ON THE CENTRALIZER OF THE SUM OF COMMUTING NILPOTENT ELEMENTS

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To Eric Friedlander, on his 60th birthday.

ABSTRACT. Let X and Y be commuting nilpotent K-endomorphisms of a vector space V, where K is a field of characteristic $p \geq 0$. If $F = K(t)$ is the field of rational functions on the projective line $\mathbf{P}^1_{/K}$, consider the $K(t)$-endomorphism $A = X + tY$ of V. If $p = 0$, or if $A^{p-1} = 0$, we show here that X and Y are tangent to the unipotent radical of the centralizer of A in $\mathrm{GL}(V)$. For all geometric points $(a : b)$ of a suitable open subset of $\mathbf{P}^1$, it follows that X and Y are tangent to the unipotent radical of the centralizer of $aX + bY$. This answers a question of J. Pevtsova.

Let G be a connected and reductive algebraic group defined over an arbitrary field K of characteristic $p \geq 0$. Write $\mathfrak{g} = \mathrm{Lie}(G)$, and consider the extension field $F = K(t)$ with t transcendental over K. For convenience, we fix an algebraically closed field k containing both K and t.

If $X, Y \in \mathfrak{g}(K)$ are nilpotent and $[X, Y] = 0$, then $A = X + tY \in \mathfrak{g}(F)$ is again nilpotent. Write C for the centralizer of A in G, and write $R_u C$ for the unipotent radical of C. Under favorable restrictions on the characteristic, the groups C and $R_u C$ are defined over $K(t)$. In this note, I want to answer – at least in part – a question put to me by Julia Pevtsova at the July 2004 meeting in Snowbird, Utah. With notation as before, this question may be stated as follows:

Question 1. When is it true that $X, Y \in \mathrm{Lie}\, R_u C$?

To begin the investigation, the first section of the paper includes some elementary results concerning G-varieties in case the algebraic group G acts with a finite number of orbits. For the most part, the use of these results could be avoided in the present application, but there is perhaps some interest in recording them.

After these preliminaries, I am mainly going to investigate Question 1 in case the K-group is $G = \mathrm{GL}(V)$, where V is a finite dimensional k-vector space defined over K; this means there is a given K-subspace $V(K)$ for which the inclusion induces an isomorphism $V(K) \otimes_K k \simeq V$.

The second section contains well-known material on nilpotent orbits, mainly for the group $\mathrm{GL}(V)$; this material is used in section three where we prove our main result –

Theorem 21 – giving a partial answer to Question 1 when G is the group $\mathrm{GL}(V)$. A final section contains some remarks about more general semisimple groups.

Let me make a few remarks about possible reasons for interest in the main result of this paper. Pevtsova's interest concerns finite group schemes over a field K of characteristic $p > 0$; see e.g. [FP]. Basic but important examples are the commutative, étale, unipotent group schemes; consider e.g. a constant finite group scheme E which "is" an elementary Abelian p-group. If (ρ, M) is a K-representation of E, the matrices $1 - \rho(g) = \rho(1 - g) \in \mathrm{End}_K(M)$ are nilpotent for $g \in E$. More generally, if x is in the augmentation ideal of the group algebra KE, then $\rho(x)$ is nilpotent, and is a linear combination of commuting nilpotent matrices $\rho(1 - g)$ for various elements $1 \neq g$ of E. Pevtsova's question was aimed at understanding properties of the Jordan block structure of suitably generic such x.

In a somewhat different direction, if G is a reductive group over K and $\mathcal{N} \subset \mathfrak{g}$ denotes the variety of nilpotent elements, one is interested in studying the subvariety $\mathcal{V}_2 \subset \mathcal{N} \times \mathcal{N}$ of commuting pairs:

$$\mathcal{V}_2 = \{(X_1, X_2) \in \mathcal{N}^2 \mid [X_1, X_2] = 0\};$$

see e.g. [Pr03]. Any K-point

$$x = (X_1, X_2) \in \mathcal{V}_2(K)$$

determines a nilpotent element $A = X_1 + tX_2 \in \mathfrak{g}(F)$ with $F = K(t)$ as before. One might hope to exploit the results of this paper to study properties of the variety $\mathcal{V}_2$.

1. Groups acting with finitely many orbits

In this section, we work "geometrically" – i.e. over the algebraically closed field k. The results recorded here are elementary and without doubt are well-known; however, I don't know of an adequate reference.

Let W be an irreducible affine k-variety with coordinate algebra $A = k[W]$. [I will identify k varieties with their k-points: $W = W(k)$.] For an extension field k' of k, write $W(k')$ for the k'-points of W, and write $W_{/k'}$ for the k'-variety obtained by extension of scalars:

$$W_{/k'} = W \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k').$$

We will be concerned here with the case where a k-group acts on W with a finite number of orbits.

1.1. **Invariance of the number of orbits.** Begin with the following:

Lemma 2. *If W is the union $W = W_1 \cup W_2 \cup \cdots \cup W_n$ of locally closed subvarieties W_j, then W_i is a non-empty open subset of W for some $1 \leq i \leq n$.*

Proof. For $1 \leq j \leq n$, write $W_j = C_j \cap U_j$ where $C_j \subset W$ is closed and $U_j \subset W$ is open. Since W is contained in the union of the C_j and irreducible, we find $W \subseteq C_i$ for some i and the lemma follows. $\square$

Let G be a connected linear algebraic group over k acting by k-automorphisms on the variety W. Let $x \in W = W(k)$, and let $\mathcal{O} = G.x$. Since $\mathcal{O}$ is a k-variety, one

may speak of its k'-points $\mathcal{O}(k') \subseteq W(k')$. On the other hand, one may regard x as an element of $W(k')$ and form its $G(k')$-orbit.

Lemma 3. *Let x be as above, and suppose that the extension field k' of k is itself algebraically closed. Then we have*

$$(1) \qquad\qquad G(k')x = \mathcal{O}(k').$$

Proof. Since $\mathcal{O}$ is locally closed, we may replace W by the closure of $\mathcal{O}$, and so suppose $\mathcal{O}$ to be open in W. Since $W - \mathcal{O}$ is a union of G-orbits each of dimension $< \dim \mathcal{O}$, $W(k') - \mathcal{O}(k')$ is $G(k')$-stable, and so $\mathcal{O}(k')$ is $G(k')$-stable [1] . Since $x \in \mathcal{O}(k')$, the containment $\subseteq$ of (1) is immediate.

One finds e.g. in [Spr98, Proposition 1.9.4 and Theorem 1.9.5] the elementary proof – which goes back to Chevalley and Weil – that the image $\phi(X)$ of a dominant morphism of affine k-varieties $\phi : X \to Y$ contains a non-empty open subset of Y. That proof shows more precisely that there is some regular function $0 \neq f \in k[Y]$ such that $D(f)(k') \subseteq \phi(X(k'))$ for each algebraically closed field k' containing k; here $D(f)$ is the "distinguished open" subset of Y determined by the non-vanishing of f.

Apply this now to the (dominant) orbit map $(g \mapsto gx) : G \to W$ to find $0 \neq f \in A = k[W]$ such that $D(f) \subseteq \mathcal{O}$ and $D(f)(k') \subseteq G(k')x$. Since $\mathcal{O} = Gx$ is a Noetherian space, there are elements $g_1, \ldots, g_n \in G = G(k)$ such that Gx is the union of the $g_i D(f)$. Then $\mathcal{O}(k')$ is the union of the $g_i D(f)(k')$. On the other hand, $G(k')x$ contains $D(f)(k')$ and hence also contains each $g_i D(f)(k')$; this proves the containment $\supseteq$ of (1) and completes the proof of the lemma. $\qquad\square$

We are now going to show:

Proposition 4. *Let k' be an algebraically closed extension field of k. Assume that G has $n < \infty$ orbits on $W = W(k)$. Then each $G(k')$-orbit in $W(k')$ has a k-rational point. In particular, $G(k')$ has n orbits on $W(k')$.*

Note that if there are an infinite number of G-orbits on W, there may indeed by $G(k')$-orbits on $G(k')$ without k rational points. This phenomenon already occurs in case G acts trivially on a positive dimensional variety W.

In view of Lemma 3 and the fact that any G-orbit in W is a locally closed subvariety [Spr98, Lemma 2.3.3], it is clear that Proposition 4 follows from the Lemma which follows.

Lemma 5. *Suppose that the irreducible affine k-variety W is a union*

$$(2) \qquad\qquad W = L_1 \cup \cdots \cup L_n$$

where the L_i are non-empty, locally closed subvarieties, and that k' is any field containing k. Then

$$W(k') = L_1(k') \cup \cdots \cup L_n(k').$$

Proof. After possibly increasing n and replacing the L_i by smaller locally closed subvarieties, we may suppose for $i = 1, 2, \ldots, n$ that the closure of L_i is the closed set

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[1]One may avoid arguing the $G(k')$-stability of $\mathcal{O}(k')$ by applying [DG70, II.5.3.2(a)].

$\mathcal{V}(J_i)$ defined by an ideal $J_i = \sqrt{J_i} \lhd k[W]$, and that L_i has the form $D(f_i) \cap \mathcal{V}(J_i)$ for a non-0 regular function $f_i \in k[W]$.

The condition (2) may be restated:

(*) for each k-algebra homomorphism $\alpha : k[W] \to k$, there is some $1 \le i \le n$ with $\alpha(J_i) = 0$ and $\alpha(f_i) \ne 0$.

Any point of $W(k')$ is given by a k-homomorphism $\alpha : k[W] \to k'$. To prove the lemma, we only must argue that $\alpha(J_i) = 0$ and $\alpha(f_i) \ne 0$ for some $1 \le i \le n$, since then α determines a point of $L_i(k')$.

Let $I = \ker \alpha$. The algebra $k[W]/I$ is isomorphic to a k-subalgebra of the field k'; in particular, I is a prime ideal and so the closed subset $\mathcal{V}(I)$ of W is an irreducible k-variety.

Since $\mathcal{V}(I)$ is contained in the union of the L_i, it follows from Lemma 2 that $\mathcal{W} = \mathcal{V}(I) \cap L_m$ is a non-empty open subset of $\mathcal{V}(I)$ for some $1 \le m \le n$. If $d = \dim \mathcal{V}(I)$, the closure of $\mathcal{W} = \mathcal{V}(I) \cap L_m$ is a closed subset of $\mathcal{V}(I)$ of dimension d; by irreducibility, $\mathcal{V}(I)$ is precisely the closure of $\mathcal{W}$. On the other hand, the closure of $\mathcal{W}$ lies in the closure of L_m, which is $\mathcal{V}(J_m)$; from this we find that $\mathcal{V}(I) \subseteq \mathcal{V}(J_m)$. Since $I = \sqrt{I}$ and $J_m = \sqrt{J_m}$ we deduce from Hilbert's Nullstellensatz that $J_m \subseteq I$; thus $\alpha(J_m) = 0$. Since $\mathcal{V}(I) \cap L_m$ is non-empty, in particular $\mathcal{V}(I) \cap D(f_m)$ is non-empty; thus $f_m \notin I$. This means that $\alpha(f_m) \ne 0$, which completes the proof of the lemma. $\qquad\square$

Remark 6. A different proof of Proposition 4 due to Guralnick may be found in [GLMS, Prop. 1.1].

1.2. **Subvarieties of a linear G-representation.** Let V be a finite dimensional k-vector space on which the algebraic group G acts linearly. Let $W \subset V$ be an irreducible G-invariant subvariety on which G has finitely many orbits. Assume as well that $kx \subset W$ for each $x \in W$.

Since the set $k^\times x$ lies in W, it only meets a finite number of G-orbits; thus there is an orbit $\mathcal{O} \subset W$ such that $k^\times x \cap \mathcal{O}$ is infinite. Hence there is some $\beta \in k^\times$ such that $k^\times x \cap G(\beta x)$ is infinite. Since G acts linearly on V, it follows at once that $k^\times x \cap Gx$ is infinite.

Consider the subgroup $N(x) = \{g \in G \mid gx \in k^\times x\} \le G$; there is a homomorphism $\lambda : N(x) \to \mathbf{G}_m$ determined by the condition $gx = \lambda(g)x$ for $g \in N(x)$. Observe that the image of λ is an infinite subgroup of $\mathbf{G}_m$. Indeed, any $\alpha \in k^\times$ such that $\alpha x \in k^\times x \cap Gx$ lies in the image of λ.

Since $\mathbf{G}_m$ is a connected subgroup of dimension 1, the image of λ is in fact all of $\mathbf{G}_m$. We conclude:

(3)   if $x \in W$, then $k^\times x \subset Gx$.

Fix $v, w \in W$, and assume that

$$av + bw \in W \text{ for each } (a, b) \in k^2.$$

Since W is stable under the scalar $k^\times$ action on V, this is a "projective" condition; i.e. we may make instead the equivalent assumption:

(4) $av + bw \in W$ for each point $(a : b) \in \mathbf{P}^1$.

Proposition 7. *Let $v, w \in W$ and assume that (4) holds.*

(1) *There is a G-orbit $\mathcal{O} \subset W$ and a non-empty open subset $\mathcal{U} \subset \mathbf{P}^1_{/k}$ such that $av + bw \in \mathcal{O}$ if $(a : b) \in \mathcal{U}$ and $\dim G(av + bw) < \dim \mathcal{O}$ if $(a : b) \in \mathbf{P}^1 - \mathcal{U}$.*

(2) *Let $k' \supset k$ be an extension field and let $t \in k'$ be transcendental over k. Then $v + tw \in \mathcal{O}(k')$ so that $\mathcal{O}(k_1) = G(k_1)(v + tw)$ for any algebraically closed field k_1 containing k'.*

Proof. Let $\phi : \mathbf{A}^2 \to W$ be the morphism $(a, b) \mapsto av + bw$. The image of ϕ is a closed and irreducible subvariety S of W. Since G has finitely many orbits on W, it follows from Lemma 2 that $S \cap \mathcal{O}$ is open in S for a unique G-orbit $\mathcal{O} \subset W$. Moreover, since S is closed, it is contained in the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$.

Thus $\mathcal{U}_1 = \phi^{-1}(\mathcal{O} \cap S)$ is an open subset of $\mathbf{A}^2$ with the property that $av + bw \in \mathcal{O}$ whenever $(a, b) \in \mathcal{U}_1$ and $av + bw \in \overline{\mathcal{O}} - \mathcal{O}$ whenever $(a, b) \in \mathbf{A}^2 - \mathcal{U}_1$.

To complete the proof of (1), view $\mathbf{A}^2 - 0$ as a $\mathbf{G}_m$-bundle $\pi : \mathbf{A}^2 - 0 \to \mathbf{P}^1$. Since π is a flat morphism of finite type, it is open – e.g. by [Li02, Exerc. 4.3.9] – so that $\mathcal{U} = \pi(\mathcal{U}_1)$ is the desired open subset of $\mathbf{P}^1$.

For (2), let $\eta \in \mathbf{P}^1$ be the generic point. Identify $k(t)$ with $k(\mathbf{P}^1)$, and view

$$\bar{\eta} = (1 : t) \in \mathbf{P}^1(k')$$

as a geometric point over η. Since η is a point of $\mathcal{U}$ (in the sense of schemes), we have $\bar{\eta} \in \mathcal{U}(k')$. Thus $v + tw \in \mathcal{O}(k')$, and the remainder of (2) follows from Lemma 3. $\square$

Remark 8. In the sequel, we will apply the previous result to $G = \mathrm{GL}(V)$ acting by the adjoint representation on its Lie algebra $\mathfrak{gl}(V)$. The nilpotent variety $\mathcal{N} \subset \mathfrak{gl}(V)$ satisifes $kX \subset \mathcal{N}$ for each $X \in \mathcal{N}$, and $\mathrm{GL}(V)$ has finitely many orbits on $\mathcal{N}$. Moreover, (4) holds for any pair $X, Y \in \mathcal{N}$ for which $[X, Y] = 0$.

2. Background for $\mathrm{GL}(V)$

Let us recall how to recognize the unipotent radical of the centralizer of a nilpotent element for the group $G = \mathrm{GL}(V)$. If $A \in \mathfrak{gl}(V)$ is any nilpotent element, the A-exponent of $v \in V$ is the non-negative integer

(5) $$\mu(v) = \mu(A; v) = \min(n \geq 0 \mid A^n v = 0).$$

The vectors $v_1, \ldots, v_n \in V$ are said to be A-independent provided that the set

(∗) $\{A^j v_i \mid 1 \leq i \leq n, \ 0 \leq j \leq \mu(v_i) - 1\}$

is linearly independent over k. The vectors $v_1, \ldots, v_n \in V$ form an A-basis if (∗) is a k-basis for V.

We recall some basic results. If $A \in \mathfrak{gl}(V)$ is nilpotent, there is an A-basis of V. If $v_1, \ldots, v_n$ is an A-basis, ordered such that $\mu(v_1) \geq \mu(v_2) \geq \cdots \geq \mu(v_n)$, write $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ for the partition of $\dim V$ whose parts are $\lambda_i = \mu(v_i)$. The partition λ is independent of the choice of A-basis for V, and the $\mathrm{GL}(V)$-orbit of A depends only on the partition λ, which is thus called the partition of A.

A cocharacter of an algebraic group G is a homomorphism $\mathbf{G}_m \to G$; cocharacters of $\mathrm{GL}(V)$ may be identified with $\mathbf{Z}$-gradings of V. Indeed, if $\chi : \mathbf{G}_m \to \mathrm{GL}(V)$ is a

cocharacter, the weight spaces

$$V(m) = V(\chi; m) = \{v \in V \mid \chi(s)v = s^m v \ \forall s \in \mathbf{G}_m\}$$

determine a $\mathbf{Z}$-grading $V = \bigoplus_{m \in \mathbf{Z}} V(m)$ of V. Conversely, if $V = \bigoplus_{m \in \mathbf{Z}} V(m)$ is a $\mathbf{Z}$-grading, there is a unique cocharacter $\chi : \mathbf{G}_m \to G$ for which $V(m) = V(\chi; m)$.

We record:

Lemma 9. *Let $A \in \mathfrak{gl}(V)$ be nilpotent with partition λ. An A-basis $\{v_1, \ldots, v_n\}$ of V determines a unique cocharacter $\chi : \mathbf{G}_m \to \mathrm{GL}(V)$ for which $V(m) = V(\chi; m)$ is spanned by the vectors*

$$(6) \qquad\qquad A^j v_i \quad with \quad m = -\lambda_i + 1 + 2j$$

for $m \in \mathbf{Z}$.

Note that the cocharacter χ depends only on A and the choice of an A-basis for V; we say that χ *is a cocharacter associated to A.*

Under the adjoint action of $\mathrm{GL}(V)$ on its Lie algebra $\mathfrak{gl}(V)$, the grading determined by χ has homogeneous components

$$\mathfrak{gl}(V)(m) = \{C \in \mathfrak{gl}(V) \mid C(V(j)) \subset V(j + m) \text{ for each } j\}$$
$$= \{C \in \mathfrak{gl}(V) \mid \mathrm{Ad}(\chi(s))C = s^m C \ \forall s \in \mathbf{G}_m\}.$$

for $m \in \mathbf{Z}$. In particular, $A \in \mathfrak{gl}(V)(2)$.

The cocharacter χ determines a unique parabolic subgroup $P(\chi) < \mathrm{GL}(V)$ whose Lie algebra is

$$\mathfrak{p}(\chi) = \sum_{j \geq 0} \mathfrak{gl}(V)(j);$$

moreover, if $U = R_u P(\chi)$, then

$$\mathfrak{u} = \mathrm{Lie}(U) = \sum_{j > 0} \mathfrak{gl}(V)(j).$$

Proposition 10. *Let $A \in \mathfrak{gl}(V)$ be nilpotent. If $B \in \mathfrak{gl}(V)$ satisfies $[A, B] = 0$, then $B \in \mathfrak{p}(\chi)$, where the cocharacter χ is associated with A. Similarly, if $g \in \mathrm{GL}(V)$ satisfies $\mathrm{Ad}(g)A = A$, then $g \in P(\chi)$.*

Proof. See [Ja04, 3.10]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 11. *Any two cocharacters associated with A are conjugate by an element of $\mathrm{GL}(V)$ centralizing A.*

Proof. Indeed, any two A-bases are conjugate by an element centralizing A. $\qquad\square$

Proposition 12. *Let $A \in \mathfrak{gl}(V)$ be nilpotent, and let χ be a cocharacter associated with A. If $P = P(\chi)$, then P is the instability parabolic subgroup for the unstable vector $A \in \mathfrak{g}$ in the sense of Kempf [Ke78]. In particular, P is independent of the choice of A-basis for V.*

Proof. The fact that P is the instability parabolic follows from the discussion (and references) in §4; see also [Ja04, Pr02, Mc04]. The fact that P is independent of the choice of A-basis for V follows from general results about the instability parabolic. However, there is an elementary proof that P is independent of the choice of A-basis: if χ and χ' are two cocharacters associated with A, then by Proposition 11, the cocharacters χ and χ' are conjugate by $g \in \mathrm{GL}(V)$ with $\mathrm{Ad}(g)A = A$. Thus $P(\chi') = gP(\chi)g^{-1} = P(\chi)$ since $g \in P(\chi)$ by Proposition 10. $\square$

Remark 13. If $L \supset K$ is a field extension and if $A \in \mathfrak{gl}(V)(L)$ is nilpotent, then there is an A-basis $v_1, \ldots, v_n \in V(L)$. For such a choice of A-basis, the homogeneous components $V(m)$ and $\mathfrak{gl}(V)(m)$ are defined over L for $m \in \mathbf{Z}$. Equivalently: the cocharacter χ determined by this choice of A-basis is defined over L. Thus the parabolic subgroup $P(\chi)$ is defined over L.

The choice of cocharacter χ associated with A determines a Levi factor $L(\chi)$ in $P(\chi)$: take $L(\chi)$ to be the subgroup $\prod_{i \in \mathbf{Z}} \mathrm{GL}(V(\chi; i)) \leq \mathrm{GL}(V)$.

Denote by C the centralizer of the nilpotent $A \in \mathrm{GL}(V)$, and choose a cocharacter χ associated with A. We have:

Proposition 14. *Let $C_\chi = C \cap L(\chi)$ and $R = C \cap R_u P(\chi)$. Then $C = C_\chi \cdot R$ is a semidirect product, C_χ is a reductive group isomorphic to a product of groups GL_r for various r, and R is the unipotent radical of C.*

Proof. [Ja04, Prop. 3.10 and Prop. 3.8.1]. $\square$

3. The main result

We begin with a few preliminary results.

3.1. **Modifying an A-basis.** Let A be a nilpotent endomorphism of V, choose an A-basis $\{v_1, \ldots, v_n\}$ of V; put $\lambda_i = \mu(v_i)$ for $1 \leq i \leq n$, and assume that $\lambda_1 \geq \cdots \geq \lambda_n$.

Assume that B is a second nilpotent endomorphism of V and that $[A, B] = 0$. The choice of A-basis made above determines a cocharacter χ as in Lemma 9. By Proposition 10, we may write $B = \sum_{i \geq 0} B_i$ with $B_i \in \mathfrak{g}(V)(\chi; i)$.

Since $\chi(\mathbf{G}_m)$ normalizes the centralizer of A, we find that $[A, B_0] = 0$ as well. It follows that the endomorphism B_0 is determined by its values on the A-basis vectors $v_1, \ldots, v_n$. In particular, if $B_0 \neq 0$, then $B_0 v_i \neq 0$ for some $1 \leq i \leq n$.

Lemma 15. *Fix $1 \leq i \leq n$, and assume that $B_0 v_i \neq 0$. Then*

  (1) $\mu(Bv_i) = \mu(A; Bv_i) = \lambda_i$, *and*
  (2) *for some $j \neq i$ with $1 \leq j \leq n$ and $\lambda_j = \lambda_i$, the vectors*

$$v_1, \ldots, v_{j-1}, Bv_i, v_{j+1}, \ldots, v_n$$

  form an A-basis for V.

Proof. Since A and B commute, it is clear that $A^{\lambda_i} Bv_i = 0$. To complete the proof of (1), we must argue that $A^{\lambda_i - 1} Bv_i \neq 0$. According to [Ja04, 3.1(1)], we have

$$(7) \qquad Bv_i = \sum_{j=1}^{n} \sum_{\ell = \max(0, \lambda_j - \lambda_i)}^{\lambda_j - 1} c_{\ell, j} A^\ell v_j$$

for certain $c_{\ell,j} \in k$. It follows that

$$(8) \qquad\qquad B_0 v_i = \sum_{\lambda_j = \lambda_i} a_j v_j \quad \text{with } a_j = c_{0,j} \in k;$$

moreover, with notation as in (7)

$$Bv_i = B_0 v_i + w + Ax$$

$$(9) \qquad\qquad \text{where } w = \sum_{\lambda_j < \lambda_i} c_{0,j} v_j \text{ so that } A^{\lambda_i - 1} w = 0 \text{ and } A^{\lambda_i + 1} x = 0.$$

Indeed, to verify (9), notice that if $\lambda_j < \lambda_i$, then $A^{\lambda_i - 1} v_j = 0$ so that $A^{\lambda_i - 1} w = 0$. Now notice that $Bv_i - B_0 v_i - w$ has the form Ax for some $x \in V$. Finally, since Bv_i, $B_0 v_i$ and w lie in the kernel of A^{λ_i}, so does Ax.

It follows that

$$A^{\lambda_i - 1} B v_i \equiv A^{\lambda_i - 1} B_0 v_i \pmod{A^{\lambda_i} V}.$$

Since $B_0 v_i$ is non-zero and is a linear combination of the v_k with $\lambda_k = \lambda_i$, it is clear that $A^{\lambda_i - 1} B_0 v_i \not\equiv 0 \pmod{A^{\lambda_i} V}$; thus $A^{\lambda_i - 1} B v_i$ is non-zero $\pmod{A^{\lambda_i} V}$. In particular, $A^{\lambda_i - 1} B v_i$ is non-zero, which completes the proof of (1).

As to (2), one knows that B_0 is nilpotent since B is nilpotent. Thus the vectors v_i and $B_0 v_i$ are linearly independent. In the above expression (8) for $B_0 v_i$, it follows that $a_j \neq 0$ for some $1 \le j \le n$ with $j \neq i$ and $\lambda_j = \lambda_i$.

We are going to prove that (2) holds for this value of j. As a preliminary step, notice that $\{v_1, \ldots, v_n\}$ remains an A-basis if we replace v_j by $B_0 v_i$; thus we may and will suppose that $B_0 v_i = v_j$.

Let us write $\lambda = \lambda_i = \lambda_j$. With notation as in (9), recall that $A^{\lambda - 1} w = 0$. Let now

$$u_s = \begin{cases} v_s & s \neq j \\ Bv_i & s = j. \end{cases}$$

We must show that $\{u_1, \ldots, u_n\}$ is an A-basis of V. To see this, let $f_1, f_2, \ldots, f_n \in K[z]$ be polynomials for which $\sum_{s=1}^{n} f_s(A) u_s = 0$. We must argue that f_s is divisible by z^{λ_s} for each $1 \le s \le n$. In fact, it is enough to argue that z^{λ} divides f_j, since then the result follows from the A-independence of the set $\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\}$.

Using (9) we have

$$0 = f_j(A) B v_i + \sum_{s \neq j} f_s(A) v_s = f_j(A) v_j + f_j(A) w + f_j(A) Ax + \sum_{s \neq j} f_s(A) v_s.$$

If $f_j = 0$, then of course z^{λ} divides f_j and the proof is complete. If $f_j \neq 0$, let $\mu \ge 0$ be maximal such that $z^{\mu} \mid f_j$, and write $f_j = z^{\mu} \cdot g$ for a polynomial $g \in K[z]$ having non-zero constant term. We find then that

$$A^{\mu} g(A) v_j \equiv -A^{\mu} g(A) w - \sum_{s \neq j} f_s(A) v_s \pmod{A^{\mu+1} V}.$$

Since w is a linear combination of v_s with $\lambda_s < \lambda_j$, the right hand side is congruent to an expression of the form $\sum\limits_{s \neq j} h_s(A)v_s$ modulo $A^{\mu+1}V$ for polynomials $h_s \in K[z]$. Since the vectors $v_1, \ldots, v_n$ are A-independent, it follows that $A^\mu g(A)v_j \equiv 0 \pmod{A^{\mu+1}V}$. Since g has non-zero constant term, this is only possible if $\mu \geq \lambda$, as required. $\qquad \square$

3.2. **Recognizing the partition of a nilpotent endomorphism A.** Let A be a nilpotent endomorphism of V. Let $v_1, \ldots, v_n \in V$ and let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ be a partition of $\dim V$. Suppose that the set of vectors

$$\mathcal{B} = \{A^j v_i \mid 1 \leq i \leq n,\ 0 \leq j \leq \lambda_i - 1\}$$

forms a k-basis for V. For $i \geq 0$ write V_ℓ for the span of $\{A^j v_i \mid 1 \leq i \leq \ell,\ 0 \leq j < \lambda_i\}$; thus $V_0 = 0$.

Lemma 16. *The following are equivalent:*

(1) $A^{\lambda_j} v_j \in A^{\lambda_j} V_{j-1}$ *for each* $1 \leq j \leq n$,
(2) *for* $1 \leq j \leq n$ *there are vectors* $w_j \in V_{j-1}$ *such that* $A^{\lambda_j}(v_j - w_j) = 0$ *for* $1 \leq j \leq n$ *and such that* $\{v_j - w_j \mid 1 \leq j \leq n\}$ *is an A-basis of V,*
(3) λ *is the partition of A.*

In particular, if λ is the partition of A, then each subspace V_ℓ, $1 \leq \ell \leq n$, is A-invariant.

Proof. To prove (1) $\implies$ (2), choose for each $j = 1, 2, \ldots, n$ a vector $w_j \in V_{j-1}$ for which

$$A^{\lambda_j} v_j = A^{\lambda_j} w_j \quad \text{hence} \quad A^{\lambda_j}(v_j - w_j) = 0.$$

To see that the vectors $v_j' = v_j - w_j$ for $1 \leq j \leq n$ form an A-basis, just note that if M is the matrix of coefficients obtained upon expressing the vectors $A^s v_t'$ in terms of the K-basis $\{A^l v_m\}$, then M is unipotent and hence invertible.

The assertions (2) $\implies$ (1) and (2) $\implies$ (3) are immediate.

We finally prove (3) $\implies$ (2). Since λ is the partition of A, $A^{\lambda_1} = 0$; in particular, $A^{\lambda_1} v_1 = 0$. Apply [La93, Lemma III.7.6] to see that $(\lambda_2 \geq \cdots \geq \lambda_n)$ is the partition of the nilpotent endomorphism $\overline{A}$ of V/V_1 induced by A; by induction on n we find vectors $w_j' \in V_{j-1}$ for $2 \leq j \leq n$ such that

$$A^{\lambda_j}(v_j - w_j') \in V_1 \quad \text{for } 2 \leq j \leq n$$

and such that $v_2 - w_2', \ldots, v_n - w_n'$ is an $\overline{A}$-basis of V/V_1. Another application of [La93, Lemma III.7.6] now gives vectors $w_2'', \ldots, w_n'' \in V_1$ for which

$$v_1, v_2 - w_2' - w_2'', \ldots, v_n - w_n' - w_n''$$

is an A-basis for V. Since $V_1 \subset V_{j-1}$ for $j \geq 2$, we have $w_j = w_j' - w_j'' \in V_{j-1}$ as desired; thus (2) indeed holds. $\qquad \square$

3.3. **A nilpotent element of $\mathfrak{gl}(V)$ over $K(t)$.** Let p denote the characteristic of K, and recall that t is transcendental over K. Let us fix nilpotent elements $X, Y \in \mathfrak{gl}(V)(K)$, and let us suppose that $[X, Y] = 0$.

Write $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ for the partition of X, and fix once and for all an X-basis $v_1, \ldots, v_n \in V(K)$ for V. Thus

$$\mathcal{B}^0 = \{X^j v_i \mid 1 \leq i \leq n,\ 0 \leq j \leq \lambda_i - 1\}$$

is a K-basis for $V(K)$.

Consider the localization $\mathcal{A} = K[t]_{(t)}$ of the polynomial ring $K[t]$ at the maximal ideal $tK[t]$; its field of fractions is $F = K(t)$, and its maximal ideal is $\mathfrak{m} = (t) = t\mathcal{A}$. Write $\mathcal{V} = V(K) \otimes_K \mathcal{A}$. Each of the vectors in the set

$$\mathcal{B}^t = \{(X + tY)^j v_i \mid 1 \leq i \leq n,\ 0 \leq j \leq \lambda_i - 1\}$$

lies in $\mathcal{V}$. By assumption, the image in $V(K) = \mathcal{V}/t\mathcal{V}$ of $\mathcal{B}^t$ is $\mathcal{B}^0$; by the Nakayama lemma, $\mathcal{B}^t$ forms an $\mathcal{A}$-basis for $\mathcal{V}$. In particular, $\mathcal{B}^t$ is an $F = K(t)$-basis for $V(F)$.

For each $1 \leq \ell \leq n$, let us write $V_\ell^0(K)$ for the K-subspace of $V(K)$ spanned by the vectors

$$\mathcal{B}_\ell^0 = \{X^j v_i \mid 1 \leq i \leq \ell,\ 0 \leq j \leq \lambda_i - 1\}.$$

Similarly, let $V_\ell^t(F)$ be the F-subspace of $V(F)$ spanned by the vectors

$$\mathcal{B}_\ell^t = \{(X + tY)^j v_i \mid 1 \leq i \leq \ell,\ 0 \leq j \leq \lambda_i - 1\},$$

and let $\mathcal{V}_\ell$ be the $\mathcal{A}$-submodule of $\mathcal{V}$ spanned by $\mathcal{B}_\ell^t$. Of course, the image of $\mathcal{V}_\ell$ in $V(K) = \mathcal{V}/t\mathcal{V}$ is V_ℓ^0.

Lemma 17. *For $1 \leq \ell \leq n$, $\mathcal{V}_\ell$ is a direct summand of $\mathcal{V}$ as an $\mathcal{A}$-module. We have in particular:*

(1) $\mathcal{V}_\ell = V_\ell^t(F) \cap \mathcal{V}$, *and*
(2) $t\mathcal{V}_\ell = \mathcal{V}_\ell \cap t\mathcal{V}$.

Proof. Since $\mathcal{B}^t$ is an $\mathcal{A}$-basis of $\mathcal{V}$, the lemma is immediate. $\square$

Lemma 18. *Assume that the partition of $X + tY$ coincides with that of X; i.e. assume that $X + tY$ and X are $\mathrm{GL}(V)$-conjugate. For each $1 \leq \ell \leq n$, we have:*

(1) $\mathcal{V}_\ell$ *is $X + tY$-invariant, and*
(2) $(X + tY)^{\lambda_\ell} v_\ell \in \mathcal{V}_{\ell-1}$.

Proof. Fix $1 \leq \ell \leq n$. Since λ is the partition of $X + tY$, Lemma 16 shows that each $V_\ell^t(F)$ is $X + tY$-invariant. Since $\mathcal{V}_\ell = V_\ell^t(F) \cap \mathcal{V}$ by Lemma 17(1), the $X + tY$-invariance of $\mathcal{V}_\ell$ results from that of $\mathcal{V}$ and of $V_\ell^t(F)$; this proves (1).

Since

$$(X + tY)^{\lambda_\ell} v_\ell \in (X + tY)^{\lambda_\ell} V_{\ell-1}^t(F) \subset V_{\ell-1}^t(F),$$

we have $(X + tY)^{\lambda_\ell} v_\ell \in \mathcal{V} \cap V_{\ell-1}^t(F)$ by another application of Lemma 17(1). $\square$

Proposition 19. *Assume that the partition of $X + tY$ coincides with that of X; i.e. assume that $X + tY$ and X are $\mathrm{GL}(V)$-conjugate. Let χ be the K-cocharacter associated with X determined by the X-basis $v_1, \ldots, v_n$, and write $Y = Y_0 + Y_+$ with*

$$Y_0 \in \mathfrak{gl}(V)(\chi; 0) \quad and \quad Y_+ \in \sum_{j > 0} \mathfrak{gl}(V)(\chi; j).$$

If $p > 0$ assume that $X^{p-1} = 0$. Then $Y_0 = 0$.

Proof. We assume that $Y_0 \neq 0$ and deduce a contradiction. Let $1 \leq \ell \leq n$ be minimal with $Y_0 v_\ell \neq 0$. After possibly re-ordering those members of the X-basis $v_1, v_2, \ldots, v_n$ for which $\lambda_k = \lambda_\ell$, we may suppose that $\lambda_k > \lambda_\ell$ whenever $k < \ell$. According to Lemma 15, we may and will assume that $Y v_\ell = v_j$ for some $j > \ell$ with $\lambda_j = \lambda_\ell$.

Since λ is the partition of $X + tY$, Lemma 18 shows that $(X + tY)^{\lambda_\ell} v_\ell \in \mathcal{V}_{\ell-1}$. Since $X^{\lambda_\ell} v_\ell = 0$, we find by Lemma 17(2):

$$(X + tY)^{\lambda_\ell} v_\ell \in \mathcal{V}_{\ell-1} \cap t\mathcal{V} = t\mathcal{V}_{\ell-1}.$$

Thus we see

$$\frac{1}{t}(X + tY)^{\lambda_\ell} v_\ell = \sum_{j=1}^{\lambda_\ell} t^{j-1} \binom{\lambda_\ell}{j} X^{\lambda_\ell - j} Y^j v_\ell \in \mathcal{V}_{\ell-1}.$$

Since the image of $\mathcal{V}_{\ell-1}$ in the quotient $V(K) = \mathcal{V}/t\mathcal{V}$ is $V_{\ell-1}^0(K)$, it follows that

$$\lambda_\ell X^{\lambda_\ell - 1} Y v_\ell = \lambda_\ell X^{\lambda_\ell - 1} v_j \in V_{\ell-1}^0(K).$$

If $p > 0$, the condition $X^{p-1} = 0$ shows that $\lambda_\ell < p$; so in every case, λ_ℓ is non-zero in K. It follows that $X^{\lambda_\ell - 1} v_j = X^{\lambda_j - 1} v_j \in V_{\ell-1}^0(K)$, contradicting the assumption that $v_1, \ldots, v_n$ is an X-basis for V. This completes the proof. $\qquad\square$

3.4. A nilpotent element of $\mathfrak{gl}(V)$ over $\mathbf{P}^1$. Let $X, Y \in \mathfrak{gl}(V)(K)$ be nilpotent with $[X, Y] = 0$, and let $\mathscr{O}$ denote the structure sheaf of $\mathbf{P}^1 = \mathbf{P}^1_{/K}$. Write $\mathcal{L} = V(K) \otimes_K \mathscr{O}$, so that $\mathcal{L}$ is a free sheaf of $\mathscr{O}$-modules on $\mathbf{P}^1$. If η denotes the generic point of $\mathbf{P}^1$, the stalk $\mathscr{O}_\eta = K(\mathbf{P}^1)$ identifies with $F = K(t)$, and the stalk $\mathcal{L}_\eta$ identifies with $V(F)$.

Choose an $A = X + tY$-basis $v_1, \ldots, v_n \in V(F)$; for $1 \leq i \leq n$ and $j \geq 0$, we may regard $A^i v_j$ as an element of $\mathcal{L}_\eta$. Thus we may choose an affine open subset $\mathcal{W} \subset \mathbf{P}^1$ such that t is regular on $\mathcal{W}$ and such that $A^j v_i \in \Gamma(\mathcal{W}, \mathcal{L})$ for $1 \leq i \leq n$ and $0 \leq j < \lambda_j$.

For a point $x \in \mathbf{P}^1$, denote by $\mathfrak{m}_x$ the maximal ideal of the stalk $\mathscr{O}_x$, and let $K(x)$ be the field of fractions of $\mathscr{O}_x/\mathfrak{m}_x$; the $K(x)$-vector space $\mathcal{L}_x \otimes_{\mathscr{O}_x} K(x)$ may be identified with $V(K(x)) = V(K) \otimes_K K(x)$. If $\bar{x} = (a : b)$ is a geometric point of $\mathcal{W}$ over x, then $X, Y, X + tY$ act on $\mathcal{L}_x$ and so on $V(K(x))$; the maps induced on $V(K(x))$ are respectively X, Y, and some non-zero multiple of $aX + bY$ [2].

We now have:

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[2]The geometric point $\bar{x} = (a : b)$ over x is determined by a field embeding $\iota : K(x) \to L$ for a separably closed field L. We have assumed that t is regular at x - i.e. $a \neq 0$ so that $t \in \mathscr{O}_x$; if $\bar{t}$ denotes the image in $K(x)$ of $t \in \mathscr{O}_x$, then $\iota(\bar{t})$ is a multiple of b/a. Now, ι determines an embedding $\iota : V(K(x)) \to V(L)$; the map $aX + bY$ leaves stable the image of ι, and coincides with some multiple

Lemma 20. *If $v_1, \ldots, v_n \in V(F)$ is an $(X + tY)$-basis for V, there is a non-empty open subset $\mathcal{U}$ of $\mathbf{P}^1$ such that*

(1) $v_1, \ldots, v_n \in \mathcal{L}(\mathcal{U})$,

(2) *the vectors $A^j v_i$ for $1 \le i \le n$ and $0 \le j < \lambda_i$ form a basis for $\mathcal{L}(\mathcal{U})$ over $\mathcal{O}(\mathcal{U})$, and*

(3) *for each $x \in \mathcal{U}$, the vectors $v_1, \ldots, v_n \in V(K(x))$ form an $(aX + bY)$-basis of V for any geometric point $(a : b)$ over x.*

Proof. With notation as before, let $\mathcal{M} = \bigwedge^{\dim V} \mathcal{L}$, and consider the element

$$\omega = \bigwedge_{j=1}^{n} \bigwedge_{i=0}^{\lambda_j - 1} A^i v_j \in \Gamma(\mathcal{W}, \mathcal{M}).$$

Let $\mathcal{U}$ be a non-empty affine open subset of $\mathcal{W}$ for which the germ ω_x does not lie in $\mathfrak{m}_x \mathcal{M}_x$ for all points $x \in \mathcal{U}$ [of course, the set of all $x \in \mathcal{W}$ having that property is non-empty and open].

By construction, the vectors $\{A^j v_i \mid 1 \le i \le n, 0 \le j < \lambda_i\}$ form an $\mathcal{O}(\mathcal{U})$-basis of $\mathcal{L}(\mathcal{U})$, and the lemma follows. $\square$

3.5. The main theorem.

Theorem 21. *Consider the nilpotent element $A = X + tY \in \mathfrak{gl}(V)(F)$ where $X, Y \in \mathfrak{gl}(V)(K)$ are nilpotent and $[X, Y] = 0$. If $p > 0$, assume that $A^{p-1} = 0$.*

(1) $X, Y \in \operatorname{Lie} R_u C$, *where C is the centralizer of $A = X + tY \in \mathfrak{gl}(V)(F)$ in $\operatorname{GL}(V)$.*

(2) *There is a non-empty open subset $\mathcal{U}$ of $\mathbf{P}^1$ such that $X, Y \in \operatorname{Lie} R_u C_{(a:b)}$ for each geometric point $(a : b)$ of $\mathcal{U}$, where $C_{(a:b)}$ is the centralizer of $aX + bY$ in $\operatorname{GL}(V)$.*

Before giving the proof, let me first give an example to demonstrate that the theorem is not correct without some hypothesis on A.

Example 22. Let $X' \in \mathfrak{gl}(V)(K)$ be a regular nilpotent element, and write $d = \dim V$. Choose $v \in V(K)$ for which $\{v_i = (X')^i v \mid i = 0, 1, \ldots, d - 1\}$ is a basis for V; we will write $v_i = (X')^i v$ for $i \ge 0$ so that $v_i = 0$ for $i \ge d$. Now let

$$X = X' \oplus X' \in \mathfrak{gl}(V \oplus V)$$

and let

$$Y = ((v, w) \mapsto (0, v)) \in \mathfrak{gl}(V \oplus V).$$

Of course, $[X, Y] = 0$. We set $A = X + tY \in \mathfrak{gl}(V \oplus V)(F)$ and write $C \le \operatorname{GL}(V \oplus V)$ for the centralizer of A.

For $m \ge 0$, we have

$$A^m = (X + tY)^m = X^m + mtX^{m-1}Y.$$

---

of $X + \bar{t}Y : V(K(x)) \to V(K(x))$. In this sense, $aX + bY$ is independent of the choice of geometric point $\bar{x}$.

If $w_1 = (v, 0)$ and $w_2 = (0, v)$ we have:

$$A^m w_1 = (v_m, mt v_{m-1}) \quad \text{and} \quad A^m w_2 = (0, v_m)$$

for $m \geq 0$, where we have put $v_{-1} = 0$.

If $d \neq 0$ in K, the reader may verify that the partition of A is $\lambda = (d+1, d-1)$. Since this partition has distinct parts, a Levi factor of C is a torus so indeed $X, Y \in \operatorname{Lie} R_u P$.

However, if $d = 0$ in K, then A has partition (d, d). To see this, observe that $A^d w_1 = (0, dt v_{d-1}) = 0$, and $A^d w_2 = 0$; now verify that w_1, w_2 is an A-basis of $V \oplus V$. It is not true that $X \in \operatorname{Lie} R_u C$. Indeed,

$$X w_1 = (v_1, 0) = (v_1, t v_0) - t(0, v_0) = A w_1 - t w_2;$$

since $w_1, w_2 \in V(-d+1)$, we find that $X_0 \neq 0$ so that $X \notin \operatorname{Lie} R_u C$ [and so, of course, also $Y \notin \operatorname{Lie} R_u C$]. If $d = p$, A is p-nilpotent, i.e. we have $A^p = 0$, but $A^{p-1} \neq 0$.

$$* \qquad * \qquad *$$

Proof of Theorem 21. First use Lemma 20 to find an $(X + tY)$-basis $v_1, \ldots, v_n$ for $V(F)$ and an open subset $\mathcal{U} \subset \mathbf{P}^1$ satisfying the conclusion of that Lemma. If $x \in \mathcal{U}$ and $(a : b)$ is a geometric point over x, the $aX + bY$-basis of $\mathcal{L}_x \otimes_{\mathscr{O}_x} K(x) = V(K(x))$ obtained from the v_i determines a cocharacter $\chi_{(a:b)}$ associated to $aX + bY$. Especially, $\chi_{(1:t)}$ is the cocharacter associated with $X + tY$ determined by the vectors $v_i \in V(F)$.

Now, write $Y = Y_0 + Y_+$ for unique elements

$$Y_0 \in \mathfrak{gl}(V)(\chi_{(1:t)}; 0) \quad \text{and} \quad Y_+ \in \sum_{j > 0} \mathfrak{gl}(V)(\chi_{(1:t)}; j).$$

Since the $\mathscr{O}(\mathcal{U})$-basis of $\mathcal{L}(\mathcal{U})$ determined by the v_i consists of weight vectors for the torus $\chi_{(1:t)}(\mathbf{G}_m)$, and since $Y(\mathcal{L}(\mathcal{U})) \subset Y(\mathcal{L}(\mathcal{U}))$, one has that

$$Y_0(\mathcal{L}(\mathcal{U})) \subset \mathcal{L}(\mathcal{U}) \quad \text{and} \quad Y_+(\mathcal{L}(\mathcal{U})) \subset \mathcal{L}(\mathcal{U}),$$

or – what is the same – one has that

$$Y_0, Y_+ \in \mathfrak{gl}(V)(\mathcal{U}) = \mathfrak{gl}(V)(K) \otimes_K \mathscr{O}(\mathcal{U}).$$

For each geometric point $(a : b)$ over $x \in \mathcal{U}$, write $(Y_0)_{(a:b)}$ and $(Y_+)_{(a:b)}$ for the images of Y_0, Y_+ in $\mathfrak{gl}(V)(K(x)) = \mathfrak{gl}(V)(\mathscr{O}_x) \otimes_{\mathscr{O}_x} K(x)$. We have:

$$(Y_0)_{(a:b)} \in \mathfrak{gl}(V)(\chi_{(a:b)}; 0)$$

and

$$(Y_+)_{(a:b)} \in \sum_{j > 0} \mathfrak{gl}(V)(\chi_{(a:b)}; j).$$

Thus the theorem will follow from Proposition 14 provided that we only show $Y_0 = 0$. Moreover, it is enough to show that $(Y_0)_{(a:b)} = 0$ for all geometric points $(a : b)$ in some dense subset of $\mathcal{U}$.

Writing $K(\mathbf{P}^1) = K(t)$, we may apply Proposition 7 to find a non-empty open subset $\mathcal{U}' \subset \mathcal{U}$ such that $aX + bY$ is $\operatorname{GL}(V)$-conjugate to $X + tY$ for each geometric point $(a : b)$ of $\mathcal{U}'$.

We are now going to show that $(Y_0)_{(1:s)} = 0$ for each point of $\mathcal{U}'$ of the form $(1 : s)$ with $s \in K$. Since we may evidently replace K by an algebraic extension, we may

and will suppose that K is infinite; thus such points are indeed dense in $\mathcal{U}'$ and hence in $\mathcal{U}$.

So fix such a point $(1:s)$. Since $(1:s)$ is a point of $\mathcal{U}'$, we know that $X + sY \in \mathrm{GL}(V)$ is conjugate to $X + tY$. Since t and $t + s$ are both transcendental over K, $X + sY$ and $X + (t+s)Y$ have evidently the same partition; thus $X + sY$ is conjugate to $X + (t+s)Y$ as well. We may now apply Proposition 19 to the elements $X + sY$ and Y to see that $(Y_0)_{(1:s)} = 0$ as desired. This completes the proof of the theorem. $\square$

4. Other semisimple groups

Consider now more general groups G; for ease of exposition I'll assume that G is semisimple over K, and that the characteristic of K is *very good* for G.

Let $X \in \mathfrak{g}$ be nilpotent. A cocharacter $\phi : \mathbf{G}_m \to G$ is *associated to X* provided that:

A1. $X \in \mathfrak{g}(\phi; 2) =$ the 2-weight space of the torus $\phi(\mathbf{G}_m)$ under the adjoint representation on $\mathfrak{g}$, and

A2. for some choice of maximal torus $S < C_G(X)$, the image of ϕ lies in (L, L), where L is the Levi subgroup of G defined by $L = C_G(S)$.

When $G = \mathrm{GL}(V)$, the reader may easily check that the above definition agrees with that given in §2; namely, if $X \in \mathfrak{gl}(V)$ is nilpotent, then the cocharacters determined by X-bases of V as in Lemma 9 are precisely those satisfying A1 and A2. For any G, the nilpotent element X is *distinguished* in $\mathrm{Lie}(L)$ for a Levi subgroup L as in A2; for more on this see [Ja04, §4–5].

Remark 23. When $p = 0$, the map $\tau \mapsto d\tau(1)$ is a bijection between cocharacters associated with X and the set of all $H \in [X, \mathfrak{g}]$ such that $[H, X] = 2X$; cf. [Ja04, §5.5]. Thus the cocharacters associated with X are precisely those obtained by the Jacobson-Morozov Lemma.

Under our assumptions on G, there are always cocharacters associated to X [Mc04, Prop. 16]; see also [Pr02]. If X is K-rational, one can even find a cocharacter associated to X which is defined over K; see [Mc04, Theorem 26]. Any cocharacter $\chi : \mathbf{G}_m \to G$ determines a parabolic subgroup $P(\chi)$ of G; namely, the unique parabolic whose Lie algebra is $\bigoplus_{i \geq 0} \mathfrak{g}(\chi; i)$ where

$$\mathfrak{g}(\chi; i) = \{X \in \mathfrak{g} \mid \mathrm{Ad}(\chi(s))X = s^i X \; \forall s \in k\}.$$

According to [Ja04, 5.9], the parabolic subgroup $P(\phi)$ is independent of the choice of cocharacter ϕ associated to X; it is the instability parabolic of Kempf and Rousseau [Mc04, Prop. 18].

The analogues of Propositions 10 and 14 hold. Namely,

Proposition 24. *Let $A \in \mathfrak{g}(K)$ be nilpotent, let χ be a cocharacter associated with A, let $P = P(\chi)$, let $\mathfrak{p} = \mathrm{Lie}(P)$, and let $C = C_G(A)$ be the centralizer of A. Then*

(1) *$C_G(A)$ is defined over K and $\mathrm{Lie}\, C_G(A) = \mathfrak{c}_{\mathfrak{g}}(A)$,*

(2) *$\mathfrak{c}_{\mathfrak{g}}(A) \subset \mathfrak{p}$ and $C_G(A) \subset P$, and*

(3) *if $L(\chi)$ denotes the centralizer in G of $\chi(\mathbf{G}_m)$, then $C_\chi = C \cap L(\chi)$ is reductive, $R_u C = C \cap R_u P$, and $C = C_\chi \cdot R_u C$ is a Levi decomposition.*

Proof. (1) follows from the separability of orbits for semisimple groups in very good characteristic; see [SS70, I.5.2 and I.5.6] together with [Spr98, Prop. 12.1.2]. (2) is [Ja04, Prop. 5.9]. (3) is [Ja04, Prop. 5.10 and 5.11]; see also [Mc04, Corollary 29].   □

We want to consider the following hypothesis on G:

(L) There is a representation $\rho : G \to \mathrm{GL}(V)$ defined over K such that $d\rho$ is injective, and such that for each nilpotent $X \in \mathfrak{g}$ and each cocharacter χ of G associated with X, the cocharacter $\rho \circ \chi$ is of $\mathrm{GL}(V)$ associated with $d\rho(X) \in \mathfrak{gl}(V)$.

Remark 25. It follows from Remark 23 that the condition **(L)** holds for any faithful representation (ρ, V) when char $K = 0$. Indeed, let $X \in \mathfrak{g}$ be nilpotent, let χ be a cocharacter of G associated with X, and let $H = d\chi(1)$. Then $d\rho(H) = d(\rho \circ \chi)(1)$ in $\mathfrak{gl}(V)$. Moreover, clearly $[d\rho(H), d\rho(X)] = d\rho([H, X]) = d\rho(2X) = 2d\rho(X)$ and $d\rho(H) \in [d\rho(X), \mathfrak{gl}(\mathfrak{g})]$ so that $\rho \circ \chi$ is associated with $d\rho(X)$ by Remark 23. This verifies **(L)** .

The following is an immediate consequence of Theorem 21:

Theorem 26. *Let G be semisimple algebraic group defined over K, assume that the characteristic of K is very good for G, and assume that **(L)** holds. Let $X, Y \in \mathfrak{g}(K)$ with $[X, Y] = 0$, and suppose that $d\rho(X + tY)^{p-1} = 0$.*

(1) *Then $X, Y \in \mathrm{Lie}\, R_u C$ where $C = C_G(X + tY)$ is the centralizer of $X + tY$.*
(2) *There is a non-empty open subset $\mathcal{U}$ of $\mathbf{P}^1$ such that for each geometric point $(a : b)$ of $\mathcal{U}$, we have $X, Y \in \mathrm{Lie}\, R_u C_{(a:b)}$, where $C_{(a:b)} = C_G(aX + bY)$.*

Lemma 27. *Let $X \in \mathfrak{g}$ satisfy $X^{[p]} = 0$, and suppose that χ is a cocharacter associated with X.*

(1) *There is a homomorphism $\psi : \mathrm{SL}_2 \to G$ such that*

$$d\psi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X,$$

and such that the restriction of ψ to the diagonal torus of SL_2 identifies with the cocharacter χ.
(2) *$(\mathrm{Ad} \circ \psi, \mathfrak{g})$ is a tilting module for SL_2; its indecomposable summands are indecomposable tilting modules $T(n)$ for $n \leq 2p - 2$.*
(3) *$(\mathrm{Ad} \circ \psi, \mathfrak{g})$ is a semisimple SL_2-module if and only if $\mathrm{Ad} \circ \chi$ is a cocharacter of $\mathrm{GL}(\mathfrak{g})$ associated with $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$.*
(4) *If $\mathrm{ad}(X)^{p-1} = 0$, then $\mathrm{Ad} \circ \chi$ is a cocharacter of $\mathrm{GL}(\mathfrak{g})$ associated with $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$.*

Proof. The main result of [Mc05] yields (1). For (2) see [Sei00] or [Mc05, Prop. 36].

For (3), we first assume $(\mathrm{Ad} \circ \psi, \mathfrak{g})$ is semisimple. Since $T(n)$ is semisimple if and only if $n < p$, $(\mathrm{Ad} \circ \psi, \mathfrak{g})$ is restricted as well. If we choose a high weight vector in each simple summand, it is a consequence of the well-known description of restricted semisimple SL_2-modules that this collection of vectors is an $\mathrm{ad}(X)$-basis for $\mathfrak{g}$, and that $\mathrm{Ad} \circ \chi$ is the cocharacter determined by this $\mathrm{ad}(X)$-basis.

On the other hand, if $(\mathrm{Ad} \circ \psi, \mathfrak{g})$ is not semisimple, then it has an indecomposable summand $T(n)$ for some $p \leq n \leq 2p - 2$. Thus the n-th weight space of $\chi(\mathbf{G}_m)$ on $T(n)$ is non-zero. On the other hand, note that all Jordan blocks of $\mathrm{ad}(X)$ acting on $\mathfrak{g}$ have size $\leq p$. Thus if κ is a cocharacter of $\mathrm{GL}(\mathfrak{g})$ associated with $\mathrm{ad}(X)$, then all weights μ of $\kappa(\mathbf{G}_m)$ on $\mathfrak{g}$ satisfy $-p + 1 \leq \mu \leq p - 1$. This shows that κ and $\mathrm{Ad} \circ \chi$ are not conjugate, so that $\mathrm{Ad} \circ \chi$ is not associated to $\mathrm{ad}(X)$. This proves (3).

For (4), note that each non-zero nilpotent element of $\mathfrak{sl}_2$ acts with partition (p, p) on $T(n)$ for $p \leq n \leq 2p - 2$. Thus $\mathrm{ad}(X)^{p-1} = 0$ implies that $(\mathrm{Ad} \circ \psi, \mathfrak{g})$ is semisimple as an SL_2-module so that (4) follows from (3). $\square$

Proposition 28. *Assume that the characteristic p of K is 0 or $p > 2h - 2$ where h is the maximal Coxeter number of a simple component of G. Then **(L)** holds for G using the adjoint representation $(\mathrm{Ad}, \mathfrak{g})$. Moreover, if $p > 0$ and if $X \in \mathfrak{g}$ is nilpotent, then $Ad(X)^{p-1} = 0$.*

Proof. Since p is very good for G, $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is injective. If $A \in \mathfrak{g}$ is regular nilpotent, and if χ is a cocharacter associated with A, then each weight n of $\chi(\mathbf{G}_m)$ on $\mathfrak{g}$ satisfies

$$-2h + 2 \leq n \leq 2h - 2.$$

If $p > 0$, our assumption on p means $p - 1 \geq 2h - 2$; together with the condition $A \in \mathfrak{g}(\chi; 2)$, it follows that $\mathrm{ad}(A)^{p-1} = 0$. Since the regular nilpotent elements are dense in the nilpotent variety, one sees that each nilpotent element $X \in \mathfrak{g}$ satisfies $\mathrm{ad}(X)^{p-1} = 0$. Part (4) of the previous lemma now shows **(L)** to hold for the action of G on $(\mathrm{Ad}, \mathfrak{g})$ as desired. $\square$

Remark 29. In general, the condition in **(L)** may fail for the adjoint representation. Indeed, let $X \in \mathfrak{g}$ be regular nilpotent, suppose that $X^{[p]} = 0$, and let $\phi : \mathrm{SL}_2 \to G$ be a homomorphism determined by X as in (1) of Lemma 27. That lemma shows **(L)** to fail in case $(\mathrm{Ad} \circ \phi, \mathfrak{g})$ is not semisimple. Semisimplicity fails e.g. in case $G = \mathrm{SL}(n+1)$ with $p > n > p/2$; indeed, in that case the indecomposable tilting SL_2-module $T(2n)$ appears as a summand of $(\mathrm{Ad} \circ \phi, \mathfrak{g})$, and $T(2n)$ is not semisimple since $2n > p$.

Remark 30. The hypothesis **(L)** holds for the symplectic group $\mathrm{Sp}(V)$ or the special orthogonal group $\mathrm{SO}(V)$ on the natural representation V provided only that $p = 0$ or $p > 2$ (so that p is good for G).

Remark 31. If G is a group of type G_2 and $p = 0$ or $p \geq 5$ (so that p is good for G), **(L)** holds using the 7 dimensional representation (ρ, V) of G. In contrast, the condition in **(L)** holds on the adjoint representation for G_2 only when $p > 2h - 2 = 10$.

Note however that if $A \in \mathfrak{g}$ is regular nilpotent, then $d\rho(A)$ is regular nilpotent in $\mathfrak{gl}(V)$ so that $d\rho(A)^{p-1}$ only when $p \geq 11$.

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