NILPOTENT CENTRALIZERS AND SPRINGER ISOMORPHISMS

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ABSTRACT. Let G be a semisimple algebraic group over a field K whose characteristic is very good for G, and let σ be any G-equivariant isomorphism from the nilpotent variety to the unipotent variety; the map σ is known as a Springer isomorphism. Let $y \in G(K)$, let $Y \in \mathrm{Lie}(G)(K)$, and write $C_y = C_G(y)$ and $C_Y = C_G(Y)$ for the centralizers. We show that the center of C_y and the center of C_Y are smooth group schemes over K. The existence of a Springer isomorphism is used to treat the crucial cases where y is unipotent and where Y is nilpotent.

Now suppose G to be quasisplit, and write C for the centralizer of a rational *regular* nilpotent element. We obtain a description of the normalizer $N_G(C)$ of C, and we show that the automorphism of $\mathrm{Lie}(C)$ determined by the differential of σ at zero is a scalar multiple of the identity; these results verify observations of J-P. Serre.

CONTENTS

1. INTRODUCTION

Let G be a reductive group over the field K and suppose G to be D-standard; this condition means that G satisfies some *standard hypotheses* which will be described in §3.2. For now, note that a semisimple group G is D-standard if and only if the characteristic of K is *very good* for G.

Consider the closed subvariety $\mathcal{N}$ of nilpotent elements of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ of G, and the closed subvariety $\mathcal{U}$ of unipotent elements of G. Since G is D-standard, one may follow the argument given by Springer and Steinberg [SS 70, 3.12] to find a G-equivariant isomorphism of varieties $\sigma : \mathcal{N} \to \mathcal{U}$. The mapping σ is called a *Springer isomorphism*. There are many such maps: the Springer isomorphisms can be viewed as the points of an affine variety whose dimension is equal to the semisimple rank of G; see the note of Serre found in [Mc 05, Appendix] which shows that despite the abundance of such maps, each Springer isomorphism induces the same bijection between the (finite) sets of G-orbits in $\mathcal{N}$ and in $\mathcal{U}$. For some more details, see §3.3 below.

Let $y \in G(K)$ and $Y \in \mathfrak{g}(K)$. Since G is D-standard, we observe in (3.4.1) – following Springer and Steinberg [SS 70] – that the centralizers $C_G(y)$ and $C_G(Y)$ are smooth group schemes over K. The first main result of this paper is as follows:

Theorem A. *Let $Z_y = Z(C_G(y))$ and $Z_Y = Z(C_G(Y))$ be the centers of the centralizers.*

(a) *Z_y and Z_Y are smooth group schemes over K.*

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(b) $Y \in \mathrm{Lie}(Z_Y)$.

See §2.6 for more details regarding the subgroup schemes $Z_y \subset C_G(y)$ and $Z_Y \subset C_G(Y)$. The existence of a Springer isomorphism plays a crucial role in the proof of Theorem A.

Keep the assumptions on G, and suppose in addition that G is *quasisplit* over K; under these assumptions, one can find a K-rational regular nilpotent element $X \in \mathfrak{g}(K)$ [Mc 05, Theorem 54]. Write $C = C_G(X)$ for the centralizer of X; it is a smooth group scheme over K (3.4.1).

Our next result concerns the normalizer of C in G; write $N = N_G(C)$.

Theorem B. (i) *N is smooth over K and is a solvable group.*
(ii) *If r denotes the semisimple rank of G, then $\dim N = 2r + \dim \zeta_G$, where ζ_G denotes the center of G.*
(iii) *There is a 1 dimensional torus $S \subset N$ which is not central in G such that $S \cdot \zeta_G^o$ is a maximal torus of N.*

Fix now a cocharacter ϕ associated with the nilpotent element X; cf. (5.2.1).

Theorem C. *Assume that the derived group of G is quasi-simple. Then the Lie algebra of N/C decomposes as the direct sum*

$$\mathrm{Lie}(N/C) = \mathrm{Lie}(S_0) \oplus \bigoplus_{i=2}^{r} \mathrm{Lie}(N/C)(\phi; 2k_i - 2),$$

where $k_1 \le k_2 \le k_3 \le \cdots \le k_r$ are the exponents *of the Weyl group of G, and where S_0 is the image of S in N/C.*

We will deduce several consequences from Theorems B and C. First,

Theorem D. *The unipotent radical of $N_{/K_{\mathrm{alg}}}$ arises by base change from a split unipotent K-subgroup of N.*

In older language, Theorem D asserts that the unipotent radical of N is *defined and split over K*. Next, fix a Springer isomorphism σ and write $u = \sigma(X)$. The unipotent radical of the group C is defined over K, and C is the product of $R_u(C)$ with the center ζ_G of G; see (5.2.4). The restriction of σ to $R_u(C)$ yields an isomorphism of varieties

$$\gamma = \sigma_{|\mathrm{Lie}(R_u C)} : \mathrm{Lie}(R_u C) \xrightarrow{\sim} R_u C$$

satisfying $\gamma(0) = \sigma_{|\mathrm{Lie}(R_u C)}(0) = 1$. So the tangent mapping $d\gamma_0$ yields a linear automorphism of the tangent space

$$\mathrm{Lie}(R_u C) = T_1(R_u C).$$

Theorem E. *Suppose that the derived group of G is quasi-simple.*
(1) *The mapping $(d\gamma)_0$ is a scalar multiple of the identity automorphism of* $\mathrm{Lie}(R_u C)$.
(2) *Let B a Borel subgroup of G with unipotent radical U. Then $\sigma_{|\mathrm{Lie}\, U} : \mathrm{Lie}\, U \to U$ is an isomorphism, and $d(\sigma_{|\mathrm{Lie}\, U})_0 : \mathrm{Lie}\, U \to \mathrm{Lie}\, U$ is a scalar multiple of the identity.*

We remark that Theorems B, C, and E confirm the observations made by Serre at the end of [Mc 05, Appendix].

The paper is organized as follows. In §2 we recall some generalities about group schemes and smoothness; in particular, we describe conditions under which the center of a smooth group scheme is itself smooth. In §3 we recall some facts about reductive groups that we require; in particular, we define D-standard groups and we recall that element centralizers in D-standard groups are well-behaved. In §4 we give the proof of Theorem A. Finally, §5 contains the proofs of Theorems B, C, D and E.

2. RECOLLECTIONS: GROUP SCHEMES

The main objects of study in this paper are group schemes over a field K. For the most part, we restrict our attention to *affine* group schemes A of finite type over K. We begin with some general definitions.

2.1. **Basic Definitions.** We collect here some basic notions and definitions concerning group schemes; for a full treatment, the reader is referred to [DG 70] or to [Ja 03, part I].

For a commutative ring Λ, let us write Alg_Λ for the category of "all" commutative Λ-algebras [1]. We will write $\Lambda' \in \mathrm{Alg}_\Lambda$ to mean that Λ' is an object of this category – i.e. that Λ' is a commutative Λ-algebra.

We are going to consider affine schemes over Λ; an affine scheme X is determined by a commutative Λ-algebra R: the algebra R determines a functor $X : \mathrm{Alg}_\Lambda \to \mathrm{Sets}$ by the rule

$$X(\Lambda') = \mathrm{Hom}_{\Lambda-\mathrm{alg}}(R, \Lambda').$$

The scheme X "is" this functor, and one says that X is represented by the algebra R. One usually writes $R = \Lambda[X]$ and one says that $\Lambda[X]$ is the coordinate ring of X. The affine scheme X has finite type over Λ provided that $\Lambda[X]$ is a finitely generated Λ-algebra.

A group valued functor A on Alg_Λ which is an affine scheme will be called an affine group scheme. If A is an affine group scheme, then $\Lambda[A]$ has the structure of a Hopf algebra over Λ.

If $\Lambda' \in \mathrm{Alg}_\Lambda$, we write $A_{/\Lambda'}$ for the group scheme over Λ' obtained by base change. Thus $A_{/\Lambda'}$ is the group scheme over Λ' represented by the Λ'-algebra $\Lambda[A] \otimes_\Lambda \Lambda'$.

Let us fix an affine group scheme A of finite type over the field K. Write $K[A]$ for the coordinate algebra of K, and choose an algebraic closure K_{alg} of K.

2.2. **Comparison with algebraic groups.** In many cases, the group schemes we consider may be identified with a corresponding algebraic group; we now describe this identification.

If the algebra $K[A]$ is *geometrically reduced* – i.e. is such that $K_{\mathrm{alg}}[A] = K[A] \otimes_K K_{\mathrm{alg}}$ has no non-zero nilpotent elements – then also $K[A]$ is reduced. The K_{alg}-points $A(K_{\mathrm{alg}})$ of A may be viewed as an affine variety over K_{alg}; since it is reduced, $K_{\mathrm{alg}}[A]$ is the algebra of regular functions on $A(K_{\mathrm{alg}})$. Moreover, $A(K_{\mathrm{alg}})$ together with the K-algebra $K[A]$ of regular functions on $A(K_{\mathrm{alg}})$ may be viewed as a variety defined over K in the sense of [Bor 91] or [Sp 98].

Conversely, an algebraic group B defined over K in the sense of [Bor 91] or [Sp 98] comes equipped with a K-algebra $K[B]$ for which $K_{\mathrm{alg}}[B] = K[B] \otimes_K K_{\mathrm{alg}}$ is the algebra of regular functions on B. The Hopf algebra $K[B]$ represents a group scheme.

The constructions in the preceding paragraphs are inverse to one another, and these constructions permit us to identify the category of linear algebraic groups defined over K with the full subcategory of the category of affine group schemes of finite type over K consisting of those group schemes with geometrically reduced coordinate algebras.

There are interesting group schemes in characteristic $p > 0$ whose coordinate algebras are not reduced. Standard examples of non-reduced group schemes include the group scheme μ_p represented by $K[T]/(T^p - 1)$ with co-multiplication given by $\Delta(T) = T \otimes T$, and the group scheme α_p represented by $K[T]/(T^p)$ with co-multiplication given by $\Delta(T) = T \otimes 1 + 1 \otimes T$. Note that μ_p is a subgroup scheme of the multiplicative group $\mathbf{G}_m$, and α_p is a subgroup scheme of the additive group $\mathbf{G}_a$.

2.3. **Smoothness.** For $\Lambda \in \mathrm{Alg}_K$, let $\Lambda[\epsilon]$ denote the algebra of *dual numbers* over Λ; thus $\Lambda[\epsilon]$ is a free Λ-module of rank 2 with Λ-basis $\{1, \epsilon\}$, and $\epsilon^2 = 0$. If A is a group scheme over K, the natural Λ-algebra homomorphisms

$$\Lambda \hookrightarrow \Lambda[\epsilon] \xrightarrow{\pi} \Lambda$$

yield corresponding group homomorphisms

$$A(\Lambda) \hookrightarrow A(\Lambda[\epsilon]) \xrightarrow{A(\pi)} A(\Lambda).$$

The Lie algebra $\mathrm{Lie}(A)$ of A is the group functor on Alg_K given for $\Lambda \in \mathrm{Alg}_K$ by

$$\mathrm{Lie}(A)(\Lambda) = \ker(A(\Lambda[\epsilon]) \xrightarrow{A(\pi)} A(\Lambda)).$$

[1] Taken in some universe, to avoid logical problems.

Abusing notation somewhat, we are going to write also $\mathrm{Lie}(A)$ for $\mathrm{Lie}(A)(K)$. We have:

(2.3.1) ([DG 70, II.4]). (a) $\mathrm{Lie}(A)$ *has the structure of a K-vector space, and the mapping* $\mathrm{Lie}(A) \to \mathrm{Lie}(A)(\Lambda)$ *induces an isomorphism*

$$\mathrm{Lie}(A)(\Lambda) \simeq \mathrm{Lie}(A) \otimes_K \Lambda$$

for each $\Lambda \in \mathrm{Alg}_K$.

(b) *For* $\Lambda \in \mathrm{Alg}_K$ *and* $g \in A(\Lambda)$, *the inner automorphism* $\mathrm{Int}(g)$ *determines by restriction a Λ-linear automorphism* $\mathrm{Ad}(g)$ *of* $\mathrm{Lie}(A)(\Lambda) \simeq \mathrm{Lie}(A) \otimes_K \Lambda$; *thus* $\mathrm{Ad} : A \to \mathrm{GL}(\mathrm{Lie}(A))$ *is a homomorphism of group schemes over K.*

(2.3.2) ([DG 70, II.5.2.1, p. 238] or [KMRT, (21.8) and (21.9)]). *One says that the group scheme A is* smooth *over K if any of the following equivalent conditions hold:*

(a) *A is geometrically reduced – i.e.* $A_{/K_{\mathrm{alg}}}$ *is reduced.*
(b) *the local ring* $K[A]_I$ *is regular, where I is the maximal ideal defining the identity element of A.*
(c) *the local ring* $K[A]_I$ *is regular for each prime ideal I of* $K[A]$.
(d) $\dim_K \mathrm{Lie}(A) = \dim A$, *where* $\dim A$ *denotes the dimension of the scheme A, which is equal to the Krull dimension of the ring* $K[A]$.

If A is a group scheme over K, we often abbreviate the phrase "A is smooth over K" to "A is smooth";

2.4. **Reduced subgroup schemes.** The following result is well known; a proof may be found in [MT 07, Lemma 3].

(2.4.1). *If K is perfect, there is a unique smooth subgroup* $A_{\mathrm{red}} \subset A$ *which has the same underlying topological space as A. If B is any smooth group scheme over K and* $f : B \to A$ *is a morphism, then f factors in a unique way as a morphism* $B \to A_{\mathrm{red}}$ *followed by the inclusion* $A_{\mathrm{red}} \to A$.

Note that if K is not perfect, the subgroup scheme $(A_{/K_{\mathrm{alg}}})_{\mathrm{red}}$ of $A_{/K_{\mathrm{alg}}}$ may not arise by base change from a subgroup scheme over K; see [MT 07, Example 4].

2.5. **Fixed points and the center of a group scheme.** For the remainder of §2, let us fix a group scheme A which is affine and of finite type over the field K. Let V denote an affine K-scheme (of finite type) on which A acts. Define a K-subfunctor W of V as follows: for each $\Lambda \in \mathrm{Alg}_K$, let

$$W(\Lambda) = \{v \in V(\Lambda) \mid av = v \quad \text{for each } \Lambda' \in \mathrm{Alg}_\Lambda \text{ and each } a \in A(\Lambda')\}.$$

We write $W = V^A$; it is the functor of *fixed points* for the action of A.

In general one indeed must define the set $W(\Lambda)$ as the fixed point set of all $a \in A(\Lambda')$ for varying Λ': e.g. if A is infinitesimal, $A(K) = \{1\}$ while $W(K)$ is typically a proper subset of $V(K)$.

Since V is affine – hence separated – and since K is a field so that $K[A]$ is free over K, we have:

(2.5.1) ([DG 70, II.1 Theorem 3.6] or [Ja 03, I.2.6(10)]). V^A *is a closed subscheme of V.*

The following assertion is somewhat related to [Ja 03, I.2.7 (11) and (12)].

(2.5.2). *Suppose in addition that A is smooth over K. Then for any commutative K-algebra K' which is an algebraically closed field, we have* $V^A(K') = V(K')^{A(K')}$. [2]

Proof. It is immediate from definitions that $V^A(K') \subset V(K')^{A(K')}$. In order to prove the inclusion $V(K')^{A(K')} \subset V^A(K')$, we will assume (for notational convenience) that $K = K'$ is algebraically closed. Suppose that $v \in V(K)$ and that v is fixed by each element of $A(K)$.

Consider now the morphism $\phi : A \to V$ given for each $\Lambda \in \mathrm{Alg}_K$ and each $a \in A(\Lambda)$ by the rule $a \mapsto av$. The result will follow if we argue that ϕ is a constant morphism. But we know that $\phi : A(K) \to V(K)$ is constant. Since A is a reduced scheme, the morphism ϕ is determined by its values on closed points; since K is algebraically closed, the closed points are in bijection with $A(K)$; the fact that ϕ is constant now follows. □

[2] Here $V(K')^{A(K')}$ denotes the subset of $V(K')$ fixed by each element of the group $A(K')$.

Consider now the action of A on itself by inner automorphisms. For any $\Lambda \in \mathrm{Alg}_K$ and any $a \in A(\Lambda)$, let us write $\mathrm{Int}(a)$ for the inner automorphism $x \mapsto axa^{-1}$ of the Λ-group scheme $A_{/\Lambda}$. The fixed point subscheme for this action is *by definition* the center Z of A; thus we have the following result (see also [DG 70, II.1.3.9]):

(2.5.3)**.** *The center Z is a closed subgroup scheme of A. For any $\Lambda \in \mathrm{Alg}_K$, we have*

$$Z(\Lambda) = \{a \in A(\Lambda) \mid \mathrm{Int}(a) \textit{ is the trivial automorphism of the group scheme } A_{/\Lambda}\}.$$

2.6. **Smoothness of the center.** Write $\mathfrak{a} = \mathrm{Lie}(A)$ for the Lie algebra of A. Recall from (2.3.1) the adjoint action Ad of A on $\mathfrak{a}$.

(2.6.1)**.** *Regarding $\mathfrak{a}$ as a K-scheme, the Lie algebra of Z is the fixed point subscheme of $\mathfrak{a}$ for the adjoint action of A.*

Proof. Since Z is the fixed point subscheme of A for the action of A on itself by inner automorphisms, the assertion follows from [DG 70, II.4.2.5]. □

In particular, $\mathrm{Lie}(Z)$ identifies with the K-points $\mathfrak{a}^{\mathrm{Ad}(A)}(K)$ of this fixed point functor, and one recovers the fixed point functor from the K-points [Ja 03, I.2.10(3)]:

$$\mathfrak{a}^{\mathrm{Ad}(A)}(\Lambda) = \mathrm{Lie}(Z) \otimes_K \Lambda.$$

(2.6.2)**.** *The center Z of A is smooth over K if and only if*

$$\dim Z = \dim_K \mathfrak{a}^{\mathrm{Ad}(A)}(K) = \dim_K \mathrm{Lie}(Z).$$

Proof. Immediate from (2.3.2) and the observation (2.6.1). □

Example. Let K be a perfect field of characteristic $p > 0$, and let A be the smooth group scheme over K for which

$$A(\Lambda) = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & t^p & s \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \Lambda^\times, s \in \Lambda \right\}$$

for each $\Lambda \in \mathrm{Alg}_K$. The Lie algebra $\mathfrak{a}$ is spanned as a K-vector space by the matrices

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Write $Z = Z(A)$ for the center of A. Since K is perfect, we may form the corresponding reduced subgroup scheme $Z_{\mathrm{red}} \subset Z$ – see e.g. [MT 07, Lemma 3]; Z_{red} is a smooth group scheme over K.

We are going to argue that Z is not smooth – i.e. that $Z \neq Z_{\mathrm{red}}$. Observe first that $\mathfrak{a}$ is an Abelian Lie algebra; thus its center $\mathfrak{z}(\mathfrak{a})$ is all of $\mathfrak{a}$.

Now, if K_{alg} is an algebraic closure of K, it is easy to check that the center of the group $A(K_{\mathrm{alg}})$ is trivial. It follows that the smooth group scheme Z_{red} satisfies $Z_{\mathrm{red}}(K_{\mathrm{alg}}) = 1$; thus Z_{red} is trivial and $\mathrm{Lie}(Z_{\mathrm{red}}) = 0$.

It is straightforward to verify that the multiples of X are the only fixed points of $\mathfrak{a}$ under the adjoint action of A. Thus $\mathrm{Lie}(Z) = \mathfrak{a}^{\mathrm{Ad}(A)}$ has dimension 1 as a K-vector space. Since $\dim Z = \dim Z_{\mathrm{red}} = 0$, it follows that Z is not smooth.

Note that for this example, both containments in the following sequence are proper:

$$\mathrm{Lie}(Z_{\mathrm{red}}) \subset \mathrm{Lie}(Z) \subset \mathfrak{z}(\mathfrak{a}).$$

2.7. **Smoothness of certain fixed point subgroup schemes.** Recall that a group scheme D over K is *diagonalizable* if $K[D]$ is spanned as a linear space by the group of characters $X^*(D)$. The group scheme D is of *multiplicative type* if $D_{/K_{\mathrm{alg}}}$ is diagonalizable.

Suppose in this section that D is either a group scheme of multiplicative type, or that D is an étale group scheme over K for which the finite group $D(K_{\mathrm{alg}})$ has order invertible in K.

Assume that D acts on the group scheme A by group automorphisms: for any $\Lambda \in \mathrm{Alg}_K$ and any $x \in D(\Lambda)$, the element x acts on the group scheme $A_{/\Lambda}$ as a group scheme automorphism.

The fixed points A^D form a closed subgroup scheme of A . Moreover, we have:

(2.7.1). *If A is smooth over K, then also the fixed point subgroup scheme A^D is smooth over K.*

Proof. According to the "Théorème de lissité des centralisateurs" [DG 70, II.5.2.8 (p. 240)] the result will follow if we know that $H^1(D, \mathrm{Lie}(A)) = 0$. It suffices to check this condition after extending scalars; thus we may and will suppose that D is diagonalizable or that D is the constant group scheme determined by a finite group whose order is invertible in K.

In each case, one knows that the cohomology group $H^n(D, M)$ is 0 for *all* D-modules M and all $n \geq 1$; for a finite group with order invertible in K, this vanishing is well-known; for a diagonalizable group, see [Ja 03, I.4.3]. □

2.8. **Possibly disconnected groups.** Let G be a smooth linear algebraic group over K.

(2.8.1). *Suppose that $1 \to G \to G_1 \to E \to 1$ is an exact sequence, where E is finite étale and $E(K_{\mathrm{alg}})$ has order invertible in K. If the center of G is smooth, then the center of G_1 is smooth.*

Proof. Write Z for the center of G, write Z_1 for the center of G_1. Note that E acts naturally on Z.

There is an exact sequence of groups

$$1 \to Z^E \to Z_1 \to H \to 1$$

for a subgroup $H \subset E$. Since Z is smooth, the smoothness of Z^E follows from (2.7.1); since H is smooth, one obtains the smoothness of Z_1 by applying [KMRT, Cor. (22.12)]. □

2.9. **Split unipotent radicals.** Fix a smooth group scheme A over K. A smooth group scheme B over K is unipotent if each element of $B(K_{\mathrm{alg}})$ is unipotent. Recall that the unipotent radical of $A_{/K_{\mathrm{alg}}}$ is the maximal closed, connected, smooth, normal, unipotent subgroup scheme of $A_{/K_{\mathrm{alg}}}$.

(2.9.1). [Sp 98, Prop. 14.4.5] *If K is perfect, there is a smooth subgroup scheme $R_u A \subset A$ such that $R_u A_{/K_{\mathrm{alg}}}$ is the unipotent radical of $A_{/K_{\mathrm{alg}}}$.*

If K is not perfect, then in general $R_u A_{/K_{\mathrm{alg}}}$ does not arise by base change from a K-subgroup scheme of A. The unipotent group B is said to be *split* provided that there are closed subgroup schemes

$$1 = B_0 \subset B_1 \subset \cdots \subset B_n = B$$

such that $B_i/B_{i-1} \simeq \mathbf{G}_a$ for $1 \leq i \leq n$.

Theorem. *Let A be a connected, solvable, and smooth group scheme over K. Let $T \subset A$ be a maximal torus, and suppose that $\phi : \mathbf{G}_m \to T$ is a cocharacter. Write S for the image of ϕ. If $\mathrm{Lie}(T)$ is precisely the set of fixed points $\mathrm{Lie}(A)^S$, and if each non-zero weight λ of S on $\mathrm{Lie}(A)$ satisfies $\langle \lambda, \phi \rangle > 0$, then $R_u A$ is defined over K and is a split unipotent group scheme.*

Proof. Write $P = P(\phi)$ for the smooth subgroup scheme of A determined by ϕ as in [Sp 98, §13.4]; it is the subgroup *contracted by* the cocharacter ϕ. Write $M = C_A(S)$; M is connected [Sp 98, p. 110] and smooth [DG 70, p. 476, cor. 2.5]. There is a smooth, connected, normal, unipotent subgroup scheme $U(\phi) \subset P$ for which the product morphism

$$M \times U(\phi) \to P$$

is an isomorphism of varieties; [Sp 98, 13.4.2]. Moreover, since $\langle \lambda, \phi \rangle > 0$ for each weight of S on $\mathrm{Lie}(A)$, it follows that $U(-\phi)$ is trivial. Thus *loc. cit.* 13.4.4 shows that $A = P$.

Evidently $T \subset M$. Since $\mathrm{Lie}(T) = \mathrm{Lie}(M)$, it follows that $M = T$. It follows that $U(\phi)_{/K_{\mathrm{alg}}}$ is the unipotent radical of $A_{/K_{\mathrm{alg}}}$ as desired.

Finally, it follows from [Sp 98, 14.4.2] that $U(\phi)$ is a K-split unipotent group, and the proof is complete. □

2.10. **Torus actions on a projective space.** Let T be a split torus over K, and let V be a T-representation. For $\lambda \in X^*(T)$, let V_λ be the corresponding weight space; thus T acts on V_λ through the character $\lambda : T \to \mathbf{G}_m$. There are distinct characters $\lambda_1, \dots, \lambda_n \in X^*(T)$ such that

$$V = \bigoplus_{i=1}^n V_{\lambda_i};$$

the λ_i are the *weights* of T on V. Let us fix a vector $0 \neq v \in V_{\lambda_1}$.

Consider now the projective space $\mathbf{P}(V)$ of lines through the origin in V; for a non-zero vector $w \in V$, write $[w]$ for the corresponding point of $\mathbf{P}(V)$. The linear action of T on V induces in a natural way an action of T on $\mathbf{P}(V)$.

Since v is a weight vector for T, the point $[v] \in \mathbf{P}(V)(K)$ determined by v is fixed by the action of T. Consider the tangent space $M = T_{[v]}\mathbf{P}(V)$; since $[v]$ is a fixed point of T, the action of T on $\mathbf{P}(V)$ determines a linear representation of T on M.

(2.10.1). *The non-zero weights of T on $M = T_{[v]}\mathbf{P}(V)$ are the characters $\lambda_i - \lambda_1$ for $1 < i \leq n$. Moreover,*

$$\dim M_0 = \dim V_{\lambda_1} - 1 \quad \textit{and} \quad \dim M_{\lambda_i - \lambda_1} = \dim V_{\lambda_i}, \quad 1 < i \leq n.$$

Proof. Choose a basis $S_1, S_2, \dots, S_r$ for the dual space of $V^\vee$ for which $S_i \in V^\vee_{-\lambda_i}$ for $1 \leq i \leq r$ – i.e. the vector S_i has weight $-\lambda_i$ for the contragredient action of T on $V^\vee$. Without loss of generality, we may and will assume that S_1 satisfies $S_1(v) \neq 0$ and that $S_i(v) = 0$ for $2 \leq i \leq n$.

Now consider the affine open subset $\mathcal{V} = \mathbf{P}(V)_{S_1}$ of $\mathbf{P}(V)$ defined by the non-vanishing of S_1. One knows that $[v]$ is a point of $\mathcal{V}$. Moreover, $\mathcal{V} \simeq \mathbf{Aff}^{r-1}$ where $r = \dim V$. Since S_1 is a weight vector for the action of the torus T, it is clear that $\mathcal{V}$ is a T-stable subvariety of $\mathbf{P}(V)$. More precisely, $\mathcal{V}$ identifies with the affine scheme $\mathrm{Spec}(\mathcal{A})$ where $\mathcal{A}$ is the T-stable subalgebra

$$\mathcal{A} = k\left[\frac{S_2}{S_1}, \frac{S_3}{S_1}, \dots, \frac{S_r}{S_1}\right]$$

of the field of rational functions $k(\mathbf{P}(V))$.

Under this identification, the point $[v] \in \mathcal{V}$ corresponds to the point $\vec{0}$ of $\mathbf{Aff}^{r-1}$; i.e. to the maximal ideal $\mathfrak{m} = \left(\frac{S_2}{S_1}, \frac{S_3}{S_1}, \dots, \frac{S_r}{S_1}\right) \subset \mathcal{A}$. Now, $\mathfrak{m}$ and $\mathfrak{m}^2$ are T-invariant; since $\frac{S_i}{S_1}$ has weight $-\lambda_i + \lambda_1$, evidently the weights of T in its representation on $\mathfrak{m}/\mathfrak{m}^2$ are of the form $-\lambda_i + \lambda_1$, and one has

$$\dim(\mathfrak{m}/\mathfrak{m}^2)_0 = \dim V_{\lambda_1} - 1 \quad \text{and} \quad \dim(\mathfrak{m}/\mathfrak{m}^2)_{-\lambda_i + \lambda_1} = \dim V_{\lambda_i}, \quad 1 < i \leq n.$$

The assertion now follows since there is a T-equivariant isomorphism between the tangent space to $\mathbf{P}(V)$ at $[v]$ – i.e. the space $M = T_{[v]}\mathbf{P}(V)$ – and the contragredient representation $(\mathfrak{m}/\mathfrak{m}^2)^\vee$. □

2.11. **Surjective homomorphisms between group schemes; normalizers.** In this section, let us fix group schemes G_1 and G_2 over K, and suppose that $f : G_1 \to G_2$ is a *surjective* homomorphism of group schemes; recall that f is surjective provided that the comorphism $f^* : K[G_2] \to K[G_1]$ is injective (cf. [KMRT, Prop. 22.3]).

The mapping f is said to be *separable* provided that $df : \mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2)$ is surjective as well.

Let $C_2 \subset G_2$ be a subgroup scheme, and let $C_1 = f^{-1}C_2$ be the scheme-theoretic inverse image.

(2.11.1). (a) *The mapping obtained by restriction $f_{|C_1} : C_1 \to C_2$ is surjective.*

(b) *If C_1 is smooth, then C_2 is smooth.*

(c) *If f is separable and C_2 is smooth, then C_1 is smooth.*
(d) *Suppose that f is separable, and that either C_1 or C_2 is smooth. Then both C_1 and C_2 are smooth, and $f_{|C_1}$ is separable.*

Proof. (a) and (b) follow from [KMRT, Prop. 22.4].

We now prove (c). Since f is separable and surjective, [KMRT, Prop. 22.13] shows that $\ker f$ is a smooth group scheme over K. Note that $\ker f \subset C_1$. If C_2 is smooth, the smoothness of C_1 now follows from [KMRT, Cor. 22.12].

We finally prove (d). The smoothness assertions have already been proved. We again know $\ker f$ to be smooth over K. In particular, $\dim \ker f = \dim \ker df$. Since $\ker f \subset C_1$, we have

$$\dim \operatorname{image}(df_{|C_1}) = \dim \operatorname{Lie}(C_1) - \dim \ker df_{|C_1} = \dim C_1 - \dim \ker f_{|C_1} = \dim C_2,$$

where we have used [KMRT, Prop. 22.11] for the final equality; since C_2 is smooth, it follows that $df_{|C_1} : \operatorname{Lie}(C_1) \to \operatorname{Lie}(C_2)$ is surjective. □

Write $N_2 = N_{G_2}(C_2)$ for the normalizer of C_2 in G_2. Thus N_2 is the subgroup functor given for $\Lambda \in \mathrm{Alg}_K$ by the rule

$$\begin{aligned} N_2(\Lambda) =& \{g \in G_2(\Lambda) \mid g \text{ normalizes the subgroup scheme } C_{2/\Lambda} \subset G_{2/\Lambda}\} \\ =& \{g \in G_2(\Lambda) \mid gC_2(\Lambda')g^{-1} = C_2(\Lambda') \text{ for all } \Lambda' \in \mathrm{Alg}_\Lambda\}. \end{aligned}$$

According to [DG 70, II.1 Theorem 3.6(b)], N_2 is a closed subgroup scheme of G_2.

As a consequence of (2.11.1), we find the following:

(2.11.2). *Set $N_1 = f^{-1}N_2$.*
(a) $N_1 = N_{G_1}(C_1)$.
(b) $f_{|N_1} : N_1 \to N_2$ *is surjective.*
(c) *If N_1 is smooth, then N_2 is smooth.*
(d) *If f is separable and N_2 is smooth, then N_1 is smooth.*
(e) *Suppose that f is separable and that either N_1 or N_2 is smooth. Then both N_1 and N_2 are smooth, and $f_{|N_1}$ is separable.*

3. RECOLLECTIONS: REDUCTIVE GROUPS

Let G be a connected and reductive group over K. Thus G is a smooth group scheme over K, or equivalently G is a linear algebraic group defined over K. To say that G is reductive means that the unipotent radical of $G_{/K_{\mathrm{alg}}}$ is trivial. We are going to write $\zeta_G = Z(G)$ for the center of G.

Some results will be seen to hold for a reductive group G in case G is *D-standard*; in the next few sections, we explain this condition. We must first recall the notions of good and bad characteristic.

3.1. **Good and very good primes.** Suppose that H is a smooth group scheme over K – i.e. an algebraic group over K – for which $H_{/K_{\mathrm{alg}}}$ is quasisimple; thus H is geometrically quasisimple. Write R for the root system of H. The characteristic p of K is said to be a bad prime for R – equivalently, for H – in the following circumstances: $p = 2$ is bad whenever $R \neq A_r$, $p = 3$ is bad if $R = G_2, F_4, E_r$, and $p = 5$ is bad if $R = E_8$. Otherwise, p is good.

A good prime p is *very good* provided that either R is not of type A_r, or that $R = A_r$ and $r \not\equiv -1 \pmod{p}$.

If H is any reductive group, one may apply [KMRT, Theorems 26.7 and 26.8] [3] to see that there is a possibly inseparable central isogeny

$$R(H) \times \prod_{i=1}^{m} H_i \to H \tag{1}$$

[3] [KMRT] only deals with the semisimple case; the extension to a general reductive group is not difficult to handle, and an argument is sketched in the footnote found in [MT 07, §2.4].

where the radical $R(H)$ of H is a torus, and where for $1 \le i \le m$ there is an isomorphism $H_i \simeq R_{L_i/K} J_i$ for a finite separable field extension L_i/K and a geometrically quasisimple, simply connected group scheme J_i over L_i; here, $R_{L_i/K} J_i$ denotes the "Weil restriction" – or restriction of scalars – of J_i to K, cf. [Sp 98, §11.4]. The H_i are uniquely determined by H up to order of the factors. Then p is *good*, respectively *very good*, for H if and only if that is so for J_i for every $1 \le i \le m$.

3.2. D**-standard.** Recall from §2.7 the notion of a diagonalizable group scheme, and of a group scheme of multiplicative type.

(3.2.1)**.** *If D is subgroup scheme of G of multiplicative type, the connected centralizer $C_G(D)^o$ is reductive.*

When D is smooth, the preceding result is well-known: the group D is the direct product of a torus and a finite étale group scheme all of whose geometric points have order invertible in K. The centralizer of a torus is (connected and) reductive, and one is left to apply a result of Steinberg [St 68, Cor. 9.3] which asserts that the centralizer of a semisimple automorphism of a reductive group has reductive identity component. In fact, the result remains valid when D is no longer smooth; a proof will appear elsewhere.

Consider reductive groups H which are direct products

$$(*) \quad H = H_1 \times T$$

where T is a torus, and where H_1 is a semisimple group for which the characteristic of K is *very good*.

Definition. A reductive group G is *D-standard* if there exists a reductive group H of the form $(*)$, a subgroup $D \subset H$ such that D is of multiplicative type, and a separable isogeny between G and the reductive group $C_H(D)^o$. [4]

(3.2.2) ([Mc 05, Remark 3])**.** *For any $n \ge 1$, the group* GL_n *is D-standard. The group* SL_n *is D-standard if and only if p does not divide n.*

In order to prove (3.2.4) below, we first observe:

(3.2.3)**.** *Let M, G_1, G_2 be affine group schemes of finite type over K. Let $f : G_1 \to G_2$ be a surjective morphism of group schemes, suppose that* $\ker f$ *is central in G_1, and let $\phi : M \to G_2$ be a homomorphism of group schemes for which $\phi^{-1}(\zeta_{G_2})$ is central in M. Consider the group scheme $\widetilde{M}$ defined by the Cartesian diagram:*

$$\begin{array}{ccc} \widetilde{M} = M \times_{G_2} G_1 & \xrightarrow{\tilde{f}} & M \\ \downarrow{\scriptstyle \tilde{\phi}} & & \downarrow{\scriptstyle \phi} \\ G_1 & \xrightarrow{f} & G_2 \end{array}$$

Then

(a) *$\widetilde{\phi}^{-1}(\zeta_{G_1})$ is central in $\widetilde{M}$.*
(b) *Suppose that G_1, G_2 are connected and reductive, that f is a separable isogeny, and that M is connected and quasisimple. Then $\widetilde{M}$ is connected and quasisimple.*

Proof. To prove (a), let $N = \widetilde{\phi}^{-1}(\zeta_{G_1})$. It is enough to show that $\widetilde{\phi}(N)$ is central in G_1 and that $\widetilde{f}(N)$ is central in M. The first of these observations is immediate from definitions, while the second follows from assumption on the mapping $\phi : M \to G_2$ once we observe that $\widetilde{f}(N) \subset \phi^{-1}(\zeta_{G_2})$.

For (b), we view $\widetilde{f}$ as arising by base change from f. Then $\widetilde{f}$ is an isogeny since $\ker(f)_{/K_{\mathrm{alg}}}$ and $\ker(\widetilde{f})_{K_{\mathrm{alg}}}$ coincide. Moreover, it follows from [Li 02, Prop 4.3.22] that $\widetilde{f}$ is separable (since it is étale). Thus $\widetilde{f}$ is a separable isogeny; since $\widetilde{M}$ is separably isogenous to a connected quasisimple group, it is itself connected and quasisimple. □

[4] This definition does not require the knowledge that $C_H(D)^o$ is reductive: if there is an isogeny between G and $C_H(D)^o$, then $C_H(D)^o$ is reductive.

(3.2.4). *Suppose that the D-standard reductive group G is split over K. There are D-standard reductive groups* $M_1, \dots, M_d$ *together with a homomorphism* $\Phi : M \to G$*, where* $M = \prod_{i=1}^d M_i$*, such that the following hold:*

(a) *The derived group of* M_i *is geometrically quasisimple for* $1 \le i \le d$.
(b) Φ *is surjective and separable.*
(c) *For* $1 \le i < j \le d$*, the image in G of* M_i *and* M_j *commute.*
(d) *The subgroup scheme* $\Phi^{-1}(\zeta_G)$ *is central in* $\prod_{i=1}^d M_i$.

Proof. We argue first that it suffices to prove the result after replacing G be a separably isogenous group. More precisely, we prove: $(*)$ if $f : G_1 \to G_2$ is a separable isogeny between D-standard reductive groups G_1 and G_2, then (3.2.4) holds for G_1 if and only if it holds for G_2.

Suppose first that the conclusion of (3.2.4) is valid for G_1. If $\Phi : M \to G_1$ is a homomorphism for which (a)–(d) hold, then evidently (a)–(d) hold for $f \circ \Phi$.

Now suppose that the conclusion of (3.2.4) is valid for G_2, and that $\Phi : M \to G_2$ is a homomorphism for which (a)–(d) hold. For each $1 \le j \le d$ write Φ_j for the composite of Φ with the inclusion of M_j in the product. Form the group $\widetilde{M_j} = M_j \times_{G_2} G_1$ as in (3.2.3). Then by (b) of *loc. cit.*, $\widetilde{M_j}$ is quasisimple. Moreover, *loc. cit.* (a) shows the kernel of $\widetilde{\Phi_j}$ it be central in $\widetilde{M_j}$.

Note that the image of $\widetilde{\Phi_j}$ is mapped to the image of Φ_j by f. Now, f is a separable isogeny, hence in particular f is central; i.e. $\ker f$ is central. It follows that the image of $\widetilde{\Phi_i}$ commutes with the image of $\widetilde{\Phi_j}$ whenever $1 \le i \ne j \le n$. We can thus form the homomorphism $\tilde{\Phi} : \prod_{j=1}^d \widetilde{M_j} \to G_1$ whose restriction to each $\widetilde{M_j}$ is just $\widetilde{\Phi_j}$, and it is clear that (a)–(d) hold for $\tilde{\Phi}$; this completes the proof of $(*)$.

In view of the definition of a D-standard group, we may now suppose that G is the connected centralizer $C_{H_1}(D)^o$ of a diagonalizable subgroup scheme $D \subset H_1 = H \times S$, where H is a semisimple group in very good characteristic and S a torus.

We may use [Sp 98, 8.1.5] to write G as a commuting product of its minimal non-trivial connected, closed, normal subgroups J_i for $i = 1, 2, \dots, n$. Fix a maximal torus $T \subset G$, so that $T_i = (T \cap J_i)^o$ is a maximal torus of J_i for each i.

Now set $T^i = \prod_{i \ne j} T_j$; then T^i is a torus in G. Moreover, J_i is the derived subgroup of the reductive group $M_i = C_G(T_i)$.

Now, M_i is the connected centralizer in H_1 of the diagonalizable subgroup $\langle T^i, D \rangle$; thus M_i is D-standard.

Finally, putting $M = \prod_i M_i$, we have a natural surjective mapping $M \to G$ for which (a)-(d) hold, as required. □

3.3. **Existence of Springer Isomorphisms.** Let G denote a D-standard reductive group. We write $\mathcal{N} = \mathcal{N}(G) \subset \mathfrak{g}$ for the *nilpotent variety* of G and $\mathcal{U} = \mathcal{U}(G) \subset G$ for the *unipotent variety* of G.

By a *Springer isomorphism*, we mean a map

$$\sigma : \mathcal{N} \to \mathcal{U}$$

which is a G-equivariant isomorphism of varieties over K.

The first assertion of the following Theorem – the existence of a Springer isomorphism – is due essentially to Springer; see e.g. [SS 70, III.3.12] for the case of an algebraically closed field, or see [Spr69]. The second assertion was obtained by Serre and appears in the appendix to [Mc 05].

Theorem (Springer, Serre). (1) *There is a Springer isomorphism* $\sigma : \mathcal{N} \to \mathcal{U}$.
(2) *Any two Springer isomorphisms induce the same mapping between the set of* $G(K_{\mathrm{alg}})$*-orbits in* $\mathcal{U}(K_{\mathrm{alg}})$ *and the set of* $G(K_{\mathrm{alg}})$*-orbits in* $\mathcal{N}(K_{\mathrm{alg}})$*, where* K_{alg} *is an algebraic closure of K.*

Proof. We sketch the argument for assertion (1) in order to point out the role of the D-standard assumption made on G.

If G is semisimple in very good characteristic, the nilpotent variety $\mathcal{N}$ and the unipotent variety $\mathcal{U}$ are both normal. Indeed, for $\mathcal{U}$, one knows [SS 70, III.2.7] that $\mathcal{U}$ is normal whenever G is simply connected (with no condition on p). Moreover, one knows that the normality of $\mathcal{U}$ is preserved by separable isogeny [5]. In positive characteristic the normality of $\mathcal{N}$ for a semisimple group G is a result of Veldkamp (for most p) and of Demazure when the characteristic is very good for G; see [Ja 04, 8.5]. Using the normality of $\mathcal{U}$ and of $\mathcal{N}$, Springer showed that [Spr69] there is a G-equivariant isomorphism as required.

To conclude that assertion (1) is valid for any D-standard groups, it suffices to observe the following: (i) if $\pi : G \to G_1$ is a separable isogeny, then there is a Springer isomorphism for G if and only if there is a Springer isomorphism for G_1, and (ii) if H is a reductive group for which there is a Springer isomorphism, and if $D \subset H$ is a subgroup of multiplicative type, then $C_H^o(D)$ has a Springer isomorphism. □

We note a related result for certain not-necessarily-connected reductive groups.

(3.3.1). *Let G be a connected reductive group for which there is a Springer isomorpism $\sigma : \mathcal{N}(G) \to \mathcal{U}(G)$. Let $D \subset G$ be a subgroup of multiplicative type, and let $M = C_G(D)$.*

(a) *σ restricts to an isomorphism $\mathcal{N}(M) \to \mathcal{U}(M)$.*
(b) *The finite group $M(K_{\mathrm{alg}})/M^o(K_{\mathrm{alg}})$ has order invertible in K.*

Proof. Assertion (a) follows from the observations: $\mathcal{N}(M) = \mathcal{N}(G)^D$ and $\mathcal{U}(M) = \mathcal{U}(G)^D$. To prove (b), note that $\mathcal{N}(M) = \mathcal{N}(M^o)$ is connected, so that by (a), also $\mathcal{U}(M)$ is connected. Thus $\mathcal{U}(M) \subset M^o$ and (b) follows at once. □

3.4. **Smoothness of some subgroups of D-standard groups.** For any algebraic group, and any element $x \in G$, let $C_G(x)$ denote the centralizer subgroup scheme of G. Then by definition $\operatorname{Lie} C_G(x) = \mathfrak{c}_{\mathfrak{g}}(x)$, where $\mathfrak{c}_{\mathfrak{g}}(x)$ denotes the centralizer of x in the Lie algebra $\mathfrak{g}$, but since the centralizer may not reduced, the dimension of $\mathfrak{c}_{\mathfrak{g}}(x)$ may be larger than the dimension $\dim C_G(x) = \dim C_G(x)_{\mathrm{red}}$, where $C_G(x)_{\mathrm{red}}$ denotes the corresponding *reduced* – hence smooth – group scheme. Similar remarks hold when $x \in G$ is replaced by an element $X \in \mathfrak{g}$.

When G is a D-standard reductive group, this difficulty does not arise. Indeed:

(3.4.1). *Let G be D-standard, let $x \in G(K)$, and let $X \in \mathfrak{g} = \mathfrak{g}(K)$. Then $C_G(x)$ and $C_G(X)$ are smooth over K. In other words,*

$$\dim C_G(x) = \dim \mathfrak{c}_{\mathfrak{g}}(x) \quad \textit{and} \quad \dim C_G(X) = \dim \mathfrak{c}_{\mathfrak{g}}(X).$$

In particular,

$$\operatorname{Lie} C_G(x)_{\mathrm{red}} = \mathfrak{c}_{\mathfrak{g}}(x) \quad \textit{and} \quad \operatorname{Lie} C_G(X)_{\mathrm{red}} = \mathfrak{c}_{\mathfrak{g}}(X).$$

Proof. When G is semisimple in very good characteristic, the result follows from [SS 70, I.5.2 and I.5.6]. The extension to D-standard groups is immediate; the verification is left to the reader. [6] □

Similar assertions holds for the center of G, as follows:

(3.4.2). *Let G be a D-standard reductive group. Then the center ζ_G of G is* smooth.

Proof. Indeed, for any field extension L of K, the center of $G_{/L}$ is just the group scheme $(\zeta_G)_{/L}$ obtained by base change. To prove that ζ_G is smooth, it suffices to prove that $(\zeta_G)_{/L}$ is smooth. So we may and will suppose that K is algebraically closed; in particular, G is split.

Fix a Borel subgroup B of G and fix a maximal torus $T \subset B$. Let $X = \sum_\alpha X_\alpha \in \operatorname{Lie}(B)$ be the sum over the simple roots α, where $X_\alpha \in \operatorname{Lie}(B)_\alpha$ is a non-zero root vector; then X is *regular nilpotent*.

For a root $\beta \in X^*(T)$ of T on $\operatorname{Lie}(G)$, write $\beta^\vee \in X_*(T)$ for the corresponding cocharacter $\beta^\vee : \mathbf{G}_m \to T$, and consider the cocharacter $\phi : \mathbf{G}_m \to T$ given by $\phi = \sum_\beta \beta^\vee \in X_*(T)$, where the sum is over all positive roots β. Then $\operatorname{Ad}(\phi(t))X = t^2 X$ for each $t \in \mathbf{G}_m(K)$ so that the image of ϕ normalizes the centralizer $C = C_G(X)$.

[5] More precisely, if $\pi : G \to G_1$ is a separable central isogeny, the restriction of π determines an isomorphism between $\mathcal{U}(G)$ and $\mathcal{U}(G_1)$.

[6] Complete details of the reduction from the case of a D-standard group to that of a semisimple group in very good characteristic can be given along the lines of the argument used in the proof of (5.4.2).

Now, C is a smooth subgroup of G by (3.4.1). The image of ϕ is a torus, hence is a diagonalizable group. So the fixed points $C^{\mathrm{im}\,\phi}$ of the image of ϕ on C form a smooth subgroup by (2.7.1).

Finally, since X is contained in the dense B-orbit on $\mathrm{Lie}(R_u B)$, X is a *distinguished* nilpotent element; cf. [Ja 04, 4.10, 4.13]. So it follows from [Ja 04, Prop. 5.10], that $C^{\mathrm{im}\,\phi}$ is precisely ζ_G, the center of G. Thus indeed ζ_G is smooth. □

Remark. In case G is semisimple in very good characteristic one can instead apply [Hum 95, 0.13] to see that the center of the Lie algebra $\mathrm{Lie}(G)$ is trivial; this shows in this special case that ζ_G is smooth.

3.5. **The centralizer of a semisimple element of $\mathfrak{g}$.** Suppose G is D-standard, let $X \in \mathfrak{g} = \mathfrak{g}(K)$ be semisimple, and write $M = C_G(X)$. Recall that the closed subgroup scheme M is smooth over K; cf. (3.4.1).

(3.5.1)**.** (a) *X is tangent to a maximal torus T of G.*
(b) *M^o is a reductive group.*

Proof. [Bor 91, Prop. 11.8 and Prop. 13.19]. □

Now fix a maximal torus T with $X \in \mathrm{Lie}(T)$ as in (3.5.1). Let us recall the following:

(3.5.2)**.** *If $S \subset G$ is a torus, there is a finite, separable field extension $L \supset K$ and a parabolic subgroup $P \subset G_{/L}$ such that $C_G(S)_{/L}$ is a Levi factor of P.*

Proof. Let the finite separable field extension $L \supset K$ be a splitting field for S. The result then follows from [BoT 65, 4.15]. □

Suppose for the moment that the characteristic p of K is positive. Let K_{sep} be a separable closure of K, and consider the (additive) subgroup B of K_{sep} generated by the elements $d\beta(X)$ for $\beta \in X^*(T_{/K_{\mathrm{sep}}})$; since $d\beta(X) = 0$ whenever $\beta \in pX^*(T_{/K_{\mathrm{sep}}})$, B is a finite elementary Abelian p-group. Write $\Gamma = \mathrm{Gal}(K_{\mathrm{sep}}/K)$ for the Galois group; since $X \in \mathfrak{g}(K)$, the group B is stable under the action of Γ.

Let $\mu = D(B)$ be the K-group scheme of multiplicative type determined by the Γ-module B. The Γ-equivariant mapping $X^*(T_{/K_{\mathrm{sep}}}) \to B$ given by $\beta \mapsto d\beta(X)$ determines an embedding of μ as a closed subgroup scheme of T.

(3.5.3)**.** *We have $M^o = C_G(\mu)^o$.*

Sketch. Since M^o and $C_G(\mu)^o$ are smooth groups over K, it suffices to give the proof after replacing K by an algebraic closure. In that case μ is diagonalizable. Let $R \subset X^*(T)$ be the roots of G for the torus T, and for $\alpha \in R$ let $U_\alpha \subset G$ be the corresponding root subgroup of G.

Then using the Bruhat decomposition of G, one finds that

$$M^o = \langle T, U_\alpha \mid d\alpha(X) = 0 \rangle = C_G(\mu)^o;$$

the required argument is essentially the same as that given in [SS 70, II.4.1] except that *loc. cit.* does not treat infinitesimal subgroup schemes; cf. [Mc 08a] for the details. □

Theorem. *There is a finite separable field extension $L \supset K$ for which the connected centralizer $M^o_{/L} = C^o_G(X)_{/L}$ is a Levi factor of a parabolic subgroup of $G_{/L}$.*

Proof. Suppose first that K has characteristic $p > 0$. In view of (3.5.3), the reductive group M^o is D-standard, since μ is a group of multiplicative type. According to (3.4.2), the center Z of M^o is smooth. Let S be a maximal torus of Z. We have evidently $M^o \subset C_G(S)$. It follows that $\mathrm{Lie}(Z) = \mathrm{Lie}(S)$. We may now use (2.6.1) to see that $X \in \mathrm{Lie}(Z) = \mathrm{Lie}(S)$. Thus $M^o \supset C_G(S)$.

It follows that $M^o = C_G(S)$, and we conclude via (3.5.2).

The situation when K has characteristic zero is simpler. In that case, the center Z of the reductive group M^o is automatically smooth. If S is a maximal torus of Z then $M^o = C_G(S)$ as before. □

3.6. **Borel subalgebras.** Suppose that K is algebraically closed. By a Borel subalgebra of $\mathfrak{g}$, we mean the Lie algebra $\mathfrak{b} = \mathrm{Lie}(B)$ of a Borel subgroup $B \subset G$.

Proposition ([Bor 91, 14.25])**.** *$\mathfrak{g}$ is the union of its Borel subalgebras. More precisely, for each $X \in \mathfrak{g}$, there is a Borel subalgebra $\mathfrak{b}$ with $X \in \mathfrak{b}$.*

4. The center of a centralizer

For a D-standard reductive group G over K, let $x \in G(K)$ and $X \in \mathfrak{g}(K)$. We are going to consider the centralizers $C_G(X)$ and $C_G(x)$, and in particular, the centers $Z_x = Z(C_G(x))$ and $Z_X = Z(C_G(X))$ of these centralizers. As we have seen, Z_x is a closed subscheme of $C_G(x)$ and Z_X is a closed subscheme of $C_G(X)$. In this section, we will prove Theorem A from the introduction; namely, in §4.2, we prove that Z_x and Z_X are smooth. In §4.1, we establish some preliminary results under the assumption that K is perfect. Since the smoothness of Z_x and of Z_X will follow if it is proved after base change with an algebraic closure K_{alg} of K, this assumption on K is harmless for our needs.

4.1. **Unipotence of the center of the centralizer when X is nilpotent.** Suppose in this section that the field K is *perfect*; thus if A is a group scheme over K, we may speak of the reduced subgroup scheme A_{red} – cf. (2.4.1). We begin with the following observation which is due independently to R. Proud and G. Seitz. For completeness, we include a proof.

(4.1.1)**.** *Let x be unipotent, let X be nilpotent, write C for one of the groups $C_G(x)$ or $C_G(X)$, and write $Z = Z(C)$; thus Z is one of the groups Z_x or Z_X.*

(a) *C^o is not contained in a Levi factor of a proper parabolic subgroup of G.*
(b) *The quotient $(Z_{\mathrm{red}})^o/(\zeta_G)^o$ is a unipotent group, where Z_{red} is the corresponding reduced group, and $(Z_{\mathrm{red}})^o$ is its identity component.*
(c) *Let $Y \in \mathrm{Lie}(Z)$ be semisimple. Then $Y \in \mathrm{Lie}(\zeta_G)$.*

Proof. It suffices to prove each of the assertions after extending scalars; thus, we may and will suppose in the proof that K is algebraically closed. Moreover, if $\sigma : \mathcal{N} \to \mathcal{U}$ is a Springer isomorphism, then $C_G(X) = C_G(\sigma(X))$. Thus it suffices to give the proof for the centralizer of X.

We first prove (a). Suppose that L is a Levi factor of a parabolic subgroup P, and assume that C^o is a subgroup scheme of L. Then $C^o = C_L^o(X)$ so that $\mathrm{Lie}\, C = \mathrm{Lie}\, C_L(X)$. Since L is again a D-standard reductive group, we see by the smoothness of centralizers that $\mathrm{Lie}\, C_L(X)$ is the centralizer in $\mathrm{Lie}\, L$ of X (3.4.1); in particular, it follows that every fixed point of $\mathrm{ad}(X)$ on $\mathrm{Lie}(G)$ lies in $\mathrm{Lie}(L)$. If L were a proper subgroup of G, the nilpotent operator $\mathrm{ad}(X)$ would have a non-zero fixed point on $\mathrm{Lie}\, R_u P$; it follows that $L = G$.

We will now deduce (b) and (c) from (a). For (b), let $S \subset Z$ be a torus. The assertion (b) will follow if we prove that S is central in G. But $L = C_G(S)$ is a Levi factor of some parabolic subgroup P of G by (3.5.2), and $C^o \subset L$. Thus by (a) we have $P = G = L$; this shows that S is central in G, as required.

For (c), let $Y \in \mathrm{Lie}(Z)$ be semisimple. According to Theorem 3.5, $L = C_G^o(Y)$ is a Levi factor of some parabolic subgroup P, and $C^o \subset L$. So again (a) shows that $P = G = L$. Since $C_G(Y) = G$, it follows that Y is a fixed point for the adjoint action of G on $\mathrm{Lie}(G)$. But according to (2.6.1), we have $\mathrm{Lie}(\zeta_G) = \mathrm{Lie}(G)^{\mathrm{Ad}(G)}$; thus indeed $Y \in \mathrm{Lie}(\zeta_G)$ as required. □

As a consequence, we deduce the following structural results:

(4.1.2)**.** *With notation and assumptions as in (4.1.1), we have:*

(a) *Z_{red} is the internal direct product $\zeta_G \cdot R_u Z_{\mathrm{red}}$.*
(b) *The set of nilpotent elements of $\mathrm{Lie}(Z)$ forms a subalgebra $\mathfrak{u}$ for which*

$$\mathrm{Lie}\, Z = \mathrm{Lie}(\zeta_G) \oplus \mathfrak{u}.$$

Proof. Note that Z and also $\mathrm{Lie}(Z)$ are commutative; since the product of two commuting unipotent elements of G is unipotent and the sum of two commuting nilpotent elements of $\mathrm{Lie}(G)$ is nilpotent, results (a) and (b) follow from (4.1.1)(b) and (c). □

4.2. **Smoothness of the center of the centralizer.** In this section, K is again arbitrary. Let $x \in G(K)$, $X \in \mathfrak{g}(K)$ be arbitrary, write C for one of the groups $C_G(x)$ or $C_G(X)$, and write $Z = Z(C)$, so that Z is one of the groups Z_x or Z_X. We are now ready to prove the following:

Theorem. *The center $Z = Z(C)$ is a smooth group scheme over K.*

Proof. Since a group scheme is smooth over K if and only if it is smooth upon scalar extension, we may and will suppose K to be algebraically closed (hence in particular perfect). So as in §4.1, we may speak of the reduced subgroup scheme A_{red} of a group scheme A over K.

Let $x = x_s x_u$ and $X = X_s + X_n$ be the Jordan decompositions of the elements; thus $x_s \in G$ and $X_s \in \mathfrak{g}$ are semisimple, $x_u \in G$ is unipotent, $X_n \in \mathfrak{g}$ is nilpotent, and we have: $x_s x_u = x_u x_s$ and $[X_s, X_n] = 0$.

Then
$$C_G(x) = C_M(x_u) \quad \text{and} \quad C_G(X) = C_M(X_n)$$
where $M = C_G(x_s)$ resp. $M = C_G(X_s)$.

Now, the Zariski closure of the group generated by x_s is a smooth diagonalizable group whose centralizer coincides with $C_G(x_s)$. And according to §3.5 the centralizer of X_s is reductive and is the centralizer of a (non-smooth) diagonalizable group scheme. Thus in both cases, the connected component of M is itself a D-standard reductive group.

Moreover, (3.3.1) shows that x_u is a K-point of M^o. There is an exact sequence
$$1 \to C_{M^o}(x_u) \to C_M(x_u) \to E \to 1$$
resp.
$$1 \to C_{M^o}(X_N) \to C_M(X_N) \to E' \to 1$$
for a suitable subgroup E resp. E' of M/M^o. Since M/M^o has order invertible in K (3.3.1), apply (2.8.1) to see that the smoothness of Z follows from the smoothness of the center of $C_{M^o}(x_u)$ resp. $C_{M^o}(X_n)$; thus the proof of the theorem is reduced to the case where x is unipotent and X is nilpotent. Since in that case $C_G(X) = C_G(\sigma(X))$ where $\sigma : \mathcal{N} \to \mathcal{U}$ is a Springer isomorphism, we only discuss the centralizer of a nilpotent element $X \in \mathfrak{g}$.

We must argue that $\dim Z = \dim \operatorname{Lie} Z$. Since it is a general fact that $\dim \operatorname{Lie} Z \geq \dim Z$, it suffices to show the following:
$$(*) \quad \dim \operatorname{Lie} Z \leq \dim Z.$$

By (4.1.2) we have $\operatorname{Lie} Z = \operatorname{Lie}(\zeta_G) \oplus \mathfrak{u}$ where $\mathfrak{u}$ is the set of all nilpotent $Y \in \operatorname{Lie} Z$. According to (3.4.2), the center ζ_G of G is smooth. Thus $\dim \zeta_G = \dim \operatorname{Lie} \zeta_G$. In view of (4.1.2), the assertion $(*)$ will follow if we prove that
$$(**) \quad \dim \mathfrak{u} \leq \dim R_u Z_{\mathrm{red}}.$$

In order to prove $(**)$, we fix a Springer isomorphism $\sigma : \mathcal{N} \to \mathcal{U}$ – see Theorem 3.3 –, and we consider the restriction of σ to $\mathfrak{u}$.

We first argue that σ maps $\mathfrak{u}$ to $R_u Z_{\mathrm{red}}$. Since $\mathfrak{u}$ is smooth – hence reduced – and since K is algebraically closed, it suffices to show that σ maps the K-points of $\mathfrak{u}$ to $R_u Z_{\mathrm{red}}$. Fix $Y \in \mathfrak{u}(K)$.

If $g \in C_G(X)(K)$, the inner automorphism $\operatorname{Int}(g)$ of C is trivial on Z; thus, the automorphism $\operatorname{Ad}(g)$ of $\operatorname{Lie} C$ is trivial on $\operatorname{Lie} Z$. It follows that
$$g\sigma(Y)g^{-1} = \sigma(\operatorname{Ad}(g)Y) = \sigma(Y).$$
Since K is algebraically closed, it now follows from (2.5.2) that
$$\sigma(Y) \in Z(K) = C_G(X)^{\operatorname{Int}(C_G(X))}(K).$$

Since $\mathfrak{u}$ is reduced, one knows $\sigma(Y) \in Z_{\mathrm{red}}(K)$. Since $\sigma(Y)$ is unipotent, it follows that $\sigma(Y) \in R_u Z_{\mathrm{red}}(K)$.

Thus the restriction of the Springer isomorphism σ gives a morphism $\sigma_{|\mathfrak{u}} : \mathfrak{u} \to R_u Z_{\mathrm{red}}$. Since σ is a closed morphism, it follows that the image of $\sigma_{|\mathfrak{u}}$ is a closed subvariety of $R_u Z_{\mathrm{red}}$ whose dimension is $\dim \mathfrak{u}$, so that indeed $(**)$ holds.

□

With notation as in the preceding proof, we point out a slightly different argument. Namely, reasoning as above, one can show that the inverse isomorphism $\tau = \sigma^{-1} : \mathcal{U} \to \mathcal{N}$ maps $R_u Z_{\text{red}}$ to $\mathfrak{u}$. It follows that $R_u Z_{\text{red}}$ and $\mathfrak{u}$ are isomorphic varieties, hence they have the same dimension.

Note that we have now proved Theorem A from the introduction.

5. Regular nilpotent elements

In this section, we are going to prove Theorems B, C, and E from the introduction. We denote by G a D-standard reductive group over the field K. Let $T \subset G$ be a maximal torus, and let $T_0 \subset T$ where T_0 is a maximal torus of the derived group $G' = (G, G)$ of G. Let us write $r = \dim T_0$ for the semisimple rank of G. Finally, let $W = N_G(T)/T \simeq N_{G'}(T_0)/T_0$ be the corresponding Weyl group.

5.1. **Degrees and exponents.** We give here a quick description of some well-known numerical invariants associated with the Weyl group W. *We suppose that the derived group G' of G is quasi-simple, and we suppose that T (and hence G) is split over K.*

Let $V = X^*(T_0) \otimes_{\mathbf{Z}} \mathbf{Q}$ and note that the action of the Weyl group W on T_0 determines a linear representation (ρ, V) of W. The algebra of polynomials (regular functions) on V may be graded by assigning the degree 1 to each element of the dual space $V^\vee \subset \mathbf{Q}[V]$. The action via ρ of W on V determines an action of W on $\mathbf{Q}[V]$ by algebra automorphisms, and it is known that the algebra $\mathbf{Q}[V]^W$ of W-invariant polynomials on V is generated as a $\mathbf{Q}$-algebra by r algebraically independent homogeneous elements of positive degree [Bou 02, V.5.3 Theorem 3]. The *degrees* of W are the degrees $d_1, d_2, \dots, d_r$ of a system of homogeneous generators for $\mathbf{Q}[V]^W$. The degrees depend – up to order – only on W; see [Bou 02, V.5.1]. The *exponents of* W are the numbers $k_1, k_2, \dots, k_r$ where $k_i = d_i - 1$ for $1 \le i \le r$.

Recall that the "exponents" earn their name as follows. Let $c \in W$ be a Coxeter element [Bou 02, V.6.1], and write h for the order of c. If E is a field of characteristic 0 containing a primitive h-th root of unity $\varpi \in E^\times$, then [Bou 02, V.6.2 Prop. 3] the eigenvalues of $\rho(c)$ on $V \otimes_{\mathbf{Q}} E$ are the values

$$\varpi^{k_1}, \varpi^{k_2}, \cdots, \varpi^{k_r}.$$

The exponents and degrees are known explicitly; cf. [Bou 02, Plate I – IX].

5.2. **The centralizer of a regular nilpotent element.** In this section, G is again a D-standard reductive group (whose derived group is not required to be quasisimple) which we assume to be quasisplit over K.

If $\phi : \mathbf{G}_m \to G$ is a cocharacter and $i \in \mathbf{Z}$, we write $\mathfrak{g}(\phi; i)$ for the i-weight space of the action of $\phi(\mathbf{G}_m)$ on $\mathfrak{g}$ under the adjoint action of $\phi(\mathbf{G}_m)$; thus

$$\mathfrak{g}(\phi; i) = \{Y \in \mathfrak{g} \mid \operatorname{Ad}(\phi(t))Y = t^i Y \quad \forall t \in K_{\text{alg}}^\times\}.$$

Any cocharacter ϕ determines a unique parabolic subgroup $P = P(\phi)$ whose K_{alg} points are given by:

$$P(K_{\text{alg}}) = \{g \in G(K_{\text{alg}}) \mid \lim_{t \to 0} \operatorname{Int}(\phi(t))g \text{ exists}\}.$$

One knows that $\mathfrak{p} = \operatorname{Lie}(P) = \sum_{i \ge 0} \mathfrak{g}(\phi; i)$.

Let $X \in \mathfrak{g}(K)$ be nilpotent. Following [Ja 04, §5.3], we say that a cocharacter $\psi : \mathbf{G}_m \to G$ is said to be *associated to* a nilpotent element X in case (i) $X \in \mathfrak{g}(\psi; 2)$, and (ii) there is a maximal torus S of the centralizer $C_G(X)$ such that the image of ψ lies in (L, L), where $L = C_G(S)$.

(5.2.1). (a) *There are cocharacters associated to X.*
(b) *If ϕ and ϕ' are cocharacters associated to X, then $P(\phi) = P(\phi')$.*
(c) *The centralizer $C_G(X)$ is contained in $P = P(\phi)$ for a cocharacter ϕ associated to X.*
(d) *The unipotent radical R of $C_G(X)_{/K_{\text{alg}}}$ is defined over K and is a K-split unipotent group.*
(e) *Any two cocharacters associated to X are conjugate by a unique element of $R(K)$.*

Proof. In the geometric setting, these assertions may be found in [Ja 04]; the existence of an associated cocharacter is an essential part of the Bala-Carter, a conceptual proof of which may be found [Pr 03]. Over the ground field K, (a) and (c) follow from [Mc 04, Theorem 26 and Theorem 28]. (b) follows since associated cocharacters are optimal for the unstable vector X in the sense of Kempf; see [Pr 03]. Finally, (d) and (e) follow from [Mc 05, Prop/Defn 21]. □

Finally, recall that a nilpotent element $X \in \mathfrak{g}$ is *distinguished* provided that a maximal torus of the centralizer $C_G(X)$ is central in G.

(5.2.2). *Let $X \in \mathfrak{g}$ be nilpotent. The following are equivalent:*

(a) *X is regular – i.e. $\dim C_G(X)$ is equal to the rank of G.*
(b) *$X \in \mathrm{Lie}(B)$ for precisely one Borel subgroup of G.*

Moreover, if X is regular then X is distinguished, and if ϕ is a cocharacter associated with X, then $B = P(\phi)$ is the unique Borel subgroup with $X \in \mathrm{Lie}(B)$.

Proof. The equivalence of (a) and (b) can be found in [Ja 04, Cor. 6.8]. Note that in *loc. cit.* it is assumed that K is algebraically closed. But, it suffices to prove that (b) implies (a) after replacing K by an extension field. It remains to argue that (a) implies (b). But given (a), one knows there to be a unique Borel subgroup $B \subset G_{/K_{\mathrm{alg}}}$ with $X \in \mathrm{Lie}(B)$, where K_{alg} is an algebraic closure of K. It now follows from [Mc 04, Prop. 27] that B is a parabolic subgroup of G [i.e. that B is defined over K], and (b) follows.

That a regular element is distinguished follows from the Bala-Carter theorem; it can be seen perhaps more directly just by observing that B is a distinguished parabolic subgroup, so that an elment of the dense orbit of B on $\mathrm{Lie}\, R_u B$ is distinguished by [Ca 93, 5.8.7].

Finally, write $P = P(\phi)$. It follows from [Ja 04, 5.9] that X is in the dense P-orbit on $\mathrm{Lie}(R_u P)$ and that $C_P(X) = C_G(X)$; thus $\dim \mathrm{Ad}(G)X = 2 \dim R_u P$ so that indeed P must be a Borel subgroup. □

Since G is assumed to be quasisplit, we have

(5.2.3) ([Mc 05, Theorem 54]). *There is a regular nilpotent element $X \in \mathfrak{g}(K)$.*

We fix now a regular nilpotent element X. Let $C = C_G(X)$ be the centralizer of X, and write ζ_G for the center of G.

(5.2.4). *For the group $C = C_G(X)$ we have:*

(a) *the maximal torus of C is the identity component of the center ζ_G of G.*
(b) *$C = \zeta_G \cdot R_u(C)$.*
(c) *C is commutative.*

Proof. Assertions (a) and (b) follow from [Ja 04, §4.10, §4.13] precisely as in the proof of (3.4.2).

For (c), use a Springer isomorphism $\sigma : \mathcal{N} \to \mathcal{U}$, to see that C is the centralizer of the regular unipotent element $u = \sigma(X)$. Then the commutativity of C follows from a result of Springer – see [Hum 95, Theorem 1.14] – which implies that the centralizer of u contains a commutative subgroup of dimension equal to the rank of G. This shows that the identity component of C is commutative. Since $R_u C$ is connected and since $C = \zeta_G R_u C$, the group C is itself commutative. □

We now fix a cocharacter ϕ of (G, G) associated to X.

(5.2.5). *The image ϕ normalizes C. Suppose that the derived group of G is quasisimple. We have*

(a)
$$\mathrm{Lie}(R_u C) = \bigoplus_{i=1}^{r} \mathrm{Lie}(C)(\phi; 2k_i)$$
where $1 = k_1 \leq \cdots \leq k_r$ are the exponents of the Weyl group of G.

(b) $\dim \mathrm{Lie}(R_u C)(\phi; 2) = 1$.

Proof. First suppose that K has characteristic 0. In that case, the assertions are a consequence of results of [Ko 59]. One deduces (a) immediately from [Ko 59, §6.7]. For (b), one knows that the integers $2k_i$ are the highest weights for the action of a principal $\mathfrak{sl}_2$ on $\mathfrak{g}$. Examining the roots of $\mathfrak{g}$, one knows that the largest weight $2k_r$ occurs precisely once; thus $\dim V(\phi; 2k_r) = 1$.

Now the duality of the exponents [Ko 59, Theorem 6.7] shows that

$$\dim V(\phi; 2) = \dim V(\phi; 2k_1) = \dim V(\phi; 2k_r) = 1$$

as required.

For general K, consider a discrete valuation ring $\mathcal{A}$ whose residue field is K and whose field of fractions L has characteristic 0, and denote by $\mathcal{G}$ a split reductive group scheme over $\mathcal{A}$ such that upon base change with K one has $\mathcal{G}_{/K} \simeq G$. Of course, the Weyl groups of $\mathcal{G}_{/K}$ and of $\mathcal{G}_{/L}$ are isomorphic.

According to [Mc 08, Theorems 5.4 and 5.7] we may find a suitable such $\mathcal{A}$ for which there is a nilpotent section $X_0 \in \mathrm{Lie}(\mathcal{G})(\mathcal{A})$ and a homomorphism of $\mathcal{A}$-group schemes $\phi : \mathbf{G}_m \to \mathcal{G}$ with the following properties:

(i) the image $X_0(K)$ of X_0 in $\mathfrak{g} = \mathrm{Lie}(G) = \mathrm{Lie}(\mathcal{G}_{/K})$ coincides with X,
(ii) the image $X_0(L)$ of X_0 in $\mathrm{Lie}(\mathcal{G}_{/L})$ is regular nilpotent,
(iii) the cocharacter $\phi_{/K}$ of $G = \mathcal{G}_{/K}$ is associated to $X = X_0(K)$, and
(iv) the cocharacter $\phi_{/L}$ of $\mathcal{G}_{/L}$ is associated to $X_0(L)$.

Moreover, it follows from [Mc 08, Prop. 5.2] that the centralizer subgroup scheme $C_{\mathcal{G}}(X_0)$ is smooth. In particular, $\mathrm{Lie}(C_{\mathcal{G}}(X_0))$ is free as an $\mathcal{A}$-module, and $\mathrm{Lie}(C) = \mathrm{Lie}(C_{\mathcal{G}}(X_0)) \otimes_{\mathcal{A}} K$. We may regard $\mathrm{Lie}(C_{\mathcal{G}}(X_0))$ as a representation for the diagonalizable $\mathcal{A}$-group scheme $\mathbf{G}_m$ via $\mathrm{Ad} \circ \phi$. Decompose this representation as a sum of its weight subspaces:

$$\mathrm{Lie}(C_{\mathcal{G}}(X_0)) = \bigoplus_{i \in \mathbf{Z}} \mathrm{Lie}(C_{\mathcal{G}}(X_0))(\phi; i).$$

Extending scalars to L, one sees that $\mathrm{Lie}(C_{\mathcal{G}}(X_0))(\phi; i)$ is non-zero if and only if $i/2$ is one of the exponents of the Weyl group of G, and $\mathrm{Lie}(C_{\mathcal{G}}(X_0))(\phi; 2)$ has rank 1. The assertions (a) and (b) now follow by base change with K. □

5.3. **Lifting regular nilpotent elements.**

(5.3.1)**.** *Let $f : G \to H$ be a homomorphism between reductive groups such that f is surjective and central – i.e. the subgroup scheme* $\ker f$ *is contained in the center of G. Then f restricts to a surjective homomorphism $f_{|\zeta_G} : \zeta_G \to \zeta_H$.*

Proof. The assertion is geometric, so we may and will suppose the field K to be algebraically closed. Since $\ker f$ is central, the pre-image of each maximal torus S of H is a maximal torus T of G. Then $f_{|T} : T \to S$ is surjective. The result now follows because ζ_G is the (scheme theoretic) intersection of all maximal tori in G , and ζ_H is the intersection of all maximal tori in H; see [SGA3, Exp. XII Prop. 4.10]. □

Suppose now that G_1 and G_2 are D-standard reductive groups, and that $f : G_1 \to G_2$ is a separable surjective homomorphism of reductive groups which is central, as before. Recall that the separability of f means that the tangent mapping df is surjective.

(5.3.2)**.** (a) *Suppose that $X_2 \in$* $\mathrm{Lie}(G_2)(K)$ *is regular nilpotent. There is a nilpotent element $X_1 \in$* $\mathrm{Lie}(G_1)(K)$ *for which $df(X_1) = X_2$.*
(b) *If $df(Y_1) = Y_2$ for nilpotent elements $Y_i \in$* $\mathrm{Lie}(G_i)$*, then Y_1 is regular if and only if Y_2 is regular.*

Proof. Let $B \subset G_2$ be a Borel subgroup with $X \in \mathrm{Lie}(B)(K)$. The inverse image B_1 of B in G_1 is a parabolic subgroup [Bor 91, 22.6]; since B_1 is evidently solvable, B_1 is a Borel subgroup of G_1. Thus f induces a morphism $\tilde{f} : \mathcal{B}_1 = G_1/B_1 \to G_2/B$, and it is clear that the tangent map at the point B_1 of $\mathcal{B}_1$ is an isomorphism. It follows from [Sp 98, Theorem 5.3.2(iii)] that $\tilde{f}$ is an isomorphism between the flag varieties.

Write $\mathfrak{u}_1 = \operatorname{Lie} R_u B_1$ and $\mathfrak{u} = \operatorname{Lie} R_u B$. According to [Bor 91, 22.5], f induces a bijection between the roots of G_1 (with respect to some maximal torus) and the roots of G (with respect to a compatible maximal torus). In particular, $\dim R_u B_1 = \dim R_u B$. Since $\ker f$ is central in G, $\ker df$ is contained in $\operatorname{Lie}(T)$ for each maximal torus T. It follows that the restriction of df to $\mathfrak{u}_1$ is injective, so that $df(\mathfrak{u}_1) = \mathfrak{u}$. Since $X \in \operatorname{Lie}(B)$ is nilpotent, we have $X \in \mathfrak{u}$. It follows that there is a – necessarily nilpotent – element $X_1 \in \mathfrak{u}_1$ with $df(X_1) = X$. This proves (a).

Now, $\tilde{f}$ induces a bijection between the varieties $\mathcal{B}_{1,Y_1}$ and $\mathcal{B}_{2,Y_2}$, where $\mathcal{B}_{i,Y_i}$ consists of those Borel subgroups B with $Y_i \in \operatorname{Lie}(B)$. Assertion (b) now follows from (5.2.2). □

(5.3.3). *Suppose that the elements $X_i \in \operatorname{Lie}(G_i)$ are nilpotent for $i = 1, 2$, that $df(X_1) = X_2$, and that X_1 is regular, equivalently that X_2 is regular. If $C_1 = C_{G_1}(X_1)$ and $C = C_{G_2}(X_2)$, then $C_1 = f^{-1}C$. In particular, f restricts to a surjective separable mapping $f_{|C_1} : C_1 \to C$.*

Proof. As before, the assertion is geometric; thus we may and will suppose that K is algebraically closed for the proof. We only must argue that $(*)\quad C_1 = f^{-1}C$. Indeed, the remaining assertions follow from $(*)$ by using (2.11.1)(d) and the smoothness of C_1 (3.4.1).

We will argue that $f_{|C_1} : C_1 \to C$ is surjective; assertion $(*)$ will then follow since $\ker f$ is central in G_1. Recall that $C_1 = \zeta_{G_1} \cdot R_u C_1$ and $C = \zeta_{G_2} \cdot R_u C$. The restriction $f_{|\zeta_{G_1}} : \zeta_{G_1} \to \zeta_{G_2}$ is surjective (5.3.1).

It remains to argue that $f_{|R_u C_1}$ yields a surjective mapping $R_u C_1 \to R_u C$. Since G_1 and G_2 are D-standard, the centralizers C_1 and C are smooth by (3.4.1). Thus the unipotent radicals of C_1 and of C are smooth group schemes over K. So the surjectivity of $f_{|R_u C_1} : R_u C_1 \to R_u C$ will follow if we only prove that $df : \operatorname{Lie}(R_u C_1) \to \operatorname{Lie}(R_u C)$ is surjective.

But $df_{|\operatorname{Lie} R_u C_1}$ is injective since $\ker df$ is central. Moreover, $\dim R_u C_1$ is the semisimple rank of G_1, and $\dim R_u C$ is the semisimple rank of G_2. Since f is surjective with central kernel, the semisimple ranks of G_1 and G_2 coincide. Thus $df_{|\operatorname{Lie} R_u C_1}$ is bijective and the assertion follows. □

5.4. **The normalizer of C.** Let us again fix a regular nilpotent element X together with a cocharacter ϕ associated to X. Let $N = N_G(C)$ be the normalizer of C.

We will argue in (5.4.2) below that N is a smooth group scheme over K. Meanwhile, we consider in the next assertion the N-orbit of X. Viewing this orbit as a subspace of $\operatorname{Lie}(R_u C)$, we may consider its closure; that closure has a unique structure of reduced subscheme [Li 02, Prop. 2.4.2]. Since the orbit of X is open in its closure, that orbit inherits a structure as a reduced subscheme.

The following argument essentially just records observations made by Serre in his note found in [Mc 05, Appendix].

(5.4.1). (a) *The N-orbit of X is the open subset of $\operatorname{Lie}(R_u C)$ consisting of the regular elements; i.e.*

$$\operatorname{Ad}(N)X = \operatorname{Lie}(R_u C)_{\mathrm{reg}}$$

(b) *The group N/C is connected and has dimension equal to the semisimple rank r of G.*
(c) *In particular,* $\dim N = 2r + \dim \zeta_G$.

Proof. Before giving the proof, we recall that $(*)\ C = C^o \cdot \zeta_G$ where ζ_G is the center of G; see (5.2.4).

For the proof of (a), we have evidently $\operatorname{Ad}(N)X \subset \operatorname{Lie}(R_u C)_{\mathrm{reg}}$. Since $\operatorname{Ad}(N)X$ is a reduced scheme, to prove equality it suffices to show that any closed point of $\operatorname{Lie}(R_u C)_{\mathrm{reg}}$ is contained in this orbit. If K_{alg} is an algebraic closure of K and $Y \in \operatorname{Lie}(R_u C)_{\mathrm{reg}}(K_{\mathrm{alg}})$, then Y is a Richardson element for B, where B is the Borel subgroup as in (5.2.2). Since the Richardson elements form a single orbit under B, there is $x \in B(K_{\mathrm{alg}})$ for which $\operatorname{Ad}(x)Y = X$. Since C is commutative, a dimension argument shows that $C_G^o(Y) = C^o$. Since also $C_G(Y) = C_G^o(Y) \cdot \zeta_G$; it follows from $(*)$ that $C = C_G(Y)$. Since

$$xCx^{-1} = xC_G(Y)x^{-1} = C_G(\operatorname{Ad}(x)Y) = C_G(X) = C,$$

one sees that $x \in N(K_{\mathrm{alg}})$. It follows that $\operatorname{Ad}(N)X = \operatorname{Lie}(R_u C)_{\mathrm{reg}}$.

For (b), first suppose that $K = K_{\mathrm{alg}}$ is algebraically closed. By (a), $(N/C)_{\mathrm{red}}$ identifies with $\mathrm{Lie}(R_uC)_{\mathrm{reg}}$, an open subvariety of the affine space $\mathrm{Lie}(R_uC)$. It follows that $(N/C)_{\mathrm{red}}$ is an irreducible variety; thus the variety N/C is connected.

But then relaxing the assumption on K, it follows that N/C is connected in general. Since $\mathrm{Lie}(R_uC)$ has dimension equal to r, conclude that $\dim N/C = r$.

Finally, (c) follows since $\dim C = r + \dim \zeta_G$. □

We can now prove:

(5.4.2). *N is a smooth subgroup scheme of G.*

Proof. The statement is geometric; thus we may and will suppose K to be algebraically closed. Let $f : G_1 \to G_2$ be a surjective separable morphism with central kernel, and suppose that G is one of the groups G_1 or G_2.

If $G = G_1$, write X_1 for X and set $X_2 = df(X_1)$. If $G = G_2$, write X_2 for X and use (5.3.2) to find a regular nilpotent $X_1 \in \mathrm{Lie}(G_1)$ for which $df(X_1) = X_2$.

Now write $C_i = C_{G_i}(X_i)$. It follows from (5.3.3) that $C_1 = f^{-1}C_2$, so we may apply (2.11.2) to see that

$(*)$ $N_{G_1}(C_1)$ is smooth over K if and only if $N_{G_2}(C_2)$ is smooth over K.

We are now going to argue: it suffices to prove the result when G is quasisimple in very good characteristic.

Well, if the result is known for quasisimple G in very good characteristic, it follows easily for any semisimple, simply connected group in very good characteristic (since any such is a direct product of simply connected quasisimple groups). But any semisimple group in very good characteristic is separably isogenous to a simply connected one, so $(*)$ then permits us to deduce the result for any semisimple G in very good characteristic.

For a general D-standard group G, we must consider a reductive group H of the form $H = H_1 \times T$ where H_1 is semisimple in very good characteristic, together with a diagonalizable subgroup scheme $D \subset H$. We suppose that G is separable isogenous to $C_H(D)^o$. The above arguments show that the desired result holds for H, and we want to deduce the result for G. Again using $(*)$, we may suppose that $G = C_H(D)^o$.

But if $N = N_G(C)$, we see that $N = N_H(C_H(X))^D$. Our assumption means that $N_H(C_H(X))$ is smooth. But then [SGA3, Exp. XI, Cor. 5.3] shows that $N = N_H(C_H(X))^D$ is smooth, as required.

Thus, we now suppose G to be quasisimple in very good characteristic. Now, $\dim N = 2r$ by (5.4.1), where r is the rank of G. Thus to show that N is smooth, we must show that $2r = \dim \mathrm{Lie}(N)$. Since one has always $\dim \mathrm{Lie}(N) \geq \dim N$, it is enough to argue that $\dim \mathrm{Lie}(N) \leq 2r$.

Write $\mathfrak{n} = \{Y \in \mathfrak{g} \mid [Y, \mathrm{Lie}\, C] \subset \mathrm{Lie}\, C\}$ for the normalizer in $\mathfrak{g}$ of $\mathrm{Lie}(C)$. Evidently $\mathrm{Lie}(N) \subset \mathfrak{n}$; it therefore suffices to show that $\dim \mathfrak{n} \leq 2r$.

Suppose that $Y \in \mathfrak{n}$. Since C is commutative, evidently $[[Y, X], X] = 0$, so that $Y \in \ker(\mathrm{ad}(X)^2)$. Thus, it suffices to show that

$$(*) \quad \dim \ker(\mathrm{ad}(X)^2) = 2r.$$

But in view of our assumptions on the characteristic of K, $(*)$ follows from [Spr 66, Cor. 2.5 and Theorem 2.6]. □

(5.4.3). *N is a solvable group.*

Proof. Let B be the unique Borel subgroup of G with $X \in \mathrm{Lie}(B)$ as in (5.2.2). Since B is solvable, the result will follow if we argue that $N \subset B$.

Since N is smooth – in particular, reduced – it suffices to argue that B contains each closed point of N. Thus, it is enough to suppose that K is algebraically closed and prove that $N(K) \subset B(K)$.

Recall first that according to (5.2.1)(c), we have $C \subset B$. If $y \in N(K)$ it follows that $\mathrm{Int}(y)B$ contains C, hence $\mathrm{Lie}(\mathrm{Int}(y)B)$ contains X. This proves that $\mathrm{Int}(y)B = B$, so y normalizes B. Since Borel subgroups are self normalizing, we deduce $N(K) \subset B(K)$, and the result follows. □

(5.4.4). *Write S for the image of ϕ and write ζ_G^o for the connected center of G. Then $S \cdot \zeta_G^o$ is a maximal torus of N.*

Proof. Let $T \subset N$ be any maximal torus of N containing S. Since T commutes with the image of ϕ, it follows that the space $\mathrm{Lie}(C)(\phi; 2)$ is stable under T. But that space is one dimensional (5.2.5) and has X as a basis vector; it follows that X is a weight vector for T so that T lies in the stabilizer in G of the line $[X] \in \mathbf{P}(\mathrm{Lie}(G))$. We know by (5.2.4) that ζ_G^o is a maximal torus of C; applying [Ja 04, 2.10 Lemma and Remark], one deduces that $S \cdot \zeta_G^o$ is a maximal torus of that stabilizer, which completes the proof. □

Note that together (5.4.1), (5.4.3), and (5.4.4) yield Theorem B from the introduction.

(5.4.5). *Consider the line $[X] \in \mathbf{P}(\mathrm{Lie}(R_uC))$ and let $\mathcal{O}$ be the N-orbit of $[X]$.*

(a) *The orbit mapping $(a \mapsto [\mathrm{Ad}(a)X]) : N \to \mathcal{O}$ is smooth.*
(b) *The stabilizer $\mathrm{Stab}_N([X])$ of $[X]$ in N is smooth and is equal to $S \cdot C$.*
(c) *The N-orbit of $[X]$ is open and dense in $\mathbf{P}(\mathrm{Lie}(R_uC))$.*

Proof. Recall that a mapping $f : X \to Y$ between smooth varieties over K is smooth if the tangent map df_x is surjective for all closed points of X. If X and Y are homogeneous spaces for an algebraic group, it suffices to check that df_x is surjective for one point x of X.

Moreover, it follows from [Sp 98, Prop. 12.1.2] that if an algebraic group H acts on a variety X, and if $x \in X$ is a closed point, then the stabilizer $\mathrm{Stab}_H(x)$ is a smooth subgroup scheme if and only if the orbit mapping $H \to H.x$ determined by x is a smooth morphism.

Now, assertion (a) is the content of [Mc 04, Lemma 23] As to (b), first note that the fact that the orbit mapping $N \to \mathcal{O}$ is smooth shows that stabilizer $\mathrm{Stab}_N([X])$ is smooth over K. Now, according to [Ja 04, 2.10] the stabilizer in G of the line $[X]$ is $S \cdot C$. Since $S \cdot C$ is a closed subgroup of N, the remaining assertion of (b) follows.

For (c), notice that $\dim N/(S \cdot C) = \dim N/C - 1 = r - 1$ by (5.4.1). Since we have also $\dim \mathbf{P}(\mathrm{Lie}(R_uC)) = r - 1$, it follows that the N-orbit of $[X]$ is open and dense in $\mathbf{P}(\mathrm{Lie}(R_uC))$, as required. [7] □

Let us write $D = \mathrm{Stab}_N([X]) = S \cdot C$, and let $\mathbf{1}$ be the closed point of N/D determined by the trivial coset of D in N. From the adjoint action of the torus S on $\mathrm{Lie}(N)$ one deduces an action of S on the tangent space $T_{\mathbf{1}}(N/D)$; thus one may speak of the weight spaces $T_{\mathbf{1}}(N/D)(\phi; j)$ for $j \in \mathbf{Z}$.

(5.4.6). *Assume that the derived group of G is quasi-simple, and let the positive integers $k_1, k_2, \dots, k_r$ be as in 5.1. Then we have the following:*

$$T_{\mathbf{1}}(N/D) = \bigoplus_{i=2}^{r} T_{\mathbf{1}}(N/D)(\phi; 2k_i - 2)$$

Proof. Let $\mathcal{O} \subset \mathbf{P}(\mathrm{Lie}\, R_uC)$ be the N-orbit of $[X]$. By (5.4.5)(c), one knows that $\mathcal{O}$ is an open subset of $\mathbf{P}(\mathrm{Lie}(R_uC))$; in particular, $T_{[X]}\mathcal{O} = T_{[X]}\mathbf{P}(\mathrm{Lie}(R_uC))$. Also by (5.4.5)(c), one knows that the orbit mapping $\alpha : N \to \mathcal{O}$ given by $\alpha(y) = [\mathrm{Ad}(y)X]$ induces an S-equivariant isomorphism $\bar{\alpha} : N/D \to \mathcal{O}$. Since $\bar{\alpha}(\mathbf{1}) = [X]$, the tangent map to $\bar{\alpha}$ at $\mathbf{1}$ yields an S-isomorphism between $T_{\mathbf{1}}(N/D)$ and $T_{[X]}\mathcal{O} = T_{[X]}\mathbf{P}(\mathrm{Lie}(R_uC))$. The assertion now follows from (5.2.5) and the description of the S-module structure on the tangent space $T_{[X]}\mathbf{P}(\mathrm{Lie}(R_uC))$ given in (2.10.1). . □

We can now complete the proofs of Theorems C and D from the introduction.

Proof of Theorem C. Consider the quotient morphism

$$\Phi : N/C \to N/(S \cdot C) = N/D$$

[7] Alternatively, one can argue as follows. Write $\mathcal{L}$ for the tautological line bundle – corresponding to the invertible sheaf $\mathcal{O}_{\mathbf{P}(\mathrm{Lie}\, R_uC)}(-1)$ – over $\mathbf{P}(\mathrm{Lie}\, R_uC)$. Then $(\mathrm{Lie}\, R_uC) - \{0\}$ identifies with the total space of $\mathcal{L}$ with the zero-section removed. It follows that the natural mapping $(\mathrm{Lie}\, R_uC) - \{0\} \to \mathbf{P}(\mathrm{Lie}\, R_uC)$ is flat and hence open.

and again write $\mathbf{1}$ for the closed point of N/C determined by the trivial coset, and $\mathbf{1}$ for the closed point of N/D determined by the trivial coset. Then differentiating Φ gives an S-equivariant mapping

$$d\Phi_{\mathbf{1}} : T_{\mathbf{1}}(N/C) \to T_{\mathbf{1}}(N/D).$$

Evidently the kernel of $d\Phi_{\mathbf{1}}$ is the image of $\mathrm{Lie}(S)$ in $T_{\mathbf{1}}((N/C)$. Regard $T_{\mathbf{1}}(N/C)$ as an S-module; by complete reducibility one can find an S-subrepresentation $V \subset T_{\mathbf{1}}(N/C)$ which is a complement to $\ker d\Phi_{\mathbf{1}}$. Then evidently $d\Phi_{\mathbf{1}}$ yields an isomorphism between V and $T_{\mathbf{1}}(N/D)$, and the assertion of Theorem C follows. □

Proof of Theorem D. We must argue that $R_u N$ is defined over K and split. Keep the preceding notations of this section; in particular, S is the image of the cocharacter ϕ associated to the regular nilpotent element $X \in \mathrm{Lie}(G)$. According to Theorem 2.9, it will suffice to show that $\mathrm{Lie}(S) = \mathrm{Lie}(N)^S$ and that each non-0 weight of S on $\mathrm{Lie}(N)$ is positive. It suffices to prove these statements after extending scalars; thus we may and will suppose that K is algebraically closed.

If G is any D-standard reductive group, we may find D-standard groups $M_1, \dots, M_d$ together with a homomorphism $\Phi : M \to G$ where $M = \prod_{i=1}^d M_i$, satisfying (a)–(d) of (3.2.4).

Using (5.3.3) we may find a regular nilpotent element $X_1 \in \mathrm{Lie}(M)$ such that – writing $C_1 = C_M(X_1)$ – the restriction $\Phi_{|C_1} : C_1 \to C = C_G(X)$ is surjective (and separable). Moreover, we may choose a cocharacter $\phi_1 : \mathbf{G}_m \to M$ associated with X_1 such that $\phi = \Phi \circ \phi_1$ is associated with X. Write $S_1 \subset M$ for the image of ϕ_1 and $S \subset G$ for the image of ϕ.

Now, by (3.2.4)(a) each M_i has quasisimple derived group. In the case where M itself has quasisimple derived group – i.e. if $M = M_1$ – one uses (5.2.5) and Theorem C to deduce that

(i) $\mathrm{Lie}(S_1) = \mathrm{Lie}(N_1)^{S_1}$, and
(ii) the non-zero weights of S_1 on $\mathrm{Lie}(N_1)$ are positive,

where we have written $N_1 = N_M(C_1)$. Since in general M is a direct product of reductive groups each having quasisimple derived group, one sees readily that (i) and (ii) hold for M.

The normalizer $N_1 = N_M(C_1)$ is smooth by Theorem B. Since Φ is separable, it follows from (2.11.2) that $\Phi_{|N_1} : N_1 \to N$ is surjective and separable – i.e. $d\Phi_{|N_1} : \mathrm{Lie}(N_1) \to \mathrm{Lie}(N)$ is surjective. Using the fact that (i) and (ii) hold together with the surjectivity of $d\Phi_{|N_1}$, one sees that $\mathrm{Lie}(S) = \mathrm{Lie}(N)^S$ and that the non-zero weights of S on $\mathrm{Lie}(N)$ are positive, and the proof is complete. □

5.5. **The tangent map to a Springer isomorphism.** In this section, we give the proof of Theorem E. Thus we suppose in this section that the derived group of G is quasisimple. We fix a Springer isomorphism $\sigma : \mathcal{N} \xrightarrow{\sim} \mathcal{U}$, and we write $u = \sigma(X)$ where $u \in G$ is regular unipotent and $X \in \mathfrak{g}$ is regular nilpotent.

Since σ is G-equivariant, one knows that $C = C_G(X) = C_G(u)$.

(5.5.1)**.** *The restriction of σ to* $\mathrm{Lie}\, R_u C$ *determines an isomorphism* $\gamma : \mathrm{Lie}\, R_u C \xrightarrow{\sim} R_u C$. *In particular, the tangent mapping $d\gamma = (d\gamma)_0$ determines an isomorphism* $d\gamma : \mathrm{Lie}\, R_u C \xrightarrow{\sim} \mathrm{Lie}\, R_u C$.

Proof. Indeed, recall that C is a smooth group scheme, and that $C = \zeta_G \cdot R_u C$ by (5.2.4), so that $R_u C$ is the space of fixed points of $\mathrm{Int}(u)$ on $\mathcal{U}$ and $\mathrm{Lie}\, R_u C$ is the space of fixed points of $\mathrm{Ad}(u)$ on $\mathcal{N}$; the assertion is now immediate. □

Write $V = \mathrm{Lie}\, R_u C$. Then $d\gamma$ is an endomorphism of V as an N-module, where N is the normalizer in G of C. As in §5.4, we fix a cocharacter ϕ associated to X; write $S \subset N$ for the image of ϕ. We now give the

Proof of Theorem E. For (1), note first that the mapping γ is in particular an S-module endomorphism of V. Since $\dim V(\phi; 2) = 1$ by Theorem (5.2.5), one knows that X spans $V(\phi; 2)$. It follows that $d\gamma(X) = \alpha X$ for some $\alpha \in K^\times$.

If now $Y \in V_{\mathrm{reg}} = (\mathrm{Lie}\, R_u(C))_{\mathrm{reg}}$, there is an element $g \in N$ with $\mathrm{Ad}(g)X = Y$; cf. (5.4.1). Then

$$d\gamma(Y) = d\gamma(\mathrm{Ad}(g)X) = \mathrm{Ad}(g) d\gamma(X) = \alpha\, \mathrm{Ad}(g)X = \alpha Y.$$

It follows that $d\gamma$ and $\alpha \cdot 1_V$ agree on the dense subset $(\mathrm{Lie}(R_u C))_{\mathrm{reg}} \subset \mathrm{Lie}(R_u C)$ so that indeed $d\gamma = \alpha \cdot 1_V$.

For (2), recall that B is a Borel subgroup of G with unipotent radical U. That $\sigma_{|\mathrm{Lie}\, U}$ is an isomorphism onto U follows from [Mc 05, Remark 10].

Now fix a Richardson element $X \in \mathrm{Lie}(U)(K)$; then X is a regular nilpotent element of $\mathfrak{g}$, and part (1) shows that $d\sigma_{|\mathrm{Lie}\, U}(X) = \alpha X$ for some $\alpha \in K^\times$. If $Y \in \mathrm{Lie}(U)(K_{\mathrm{alg}})$ is a second Richardson element, then $Y = \mathrm{Ad}(g)X$ for $g \in B(K_{\mathrm{alg}})$, and it is then clear by the equivariance of $d(\sigma_{|\mathrm{Lie}\, U})_0$ that $d(\sigma_{|\mathrm{Lie}\, U})_0(Y) = \alpha Y$. Since the Richardson elements are dense in $\mathrm{Lie}\, U$, the result follows. □

Note that Theorem E need not hold when the derived group of G fails to be quasi-simple. Indeed, take for G the D-standard group $G = \mathrm{GL}_n \times \mathrm{GL}_m$ where $n, m \geq 2$. Then $\mathfrak{g} = \mathfrak{gl}_n \oplus \mathfrak{gl}_m$, and the mapping

$$(X, Y) \mapsto (1 + \alpha X, 1 + \beta Y)$$

defines a Springer isomorphism σ for any $\alpha, \beta \in K^\times$. If $X_0 \in \mathfrak{gl}_n$ and $Y_0 \in \mathfrak{gl}_m$ are regular nilpotent, then $X = (X_0, Y_0) \in \mathfrak{g}$ is regular nilpotent; the mapping $d\sigma$ has eigenvalues α and β on $\mathrm{Lie}\, R_u C_G(X)$ and hence is not a multiple of the identity if $\alpha \neq \beta$.

We finally conclude with an argument which gives an alternate proof of (b) of Theorem A in case G has quasi-simple derived group. This argument does not rely on the fact that $Z(C_1)$ is smooth; on the other hand, in order to make sense of $Z(C_1)_{\mathrm{red}}$, we are forced to assume K to be perfect.

(5.5.2). *Let K be perfect, let $X_1 \in \mathfrak{g}(K)$ be nilpotent, and let $C_1 = C_G(X_1)$ be its centralizer. Then the rule $t \mapsto \sigma(tX_1)$ defines a mapping $\Phi : \mathbf{Aff}^1 \to Z(C_1)_{\mathrm{red}}$, and $X_1 = c \cdot d\Phi_0(1) \in \mathrm{Lie}(Z(C_1)_{\mathrm{red}})$ for some $c \in K^\times$.*

Proof. Let $u = \sigma(X_1)$ and observe that $C_1 = C_G(u)$ by the G-equivariance of σ, so in particular, $u \in C_1$. Then for each $t \in \mathbf{Aff}^1$, and for each $g \in C_1$, we have

$$g \cdot \sigma(tX_1) \cdot g^{-1} = \sigma(t\,\mathrm{Ad}(g)X_1) = \sigma(tX_1).$$

Since $\mathbf{Aff}^1$ is reduced, it follows that $\sigma(tX_1)$ indeed lies in $Z(C_1)_{\mathrm{red}}$.

The formula for the tangent mapping of Φ is now immediate from Theorem E. □

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