deduce that $\mathcal{C}\langle\lambda_i + \rho, \alpha_0^\vee\rangle \leq \dim_k L$. Recall that the Coxeter number is given by $h = \langle\rho, \alpha_0^\vee\rangle + 1$. We thus obtain

$$\mathcal{C}\langle\lambda_i, \alpha_0^\vee\rangle \leq \dim_k L - \mathcal{C}(h-1).$$

Combining this inequality with the corresponding statement for λ_i', we get

$$\mathcal{C}\langle\lambda_i + \lambda_i', \alpha_0^\vee\rangle \leq \dim_k L + \dim_k L' - 2\mathcal{C}(h-1),$$

or, applying Condition 5.1,

$$\langle\lambda_i + \lambda_i', \alpha_0^\vee\rangle \leq p - 2(h-1). \tag{5.c}$$

Suppose first that $\Phi \neq A_2, G_2$. One has in this situation $h \geq 4$. This implies that $2(h-1) \geq 6$ so that $p - 2(h-1) \leq p - 4$; this verifies (5.b).

When $\Phi = G_2$, $h = 6$. The right-hand side of (5.c) is thus $p - 10$, which is less than $p - 6$; this again verifies (5.b).

Finally, when $\Phi = A_2$, $h = 3$. The right-hand side of (5.c) is $p - 4$ which is less than $p - 3$ as required by (5.b).

To prove (c), note that by (a) there is some $0 \leq i \leq r$ so that (5.b) fails. After possibly interchanging the roles of λ and λ', we may suppose that λ_i satisfies

$$\langle\lambda_i, \alpha_0^\vee\rangle > \frac{1}{2} \begin{cases} p - 3 & \text{if } \Phi = A_2 \\ p - 6 & \text{if } \Phi = G_2 \\ p - 4 & \text{for all other root systems} \end{cases} \tag{5.d}$$

Furthermore, by the proof of (b), we may suppose that λ_i is not allowable, whence $\lambda_i \in \mathcal{I}$ by Proposition 3.2.

Let $n = n(\Phi) = \max_{\mu \in \mathcal{I}}(\langle\mu, \alpha_0^\vee\rangle)$. In order that (5.d) hold, one must have

$$n \geq \frac{1}{2} \begin{cases} p - 3 & \text{if } \Phi = A_2 \\ p - 6 & \text{if } \Phi = G_2 \\ p - 4 & \text{for all other root systems} \end{cases}$$

Upper bounds on p may now be obtained by computing $n(\Phi)$ for each Φ. One finds, for $\Phi \neq A_2, G_2$, that $2n(\Phi) + 4 \geq p_{\max}$ for $\Phi = B_\ell$ $(\ell \geq 4), C_\ell(\ell \geq 4), D_\ell, E_\ell, F_4$. When $\Phi = A_\ell$, one has $2n(\Phi) \geq p_{\max}$ unless $\ell \leq 5$. When $\Phi = A_\ell$ and $\ell \leq 5$, one has $2n(\Phi) = 8$. However, one easily checks that every $\mu \in \mathcal{I}$ with $\langle\mu, \alpha_0^\vee\rangle \geq 4$ satisfies $\dim_k L(\mu) \geq \ell p$ whenever $p \leq 11$. It follows that we may take $p_{\max}$ as indicated by Table 5.

The arguments establishing the bound $p_{\max}$ for $\Phi = B_2, B_3, C_3, G_2, A_2$ are similar; the details are omitted. $\qquad\square$

Corollary 5.7. *Let* $\lambda, \lambda' \in X_r$. *If* $\dim_k L + \dim_k L' \leq cp$ *and*

$$\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L, L') \neq 0$$

then $p \leq p_{max}$.

Proof. Assume that $p > p_{\max}$. By (c) of Proposition 5.6 we deduce that that $\langle\lambda + \lambda', \alpha_0^\vee\rangle \leq f(r, p)$. Proposition 4.1 then shows that

$$\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L, L') \simeq \mathrm{Ext}^1_G(L, L')$$

The result now follows from Proposition 5.2. $\qquad\square$

5.1. **The set** $I(p)$**.** Proposition 5.6 reduces the proof of Theorem 1, or Theorem 2, to the situation where $p \le p_{\max}$. Furthermore, we may evidently assume that at least one p-adic term of one of the weights λ, λ' fails to be allowable. However, more is true. In this section, we verify that *all* of the p-adic terms of each weight λ, λ' actually lie in the set $\mathcal{I}$.

Definition 5.8. For Φ an indecomposable root system, and p a prime, put

$$I(p, \Phi) = I(p) = \{\mu \in X_1 \mid \dim_k L(\mu) < cp\}.$$

Proposition 5.9. *Suppose that $p \le p_{max}$, where p_{max} is given in Table 5. If $0 \ne \lambda \in I(p)$, then $\lambda \in \mathcal{I}$.*

Proof. Suppose that $\lambda \notin \mathcal{I}$. If p is not special, we have then

$$\langle \lambda + \rho, \alpha_0^{\vee} \rangle < \dim_k L(\lambda)/\mathcal{C} \le cp/\mathcal{C} \tag{5.e}$$

In particular, since $\lambda \ne 0$, we have

$$1 + \langle \rho, \alpha_0^{\vee} \rangle < cp/\mathcal{C} \tag{5.f}$$

One now examines this inequality for each indecomposable root system Φ.

When $\Phi = A_\ell$, $\langle \rho, \alpha_0^{\vee} \rangle = \ell$ and $\mathcal{C} = \begin{pmatrix} \ell + 1 \\ 2 \end{pmatrix}$; (5.f) thus yields

$$(\ell + 1)^2 < 2p.$$

Since $p \le 7$, we deduce $\ell = 2$. When $\ell = 2$, (5.e) yields $\langle \lambda, \alpha_0^{\vee} \rangle \le \dfrac{2p - 6}{3}$; since $p \le 7$ this shows that $\langle \lambda, \alpha_0^{\vee} \rangle \le 8/3 < 3$. The result now follows since $\lambda \in \mathcal{I}$ for every $\lambda \in X_+$ with $\langle \lambda, \alpha_0^{\vee} \rangle < 3$.

When $\Phi = B_\ell$ or C_ℓ, (5.f) yields

$$2\ell(\ell - 1) < p$$

When $\ell \ge 4$, we have in all cases $p \le 13$ and $2\ell(\ell - 1) \ge 24$. When $\ell = 3$, $2\ell(\ell - 1) = 12$; if $\Phi = C_3$, the result follows since $p \le 11$. If $\Phi = B_3$, one has $p \le 13$. In this case, (5.e) yields $\langle \lambda, \alpha_0^{\vee} \rangle \le 1$, hence $\lambda = \varpi_3 \in \mathcal{I}$.

Finally, when $\Phi = C_2$, $p \le 11$; thus (5.e) shows that $\langle \lambda, \alpha_0^{\vee} \rangle < p - 3 \le 8$. This inequality shows that λ is in the "lowest dominant alcove"; thus, Corollary 4.4.3 of [11] shows that $L(\lambda) = V(\lambda)$, where $V(\lambda)$ is the Weyl module. In particular, $\dim_k L(\lambda)$ is given by Weyl's degree formula – see (2.1.a) of [11]. For type C_2, this formula yields

$$\dim_k L(a\varpi_1 + b\varpi_2) = \frac{(a + b + 2)(b + 1)(a + 2b + 3)(a + 1)}{6}$$

Using this formula, one immediately verifies that $\dim_k L(\lambda) > 2p$ whenever $\langle \lambda, \alpha_0^{\vee} \rangle \le 8$ and $\lambda \notin \mathcal{I}$.

If $\Phi = D_\ell$, we have $p \le 7$. As in the analysis for $\Phi = B_\ell, C_\ell$, (5.f) yields $2\ell(\ell - 1) < p$; since $\ell \ge 4$, this inequality is impossible and the result follows.

Finally, assume Φ is exceptional. One determines the following values.

Table 6.

Φ	E_6	E_7	E_8	F_4	G_2
$1 + \langle \rho, \alpha_0^\vee \rangle$	12	19	30	12	6
$c/\mathcal{C} \leq$	$\dfrac{53}{252}$	$\dfrac{16}{63}$	$\dfrac{62}{105}$	$\dfrac{13}{33}$	1

Let $\Phi = E_\ell$, $\ell = 6, 7, 8$, or $\Phi = F_4$; one immediately sees that (5.f) fails for all p in Table 5. When $\Phi = G_2$, one applies (5.e) to learn that $\langle \lambda, \alpha_0^\vee \rangle \leq \dfrac{19}{3} < 7$. If $\lambda = a\varpi_1 + b\varpi_2$, then $\langle \lambda, \alpha_0^\vee \rangle = 2a + 3b$. One immediately sees that $a \leq 3$ and $b \leq 2$. As in the case for C_2, (5.e) shows that λ is in the lowest alcove, so that $\dim_k L(\lambda)$ is given by the Weyl degree formula. In this case, $d = \dim_k L(\lambda)$ is given by

$$d = \frac{(a+1)(b+1)(a+b+2)(a+2b+3)(a+3b+4)(2a+3b+5)}{5!}$$

Using this formula, it is straightforward to verify that $d > 2p$ if $\lambda \notin \mathcal{I}$. $\qquad \square$

Proposition 5.10. *Let Φ be an indecomposable root system of exceptional type, and let $p \leq p_{max}$ be a prime number which is not special. Then the set $I(p, \Phi)$ is given in Table 7.*

Table 7. The set $I(p)$ for exceptional types.

Φ	p	$I(p)$
E_6	$5, 7$	ϖ_1, ϖ_6
	$2, 3$	$\emptyset$
E_7	$5, 7$	ϖ_7
	$2, 3$	$\emptyset$
E_8	$5, 7$	ϖ_8
	$2, 3$	$\emptyset$
F_4	$7, 11$	ϖ_4
	$2, 3, 5$	$\emptyset$
G_2	17	$\varpi_1, \varpi_2, 2\varpi_2$
	$5, 7, 11, 13$	ϖ_1, ϖ_2
	$2, 3$	$\emptyset$

Sketch: Let $\lambda \in I(p)$. We have seen that $\lambda \in \mathcal{I}$. The tables in [7] list the dimensions of certain irreducible modules for exceptional groups; the dimensions computed handle all of the weights in $\mathcal{I}$ with the single exception of the weight $2\varpi_7$ for type E_7. It was shown in the proof of Lemma 4.10.1 of [11] that $\dim_k L(2\varpi_7)$ exceeds cp_{max}. It is a simple matter to exploit the computations for the remainder of the weights in $\mathcal{I}$ to deduce that the indicated weights are the only possibilities for λ. $\qquad \square$

5.2. **Tensor products.** Examining Steinberg's tensor product theorem (2.b), we are motivated to say that a weight λ is *tensor indecomposable* if $p^{-j}\lambda \in X_1$ for some $j \geq 0$ and *tensor decomposable* otherwise.

For each indecomposable root system Φ, the minimal dimension of a non-trivial module is given by the following: (see [10], 5.4.13)

Table 8. Minimal module dimensions.

Φ	A_ℓ	B_ℓ		C_ℓ	D_ℓ	E_6	E_7	E_8	F_4	G_2	
m	$\ell+1$	$2\ell+1$	$(p \neq 2)$	2ℓ	2ℓ	27	56	248	26	7	$(p \neq 3)$
		2ℓ	$(p = 2)$							6	$(p = 3)$

Proposition 5.11. *Let $L = L(\lambda)$ be a simple module with $\dim_k L \leq cp$. Assume that $p \leq p_{max}$ where p_{max} is given in Table 5. Then λ is tensor indecomposable unless $p^j\lambda$ or $p^j\lambda^*$ appears on the following list for some $j \geq 0$.*

Φ	λ	p
A_2	$(1+p^i)\varpi_1,\ 1 \leq i \leq r-1$	$p = 5,7$
A_2	$\varpi_1 + p^i\varpi_3,\ 1 \leq i \leq r-1$	$p = 5,7$
A_3, A_4	$(1+p^i)\varpi_1,\ 1 \leq i \leq r-1$	$p = 5,7$
A_3, A_4	$\varpi_1 + p^i\varpi_\ell,\ 1 \leq i \leq r-1$	$p = 5,7$

Proof. Let m be the minimal non-trivial module dimension; this quantity is given in Table 8. If λ is tensor decomposable, (2.b) shows that

$$\dim_k L \geq m^2.$$

If $\Phi = C_\ell, D_\ell$, then $m = 2\ell$; the hypothesis thus leads to $4\ell^2 \leq \ell p$ or $4\ell \leq p$. Since $p_{max} \leq 7$, and since $\ell \geq 2$, this is impossible.

If $\Phi = B_\ell$, then $m = 2\ell + 1$; the hypothesis thus leads to $4\ell^2 + 2\ell + 1 \leq \ell p$, or $4\ell + 2 + \frac{1}{\ell} \leq p_{max} = 13$. Since $\ell \geq 3$, this inequality can not hold.

For the exceptional groups, we record m^2 and the value for cp_{max} here; the reader may observe that $m^2 \geq cp_{max}$ in all cases.

Φ	m^2	cp_{max}
E_6	729	54
E_7	$24,192$	112
E_8	$61,504$	496
F_4	676	52
$G_2,\ p \neq 3$	49	34

When $\Phi = G_2$ and $p = 3$, one has $cp = 6$, and $m^2 = 36$; so the result holds in this case as well.

Finally, suppose that $\Phi = A_\ell$. Then $m^2 = \ell^2 + 2\ell + 1$, so that $\ell + 2 + \frac{1}{\ell} \leq p_{max} = 7$. It follows that $\ell = 2, 3, 4$. It is straightforward to see that $m^3 \geq \ell p_{max}$, thus λ has no more than two distinct non-zero p-adic terms. Suppose that μ, μ' are two such; one has then

$$\dim_k L(\mu) \leq 7\ell/\dim_k L(\mu').$$

If $\ell = 4$, then $m^2 = 25$. Evidently, $m^2 \leq 4p$ only when $p = 7$. In this case, we get $\dim_k L(\mu) \leq 28/\dim_k L(\mu') \leq 28/5 < 6$. Using the bound $\dim_k L(\mu) \geq |\mathcal{W}\mu|$, and Lemma 5.4.4 from [11], one immediately sees that $\mu = n\varpi_1$ or $n\varpi_4$ for some $n \geq 0$, since $|\mathcal{W}\mu|$ exceeds 6 for other weights. It is then a simple matter to deduce that $\mu = \varpi_1$ or ϖ_ℓ. This argument is symmetric in μ, μ'; the result now follows for $\ell = 4$

Completely similar arguments when $\ell = 2, 3$ show that $\{\mu, \mu'\} \subseteq \{\varpi_1, \varpi_2\}$, resp. $\{\varpi_1, \varpi_3\}$. The result now follows.

$\square$

Proposition 5.12. *Assume that p satisfies* (5)*, and that λ is tensor decomposable. Let $L = L(\lambda)$, and let $L' = L(\mu)$ for $\mu \in X_r$; assume that Condition 5.1 holds for the pair L, L'. Then*

$$\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L, L') = 0.$$

Proof. By proposition 5.11, we have $\Phi = A_\ell$ with $\ell = 2, 3, 4$. We must show that

$$H^1(G(\mathbb{F}_q), L(\lambda) \otimes L(\mu)^*) = 0.$$

Since $\dim_k L(\lambda) = (\ell + 1)^2$ in all cases, Condition 5.1 gives immediately the following upper bound B for $\dim_k L(\mu)$:

ℓ	2	2	3	4
p	5	7	7	7
B	1	5	5	3

It is immediate that μ must in all cases be tensor indecomposable. For the configurations $\ell = 2$, $p = 5$, and $\ell = 4$, $p = 7$, the minimal non-trivial module dimension exceeds B, hence $\mu = 0$. For the remaining cases, application of [11], Lemma 5.4.4 shows that $\mu = n\varpi_1$ or $n\varpi_\ell$ for some $n \geq 0$. It is then straightforward to check that $n = p^j$ for some j.

In view of Corollary 4.3, it suffices to assume that λ is either $(1 + p^i)\varpi_1$ or $\varpi_1 + p^i\varpi_\ell$, and that μ is one of 0, $p^j\varpi_1$, or $p^j\varpi_\ell$, for $1 \leq i < r$ and $0 \leq j < r$. Proposition 5.2 gives cohomology vanishing for the algebraic group, i.e. we have $H^1(G, L(\lambda) \otimes L(\mu)^*) = 0$. The result will now follow from Proposition 4.1 if we argue that $\langle \lambda + \mu, \alpha_0^\vee \rangle < f(r, p)$. But $\langle \lambda, \alpha_0^\vee \rangle \leq 1 + p^i + p^j \leq 1 + 2p^{r-1}$; since $p = 5, 7$ and $r \geq 2$, we have

$$f(r, p) - 2p^{r-1} - 1 = p^r - 4p^{r-1} - 3 = p^{r-1}(p - 4) - 3 > 0$$

and the result follows.

$\square$

5.3. **Extensions involving the trivial representation.**

Proposition 5.13. *Assume that (Φ, p, r) is not on the list of Table 1 or 3. Suppose that $\lambda \in X_r$, and assume $\dim_k L(\lambda) \leq cp - 1$. Then*

$$H^1(G(\mathbb{F}_q), L(\lambda)) = 0. \tag{5.g}$$

Proof. By Proposition 5.12, we may suppose that λ is tensor indecomposable. Since we are interested in the vanishing of H^1, Proposition 4.3 shows that we may suppose $\lambda \in X_1$, and that we may replace $L(\lambda)$ with its dual, i.e. we may replace λ with its conjugate under the diagram automorphism σ (note that we are excluding the root system D_4).

Furthermore, by Corollary 5.7 we may assume that $p \leq p_{\max}$ where $p_{\max}$ is given in Table 5. By (b) of Proposition 5.6 we may assume that

$$\langle \lambda, \alpha_0^{\vee} \rangle \leq \begin{cases} p - 3 & \text{if } \Phi = A_2 \\ p - 6 & \text{if } \Phi = G_2 \\ p - 4 & \text{for all other root systems} \end{cases} \tag{5.h}$$

Furthermore, we know that $\lambda \in I(p) \subseteq \mathcal{I}$.

We consider first the classical root systems; we verify in most cases that (5.h) is incompatible with the dimensional hypotheses on $L(\lambda)$; the remaining cases will be handled by results in [9].

First, suppose $p = 2$. Since $p = 2$ is special for $\Phi = B_{\ell}, C_{\ell}$, we only need to consider this prime when $\Phi = A_{\ell}, D_{\ell}$. However, when $\Phi = D_{\ell}$, the minimum non-trivial module dimension is 2ℓ so the result is vacuous in this case. When $\Phi = A_{\ell}$, (b) of Proposition 3.3 shows that (up to diagram automorphism) $\lambda = \varpi_1$. The vanishing of the cohomology then follows from [9] 6.B unless $\ell = 2$ and $q = 2$; this situation was excluded in Table 1 (see section 7.2 below.)

Next suppose that $p = 3$. When $\Phi = A_{\ell}$, Table 1 excludes $q = 3, 9$. Hence $f(r, p) \geq 7$. Examining the weights in $\mathcal{I}$, it is evident that $\langle \lambda, \alpha_0^{\vee} \rangle \leq 13$. When $\Phi = B_{\ell}, C_{\ell}, D_{\ell}$, Propositions 3.4 (b), 3.5 (b), and 3.6 (b) give the possibilities for λ explicitly. For these λ, the vanishing of cohomology was verified in [9] 6B.

Now assume $p = 5$. For all classical root systems, we have excluded $q = 5$ in Table 3. In particular, $f(r, p) \geq 13$. Examining each possible weight $\lambda \in \mathcal{I}$, one sees that $\langle \lambda, \alpha_0^{\vee} \rangle \leq f(r, p)$.

Now suppose that $p \geq 7$. When $\Phi = A_{\ell}$ we must consider only $p = 7$. Of the weights $\mu \in \mathcal{I}$, all satisfy $\langle \mu, \alpha_0^{\vee} \rangle \leq 3 = p - 4$ with the exception of $\mu = 4\varpi_1$ or $4\varpi_{\ell}$. These weights occur only when $\ell = 4, 5$. By [11], Proposition 4.2.2 (b), one knows that $\dim_k L(4\varpi_1) = 35, 70$ when $\ell = 4, 5$; these dimensions exceed the required bound. Thus, the result holds in case $p = 7$.

Now consider $\Phi = B_{\ell}$, $\ell \geq 3$. We must consider $7 \leq p \leq 13$. When $p = 13, 11$, observe that $p - 4 \geq 7 \geq \langle \lambda, \alpha_0^{\vee} \rangle$ for all $\lambda \in \mathcal{I}$. When $p = 7$, Proposition 3.4 (a) shows that $\dim_k L(\lambda) > 7\ell$ whenever $\langle \lambda, \alpha_0^{\vee} \rangle > 2$.

Next suppose that $\Phi = C_{\ell}$. We only need consider $p = 7$ when $\ell \geq 4$, and $7 \leq p \leq 11$ when $\ell = 2, 3$. According to (a) of Proposition 3.5, $\langle \lambda, \alpha_0^{\vee} \rangle \leq 2$ when $\dim_k L(\lambda) \leq \ell p$; this verifies the result for $p \geq 7$.

Finally, suppose that $\Phi = D_{\ell}$. It only remains to consider $p = 7$. According to (a) of Proposition 3.6, $\langle \lambda, \alpha_0^{\vee} \rangle \leq 2$ whenever $\dim_k L(\lambda) \leq 7\ell$.

Assume now that Φ is of exceptional type. We have listed in Table 7 the set $I(p)$ for the exceptional groups; by definition, $I(p)$ contains the weight λ. We now verify the proposition by considering the weights in $I(p)$ for each Φ.

If Φ is of type E_{ℓ} for $\ell = 6, 7, 8$, then $I(p) = \emptyset$ if $p = 2, 3$. For $p = 5, 7$, (5.g) is verified for each $\lambda \in I(p)$ in [9], see especially 6.B,C (Tables).

If Φ is of type F_4, then $I(p) = \emptyset$ if $p = 2, 3, 5$. When $p = 7, 11$, (5.g) is verified for each $\lambda \in I(p)$ in [9], 6.B.

Let $\Phi = G_2$. Then $I(p) = \emptyset$ if $p = 2, 3$. When $7 \leq p \leq 17$, we have always $\varpi_1, \varpi_2 \in I(p)$; (5.g) is verified for these λ in [9], 6.B. When $p = 5, 7, 11, 13$, these two weights exhaust $I(p)$; when $p = 17$, however, we have also $2\varpi_2 \in I(p)$. Notice that $\langle 2\varpi_2, \alpha_0^{\vee} \rangle = 6$, whereas $p - 6 \geq 11$. Thus, (5.g) follows in this case from Proposition 5.6 (a). $\qquad \square$

5.4. **Establishing Theorem 2.**

Proof of Theorem 2. By Corollary 5.7, we may suppose that $p \leq p_{\max}$ where $p_{\max}$ is given in Table 5. By Proposition 5.12, we may suppose that λ, λ' are tensor indecomposable. According to Proposition 5.13, we may suppose that $\lambda \neq 0$ and $\lambda' \neq 0$.

Invoking Corollary 4.3, we may apply a power of the Frobenius automorphism F to the modules L and L' without changing the dimension of $\operatorname{Ext}^1_{G(\mathbb{F}_q)}(L, L')$; thus, we may assume that $\lambda \in X_1$ and $\lambda' = p^s \lambda'' \in X_s$ with $1 \leq s \leq \dfrac{r}{2}$. Furthermore, λ, λ'' are by definition in $I(p)$; by Proposition 5.9, we have thus $\lambda, \lambda'' \in \mathcal{I}$.

Finally, invoking 5.d, we may assume if $s \neq 0$,

$$\langle \mu, \alpha_0^\vee \rangle > \begin{cases} p - 3 & \text{when } \Phi = A_2 \\ p - 6 & \text{when } \Phi = G_2 \text{ for } \mu = \lambda, \lambda'' \\ p - 4 & \text{otherwise} \end{cases} \tag{5.i}$$

while if $s = 0$,

$$\langle \lambda + \lambda', \alpha_0^\vee \rangle > \begin{cases} p - 3 & \text{when } \Phi = A_2 \\ p - 6 & \text{when } \Phi = G_2 \\ p - 4 & \text{otherwise} \end{cases} \tag{5.j}$$

We first handle Φ of classical type. We consider all primes $p \leq p_{\max}$.

Consider $p = 2$. By Propositions 3.6(b) and 5.5, we need only consider $\Phi = A_\ell$. Since $\ell + 1$ is the minimal dimension of a non-trivial module, we obtain $\dim_k L + \dim_k L' \geq 2\ell + 2 > 2\ell$, contrary to the hypothesis.

Now suppose $p = 3$. If Φ is classical but different than A_ℓ, then the minimum dimension of a non-trivial module is at least 2ℓ; whence $\dim_k L + \dim_k L' \geq 4\ell > 3\ell$, contrary to hypothesis. Thus we need only consider $p = 3$ when $\Phi = A_\ell$. The restrictions in Table 1 then require that $r \geq 3$. Since we have $\langle \mu, \alpha_0^\vee \rangle \leq 3$ for $\mu = \lambda, \lambda''$, we obtain

$$\langle \lambda + \lambda', \alpha_0^\vee \rangle \leq 3(1 + 3^{r/2}) = 3 + 3^{(r+2)/2}.$$

When $r \geq 4$, we have $3^{(r+2)/2} \leq 3^{r-2}$ so that

$$f(r, p) - \langle \lambda + \lambda', \alpha_0^\vee \rangle \geq 3^r - 2 \cdot 3^{r-1} - 3^{r-2} - 5 \geq 0.$$

When $r = 3$, it is straightforward to verify that the condition $f(3,3) - \langle \lambda + \lambda', \alpha_0^\vee \rangle \geq 0$ continues to hold. The result now follows when $p = 3$ by Proposition 4.1 and 5.2.

When $p = 5$ and Φ is classical, we have in all cases ruled out $q = 5$. Applying Propositions 3.3, 3.4, 3.5, 3.6, one has $\langle \lambda, \alpha_0^\vee \rangle \leq 2$ for all $\lambda \in \mathcal{I}$. Thus the result holds when $s \neq 0$ by (5.i). When $s = 0$, one observes that $f(r, 5) \geq 13$ and that $\langle \lambda + \lambda', \alpha_0^\vee \rangle \leq 13$ for every pair $\lambda, \lambda' \in \mathcal{I}$. Thus, the result follows in this case from Propositions 4.1 and 5.2.

When $p = 7$ and Φ is classical, Propositions 3.3, 3.4, 3.5, 3.6 show that $\langle \lambda, \alpha_0^\vee \rangle \leq 2$ for all $\lambda \in \mathcal{I}$ with the exception of $3\varpi_1$ when $\Phi = A_2, A_3$. When $s \neq 0$, the result now follows immediately from (5.i). When $s = 0$ and $\Phi \neq A_2, A_3$, the result follows from (5.j). When $\Phi = A_2, A_3$, suppose that $\lambda = 3\varpi_1$. Then $\dim_k L(\lambda) = 10, 20$ resp.; it follows that $\dim_k L(\lambda') \leq 4, 1$ resp. Since $\lambda' \neq 0$ we are reduced to consideration of $\ell = 2$ It is then straightforward to check that the only weights $\lambda' \in \mathcal{I}$ with $\dim_k L(\lambda') \leq 4$ are $\lambda' = \varpi_1, \varpi_2$; one then verifies easily that (5.j) holds.

Let us now suppose that Φ is of exceptional type. Let M denote the minimal non-trivial module dimension for the group. Then $\dim_k L + \dim_k L' \geq 2M$. For the root systems $\Phi = E_6, E_7, E_8, F_4$ one sees, by comparing the value of M with the definition of c, that $2M > p_{\max}c$ contrary to our assumption.

When $\Phi = G_2$, the dimensional condition implies that $7 \leq p$ since 7 is the minimum non-trivial module dimension (for $p \neq 3$). Thus we consider $7 \leq p \leq 17$. For $p \geq 7$, every weight $\mu \in I(p)$ satisfies

$$\langle \mu, \alpha_0 \check{\ } \rangle \leq 3$$

with the exception of $2\varpi_1 \in I(17)$. Thus (5.i) yields the result when $s \neq 0$ (even when $p = 17$). When $s = 0$, one observes that (5.j) holds unless: $p = 11$ and $\lambda = \lambda' = \varpi_2$, $p = 17$ and $\lambda = \lambda' = 2\varpi_2$. One easily checks in each case that $\dim_k L + \dim_k L' > 2p$, and the result follows. $\qquad\square$

6. Handling the small field cases.

In this section, we provide the proof of Theorem 1 for the triples (Φ, p, r) which appear in Table 3. The essential tools here are the "generic cohomology" arguments of [5] and [2]. We state the results for first cohomology groups (see Conditions 6.1 and 6.2 and Proposition 6.4), and provide some techniques for applying these results (see Remarks 6.7 and 6.3). In section 6.2, we apply these results to deal with those triples in Table 3. Applying Conditions 6.1 and 6.2 can be fairly labor intensive; we do not provide all details here.

We first formulate the conditions from [5] which guarantee that (4.a) is an isomorphism. Fix a G module V.

Condition 6.1. Assume for every dominant weight μ of V that whenever

$$\mu \equiv p^i \alpha \quad (\mathrm{mod}\ (q - \sigma)X) \quad \text{where } \alpha \in \Phi, 1 \leq p^i < q,$$

then $\mu = p^i \alpha$.

Condition 6.2. Assume for every dominant weight μ of V that whenever either

$$\mu = p^i \alpha \quad (\alpha \in \Phi, 1 \leq i)$$

or

$$\mu = p^i \alpha + p^j \beta \quad (\alpha, \beta \in \Phi, w\alpha, w\beta \in \Phi_+ \text{for some } w \in \mathcal{W}, 0 \leq i, j)$$

then $p^i < q$ and (if applicable) $p^j < q$.

Remark 6.3. Observe that Condition 6.2 is independent of σ. Furthermore, we remark that Condition 6.2 is automatic when $\mu = 0$. For $\mu \neq 0$ and $q = p$, Condition 6.2 is equivalent to the following two conditions:

$$\mu \notin pX$$

and

$$\mu - \alpha \notin pX \setminus \{0\} \text{ for any } \alpha \in \Phi$$

Proposition 6.4. *Let V be a rational G module. Assume that $p \neq 2$. Suppose that Condition 6.1 and 6.2 hold. Then the homomorphism* (4.a) *is an isomorphism.*

Proof. Condition 6.1 is the isomorphism condition 5.5 of [5] (or its twisted analogue in [2]) for n=1. Condition 6.2 is the injectivity condition 5.4 of [5] resp [2] for $n = 2$. (We have here rephrased the conditions in terms of dominant weights rather than positive roots.)

According to lemma 5.1 of [5] these conditions yields an isomorphism

$$H^1(B, V) \to H^1(B(q), V). \tag{6.a}$$

The proposition then follows by [5] Lemma 5.6. $\square$

6.1. **The image of** $\mathbf{q} - \sigma$.

Suppose that $\sigma \neq 1$; we would like to characterize the image of $q - \sigma$.

First consider the case where the order is 2. Then σ has eigenvalues ± 1 on $X_{\mathbb{Q}} = X \otimes_{\mathbb{Z}} \mathbb{Q}$. We describe maps from X to the eigenspaces of σ: for any $x \in X$, define $\mathrm{sym}(x) = x + \sigma(x)$ and $\mathrm{alt}(x) = x - \sigma(x)$. Observe that these maps are not projections; indeed, $\mathrm{sym}(x_+) = 2x_+$ if x_+ is in the 1-eigenspace.

Suppose now that $x = (q - \sigma)(y)$. It is then a simple matter to check that

$$\mathrm{sym}(x) = (q + 1)\,\mathrm{sym}(y) \text{ and } \mathrm{alt}(x) = (q - 1)\,\mathrm{alt}(y). \tag{6.b}$$

We thus get the following result:

Lemma 6.5. *Let σ have order 2. In order that $x \in X$ be in the image of $q - \sigma$, it is necessary that* $\mathrm{sym}(x) \in (q + 1)X$ *and* $\mathrm{alt}(x) \in (q - 1)X$

Let us now consider the case where σ has order 3. Let $K = \mathbb{Q}(\zeta)$ where ζ is a primitive 3rd root of unity. Let $A = \mathbb{Z}[\zeta]$; the ring A is a free $\mathbb{Z}$-module with basis $1, \zeta$. The eigenvalues of σ on $X_K = X \otimes_{\mathbb{Z}} K$ are $1, \zeta, \zeta^2$. We define maps $\theta_i : X_A \to X_A, i = 0, 1, 2$, by the rule

$$\theta_i(x) = x + \zeta^i \sigma(x) + \zeta^{2i} \sigma^2(x). \tag{6.c}$$

It is clear that $\theta_i(X)$ lies in the ζ^i eigenspace of σ on $X_A \subset X_K$. Suppose that $x = (q - \sigma)(y)$ for $y \in X$. One then has

$$\theta_i(x) = (q - \zeta^i)\theta_i(y) \tag{6.d}$$

The following result is now evident:

Lemma 6.6. *Let σ have order 3. In order that $x \in X$ be in the image of $q - \sigma$, it is necessary that $\theta_i(x) \in (q + \zeta^i)X_A$ for $i = 0, 1, 2$.*

Remark 6.7. Suppose that $\mu = 0$ and that σ is trivial or has order 2. Then Condition 6.1 is immediate. Indeed, this is easy to see when $\sigma = 1$, since $p^i \alpha \in (q - 1)X$ is clearly impossible. When σ has order 2, note that

$$\mathrm{sym}(p^i\alpha) = p^i\,\mathrm{sym}(\alpha) \notin (q + 1)X;$$

the assertion now follows from Lemma 6.5.

When σ has order 3, assume that $p > 3$. Note that

$$\theta_0(p^i\alpha) = p^i\theta_0(\alpha).$$

Observe that, for $x \in X$, $p^ix \in (p + 1)X_A$ if and only if $x \in (p + 1)X$, and that $\theta_0(\alpha) = \alpha + \sigma(\alpha) + \sigma^2(\alpha) \notin (p + 1)X$ for any α (since $p > 3$). The result now follows by applying Lemma 6.6.

6.2. **Some of the remaining values of q.**

Proposition 6.8. *Let* $\Phi = G_2$, $q = p = 7$. *Suppose that* $\lambda, \lambda' \in X_1$ *satisfy*

$$\dim_k L(\lambda) + \dim_k L(\lambda') \leq 14$$

Then $\mathrm{Ext}^1_{G_2(\mathbb{F}_7)}(L(\lambda), L(\lambda')) = 0$.

Proof. According to Propositions 5.3 and 5.13, the result is known if either $\lambda = 0$ or $\lambda' = 0$; thus, we may suppose that neither weight is 0. The prime 7 is less than $p_{\max}$; it follows from 5.9 that $\lambda, \lambda' \in \mathcal{I}$. The dimensions of the simple modules $L(\lambda)$ for $\lambda \in \mathcal{I}$ are known; see [7]. It is straightforward, using this data, to deduce that $\lambda = \lambda' = \varpi_1$. The result will follow from Proposition 5.2 and Proposition 6.4 provided we check that Conditions 6.2 and 6.1 hold for the weights $\mu \in \Pi_+$, where $\Pi_+ = \{2\varpi_1, \varpi_2, \varpi_1, 0\}$.

One can easily check by hand (or using a computer) that $\mu - \alpha \not\equiv 0 \pmod{6X}$ for $\mu \in \Pi_+$ and for any root α. We point out that the tables in [3], Planche IX, are useful references for these verifications. Condition 6.1 follows immediately from these observations.

To verify 6.2, one employs the techniques described in Remark 6.3. One checks the following for each $0 \neq \mu \in \Pi_+$:

$$\mu \notin 7X;$$
$$\mu - \alpha \notin 7X \setminus \{0\} \text{ for any root } \alpha.$$

Condition 6.2 now follows at once. $\qquad\square$

Proposition 6.9. *Let* Φ *be a classical root system other than* C_ℓ, *and let* $q = p = 5$. *Suppose that* $\lambda, \lambda' \in X_1$ *satisfy*

$$\dim_k L(\lambda) + \dim_k L(\lambda') \leq 5\ell.$$

Then $\mathrm{Ext}^1_{G(\mathbb{F}_5)}(L(\lambda), L(\lambda')) = 0$.

Proof. As in the G_2 result above, Propositions 5.3 and 5.13 give the result unless $\lambda \neq 0$ and $\lambda' \neq 0$; thus, we assume these conditions hold. Furthermore, Proposition 5.6 shows that $\lambda, \lambda' \in \mathcal{I}$. We verify Conditions 6.2 and 6.1; the result then follows from Propositions 6.4 and 5.2.

Assume first that $\Phi = A_\ell$. We discuss here the case where $\ell \geq 9$. Then $\dim_k L \leq 4\ell - 1$ and $\dim_k L' \leq 4\ell - 1$. When $\ell \geq 4$, one can apply Lemma 5.4.4 of [11] to λ and λ; one has

$$2\mathcal{C} \geq 5\ell \geq \dim_k L \geq |\mathcal{W}\mu| \qquad (\mu = \lambda, \lambda'.)$$

Now applying the condition $\ell \geq 9$, it is routine to verify that the only possibilities for λ are ϖ_1 and ϖ_ℓ.

Now let $\mu = \lambda + \lambda'$, so $\mu \in \{2\varpi_1, \varpi_1 + \varpi_\ell, 2\varpi_\ell\}$. Let

$$\Pi_+ = \{2\varpi_1, \varpi_1 + \varpi_\ell, 2\varpi_\ell, \varpi_2, \varpi_{\ell-1}, 0\}.$$

If $\gamma \in X_1$ satisfies $\gamma \leq \mu$, one knows that $\gamma \in \Pi_+$.

The tables in [3], Planche I, show that we may view X as a lattice in a Euclidean space $E \simeq \oplus_{i=1}^{\ell+1} \mathbb{R}\varepsilon_i$. Let $e = \frac{1}{\ell+1} \sum_{i=1}^{\ell+1} \varepsilon_i$ and $\tilde{\varepsilon}_i = \varepsilon_i - e$ for $i = 1, 2, \ldots, \ell+1$; one knows that

$\{\tilde{\varepsilon}_i \mid i = 1, 2, \ldots, \ell\}$ is a $\mathbb{Z}$ basis of X. Observe that $\sum_{j=1}^{\ell+1} \tilde{\varepsilon}_j = 0$. One has expressions for the fundamental dominant weights and the roots in terms of this basis:

$$\varpi_i = \sum_{j=1}^{i} \tilde{\varepsilon}_i$$

$$\Phi = \{\tilde{\varepsilon}_i - \tilde{\varepsilon}_j \mid 1 \leq i \neq j \leq \ell + 1\}$$

Let $\phi_i : X \to \mathbb{Z} \simeq \mathbb{Z}\tilde{\varepsilon}_i$ be the projection on $\mathbb{Z}\varepsilon_i$, for $i = 1, 2, \ldots \ell$. Evidently, for $k \in \mathbb{Z}$, $x \in kX$ if and only if $\phi_i(x) \in k\mathbb{Z}$ for $i = 1, 2, \ldots, \ell$.

Observe that Conditions 6.2 and 6.1 are invariant with respect to the diagram automorphism, and are automatic for $\gamma = 0$. Hence we need only consider $\gamma = 2\varpi_1, \varpi_2, \varpi_1 + \varpi_\ell$.

We describe the argument fully for $\gamma = 2\varpi_1$; we leave the verification for the remaining γ for the interested reader. Let $\alpha = \tilde{\varepsilon}_i - \tilde{\varepsilon}_j$ where $i \neq j$. We claim that $\gamma - \alpha \notin (5 - \sigma)X$. Let $k = \max\{i, j\}$. Suppose first that $\sigma = 1$. When $k < \ell + 1$, notice

$$\phi_k(\gamma - \alpha) = \pm 1.$$

If $k = \ell + 1$, then

$$\phi_\ell(\gamma - \alpha) = \pm 1.$$

When $\sigma = 1$, Condition 6.1 now follows. When $\sigma \neq 1$ we utilize Lemma 6.5 to verify Condition 6.1. Notice that for *all* k

$$|\phi_k(\mathrm{sym}(\gamma - \alpha))| \leq 4.$$

If $\mathrm{sym}(\gamma - \alpha) \neq 0$, it is now clear that $\mathrm{sym}(\gamma - \alpha) \not\equiv 0 \pmod{6X}$. If $\mathrm{sym}(\gamma - \alpha) = 0$, one easily sees that $\alpha = \varpi_1 + \varpi_\ell$; in this case one observes that $\mathrm{alt}(\gamma - (\varpi_1 + \varpi_\ell)) \not\equiv 0 \pmod{4X}$. Condition 6.2 follows from Remark 6.3 and the following observations:

$$\gamma \notin 5X \quad |\phi_1(\gamma - \alpha)| \leq 4 \text{ for all } \alpha \in \Phi.$$

When $2 \leq \ell \leq 9$, in addition to the weights ϖ_1 and ϖ_ℓ, one must also consider the weights ϖ_2 and $\varpi_{\ell-1}$. Arguing as above, one can verify that Conditions 6.2 and 6.1 hold in all needed cases. Since here the rank is bounded, computer verification of this result is straightforward to implement and easily yields the required result.

Now suppose that $\Phi = B_\ell$ or $\Phi = D_\ell$. We treat here the case where $\ell \geq 5$ for D_ℓ and $\ell \geq 4$ for B_ℓ. Since $\dim_k L + \dim_k L' \leq 5\ell$, and since L and L' are non-trivial, one deduces that $\dim_k L \leq 3\ell$ and $\dim_k L' \leq 3\ell$; it then follows from Propositions 3.4 and 3.6 that $\{\lambda, \lambda'\} = \{\varpi_1\}$. The dominant weights lying below $2\varpi_1$ are $\Pi_+ = \{2\varpi_1, \varpi_2, \varpi_1, 0\}$ for type B_ℓ and $\Pi_+ = \{2\varpi_1, \varpi_2, \varpi_1\}$ for type D_ℓ.

Once again, [3] Planches II,IV may be used to realize X as a $\mathbb{Z}$ lattice in the Euclidean space $E = \oplus_{i=1}^{\ell} \mathbb{R}\varepsilon_i$. In these cases X is spanned over $\mathbb{Z}$ by the ε_i and the additional element

$$\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_\ell).$$

One has in each case $\varpi_1 = \varepsilon_1$; furthermore, the root system has a simple description in terms of the ε_i.

For $x \in X$, let $\phi_i(x)$ be the projection of x on $\mathbb{Z}\varepsilon_i$. Given an integer d, in order to show that $x \notin dX$ it suffices to check that $2\phi_i(x) \not\equiv 0 \pmod{d}$; this is actually a rather crude estimate. However, using this test, it is routine to check for every $\gamma \in \Pi_+ \setminus \{0\}$ and every root α that

$$\gamma - \alpha \not\equiv 0 \pmod{4X}$$
$$\gamma \not\equiv 0 \pmod{5X}$$
$$\gamma - \alpha \not\equiv 0 \pmod{5X}$$
$$\mathrm{sym}(\gamma - \alpha) \not\equiv 0 \pmod{6X} \quad \text{when } \Phi = D_\ell \text{ and } \sigma \text{ has order 2}$$

Conditions 6.2 and 6.1 now follow from Remarks 6.3 and 6.7 and Lemma 6.5.

The situations where $\Phi = D_4$ and $\Phi = B_3$ are more complex. For type D_4 there are 3 possible simple modules which must be considered, and for type B_3 there are 2. Note also that for type D_4, there are 3 possibilities for σ; when σ has order 1 or 2, the argument proceeds as above. When σ has order 3, one may use Lemma 6.6. Using these techniques, it is routine to verify the results by computer in these cases. $\qquad\square$

Remark 6.10. We now describe some cases where Condition 6.1 fails to hold. In these case, one knows that the algebraic group G possesses no length two indecomposable rational modules with dimensions smaller than ℓp; however, different techniques would be required to compute the relevant Ext groups for $G(\mathbb{F}_q)$.

Let $\Phi = A_\ell$ and let $q = p = 3$ so that $q - 1 = p - 1 = 2$. In this situation, there are non-trivial simple modules L and L' with the property that $\dim_k L + \dim_k L' \leq 3\ell$. However, the conditions given in Proposition 6.4 above, i.e. the conditions from [5], do not provide enough data to demonstrate that $\mathrm{Ext}^1_{G(\mathbb{F}_3)}(L, L') = 0$. Indeed, take for L the simple module with high weight ϖ_1 and for L' the simple module with high weight ϖ_ℓ. Then $\varpi_1 + \varpi_\ell = \tilde{\alpha} \in \Phi$. Observe that $\varpi_1 + \varpi_\ell - (-\tilde{\alpha}) \in 2X$ whence $\varpi_1 + \varpi_\ell \equiv -\tilde{\alpha} \pmod{2X}$, but $\varpi_1 + \varpi_\ell \neq -\tilde{\alpha}$. Thus, Condition 6.1 fails to hold.

Let $\Phi = C_\ell$, and $q = p = 5$. Let $L = L(\varpi_1)$, then $\dim_k L = 2\ell$ so that $2 \dim_k L < 5\ell$. Now, $2\varpi_1 - (-\tilde{\alpha}) = 4\varpi_1 \in 4X$ so Condition 6.1 fails to hold.

7. CONCLUSIONS

7.1. **Proof of Theorem 1.** We are now in a position to prove the main result of this paper.

Proof of Theorem 1. Suppose that V is a $kG(\mathbb{F}_q)$ module with $\dim_k V \leq cp$. Noting the restrictions made in Table 1, we may suppose that $\ell > 1$.

Assume (Φ, p, r) is not among the triples listed in Table 3. If L, L' are any two composition factors of V, then $\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L, L') = 0$ by Theorem 2; it follows immediately that V is completely reducible.

For the remaining triples (Φ, p, r), i.e. those appearing in Table 3 but not in Table 1, the Theorem follows from Propositions 6.8 and 6.9. $\qquad\square$

7.2. **Extensions occurring for small fields.** We conclude with some remarks discussing low dimensional indecomposable modules. These indecomposables necessitate some of the restrictions on q given in Table 1.

Remark 7.1. Let p be any prime and consider the group $\mathcal{G} = \mathrm{SL}_2(\mathbb{F}_p)$. It was pointed out in [8] that $\mathcal{G}$ possesses indecomposable modules of length two with dimension $p - 1$; see [8] Remark 2.3 and [1], p. 49.

Remark 7.2. Let $p = 2$, and consider the group $\mathcal{G} = \mathrm{SL}_3(\mathbb{F}_2)$. We claim that $\mathcal{G}$ possesses indecomposable modules of length two with dimension less than $\ell p = 4$. It is known that $\mathrm{SL}_3(\mathbb{F}_2) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$; see [13], §10, especially the remarks in the Exercise on p. 168. $\mathcal{G}$ is a simple group of order $168 = 3 \cdot 7 \cdot 8$.

Regarding $\mathcal{G}$ as a projective linear group in characteristic 7, one obtains an action of $\mathcal{G}$ on the 8 element set $\Omega = \mathbb{P}^2\mathbb{F}_7$, the set of lines through 0 in $\mathbb{F}_7{}^2$. Let B_7 be the stabilizer of an element $\omega \in \Omega$; then $|B_7| = 21$. Denote by $V = k\Omega = \mathrm{Ind}_{B_7}^{\mathcal{G}}(k)$ the permutation representation (over the field k of characteristic 2) of this $\mathcal{G}$-set.

Let $P_2 < G$ be a 2-Sylow subgroup of $\mathcal{G}$. Since $P_2 \cap B_7 = \{1\}$, it is clear that P_2 is a full set of coset representations for $\mathcal{G}/B_7$. In particular, the restriction of the permutation representation $\mathrm{res}_{P_2}^{\mathcal{G}}(V)$ affords the regular representation for kP_2. Applying [6], Proposition 19.5 (ix) and (viii), one deduces that V is *projective* as a $k\mathcal{G}$ module.

It is straightforward to verify that the lattice of submodules of V is as follows:
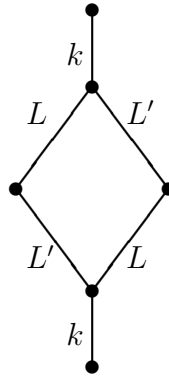
where k denotes the trivial representation, L is a 3 dimensional simple representation, and L' is the dual of L. Thus, $V = P(0) = I(0)$ is the projective cover, and the injective hull, of the trivial module k. According to [9], $H^1(\mathcal{G}, L(\varpi_1)) \simeq k$, hence we have

$$\mathrm{Ext}_{\mathcal{G}}^1(k, L) \simeq \mathrm{Ext}_{\mathcal{G}}^1(k, L') \simeq \mathrm{Ext}_{\mathcal{G}}^1(L, k) \simeq \mathrm{Ext}_{\mathcal{G}}^1(L', k) \simeq k;$$

thus, sections of V give all of the indecomposable length 2 modules with composition factors $\{k, L\}$ or $\{k, L'\}$.

We remark that one can locate the projective covers $P(\varpi_1)$ and $P(\varpi_2)$ for L and L' as summands of $P(0) \otimes_k L$ and $P(0) \otimes_k L'$. Using these modules, one can compute the first few terms of a projective resolution of k and verify the cohomology calculation from [9] cited above.

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