LINEARITY FOR ACTIONS ON VECTOR GROUPS

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ABSTRACT. Let k be an arbitrary field, let G be a (smooth) linear algebraic group over k, and let U be a vector group over k on which G acts by automorphisms of algebraic groups. The action of G on U is said to be *linear* if there is a G-equivariant isomorphism of algebraic groups $U \simeq \operatorname{Lie}(U)$.

Suppose that G is connected and that the unipotent radical of G is defined over k. If the G-module $\operatorname{Lie}(U)$ is simple, we show that the action of G on U is linear. If G acts by automorphisms on a connected, split unipotent group U, we deduce that U has a filtration by G-invariant closed subgroups for which the successive factors are vector groups with a linear action of G. When G is connected and the unipotent radical of G is defined and split over k, this verifies an assumption made in earlier work of the author on the existence of Levi factors.

On the other hand, for any field k of positive characteristic we show that if the category of representations of G is not semisimple, there is an action of G on a suitable vector group U which is not linear.

CONTENTS

1. INTRODUCTION

Let k be a field, and let G denote a linear algebraic group over k; otherwise said, G is a smooth affine group scheme of finite type over k. A vector group U is a linear algebraic group (over k) isomorphic to the product of (finitely many) copies of the additive group $\mathbf{G}_a$.

In this paper, we are interested the action of G by algebraic group automorphisms on a vector group U. If the linear algebraic group G acts on any linear algebraic group H by algebraic group automorphisms, the induced action of G on the Lie algebra of H makes $\operatorname{Lie}(H)$ a G-module. We say that the action of G on U is *linear* if there is a G-equivariant isomorphism of algebraic groups $U \simeq \operatorname{Lie}(U)$ [1].

If k has characteristic 0, view U as a closed subgroup of $\operatorname{GL}(V)$ for some faithful finite dimensional U-module V. Then every vector in $\operatorname{Lie}(U)$ is a nilpotent endomorphism of V, and the exponential mapping $X \mapsto \exp(X)$ defines a G-equivariant isomorphism of algebraic groups $\operatorname{Lie}(U)_a \xrightarrow{\sim} U$. On the other hand, if k has characteristic $p > 0$, in §5, we give examples of non-linear actions of G whenever there are G-modules which are not completely reducible (in particular, for semisimple groups G). *Thus, our results are only interesting when k has characteristic $p > 0$, which we assume from now on.*

Our main result gives a sufficient condition for linearity of the action of G on U which holds under some hypothesis which we now discuss.

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[1]A finite dimensional k-vector space V may be viewed as a linear algebraic group – in fact, a vector group – in a natural way. In what follows, we will write V_a when we view V as an algebraic group. With this notation, the action of G on the vector group U is linear if there is a G-equivariant isomorphism $U \simeq \operatorname{Lie}(U)_a$ of algebraic groups.

1.1. **Assumptions on G.** When the ground field k is imperfect, the geometric unipotent radical of G – i.e. the unipotent radical of $G_{/k_{\mathrm{alg}}}$ – may not arise by base-change from any subgroup of G – see e.g. [CGP 10, Example 1.1.3]. We are going to sidestep this issue here. Consider the conditions

(R) there is a subgroup $R \subset G$ such that $R_{/k_{\mathrm{alg}}}$ is the unipotent radical of $G_{/k_{\mathrm{alg}}}$, and

(RS) Condition **(R)** holds and R is split over k.

Recall that a connected unipotent group U is *split* provided that there is a filtration

$$U = U^0 \supset U^1 \supset \cdots \supset U^r = 1$$

by closed normal subgroups for which each subquotient U^i/U^{i+1} is a *vector group*. When k is imperfect, there are so-called *wound* unipotent groups which are not split – see e.g. [CGP 10, Example B.2.3] or Example (2.2.4) below.

If **(R)** holds, we refer to the group $R \subset G$ as the unipotent radical of G. In the language used in [Sp 98], condition **(R)** means that the unipotent radical of G is defined over k, and **(RS)** means that the unipotent radical of G is defined and split over k. Observe that conditions **(R)** and **(RS)** are automatic for any G when k is perfect; see e.g. [Sp 98, 14.4.5(v) and 14.3.10]. If **(R)** holds, the quotient G/R is a (not necessarily connected) *reductive* algebraic group over k.

1.2. **The main result: a condition for linearity.** If the linear algebraic group G acts by group automorphisms on the vector group U, then $\mathrm{Lie}(U)$ is a G-module and hence a module for the identity component G^0 of G. The following condition for the linearity of the action of G on U will be obtained in Theorem (3.2.6):

Theorem A. *Assume that condition **(R)** holds for the linear algebraic group G, and that G acts by group automorphisms on the vector group U. If* $\mathrm{Lie}(U)$ *is a simple module for the identity component G^0 of G, then the action of G on U is linear.*

In the hypothesis of Theorem A, observe that we do *not* require the G^0-module $\mathrm{Lie}(U)$ to remain simple after scalar extension.

1.3. **Consequences: actions on connected, split, unipotent groups.** Let U be a split unipotent group on which G acts by group automorphisms. We say that the action of G on U is *linearly filtered* if there is a filtration of U

$$U = U^0 \supset U^1 \supset \cdots \supset U^r \supset U^{r+1} = 0$$

by G-invariant closed subgroups U^i such that for each $0 \le i \le r$, U^i/U^{i+1} is a vector group on which G acts linearly.

As an application of Theorem A we obtain the following result; see Theorem (4.3.1) for the proof.

Theorem B. *Suppose that the connected linear algebraic group G acts by group automorphisms on the connected, split, unipotent group U. If condition **(R)** holds for G, then the action of G on U is linearly filtered.*

For a group G satisfying condition **(R)**, we wish to apply the preceding result to the unipotent radical R of G. If the conclusion of Theorem B is valid for the action of G on R, then R is automatically split unipotent. Thus we are led to suppose at the outset that R is split unipotent; i.e. we require that **(RS)** holds for G.

The preceding Theorem then shows that the action of G on R is linearly filtered when G is connected. The main results of the author's earlier investigation of the existence and conjugacy of Levi factors of G made in [Mc 10], were proved under an assumption formulated as "condition **(L)**" of [Mc 10, §2.3]. Condition **(L)** holds provided the unipotent radical R of G possesses a filtration by closed subgroups R_i which are normal in G and for which each quotient group R_i/R_{i+1} is a vector group with a linear action of G/R.

According to Theorem B, the action of G on the connected, split unipotent group R is linearly filtered. Thus, [Mc 10](2.2.3) yields the following result:

Theorem C. *Assume that the linear algebraic group G is connected and satisfies condition **(RS)**. Then condition **(L)** of* [Mc 10, (2.2.3)] *holds for G.*

In [Mc 10, §6.3], condition **(L)** was verified directly for the special fiber of a parahoric group scheme attached to a split reductive group over a local field. Since the special fiber of a parahoric group scheme is connected, Theorem C gives a simpler proof that **(L)** holds in this case.

Write $\pi : G \to G/R$ for the natural mapping. A *Levi factor* of G is a k-subgroup $M \subset G$ for which the restriction $\pi_{|M} : M \to G/R$ is an isomorphism of algebraic groups. In view of Theorem C, we deduce the following result(s) from [Mc 10, Theorem 5.1 and 5.2]:

Corollary D. *Assume that the linear algebraic group G is connected and satisfies condition* **(RS)***. Choose a filtration*

$$R = R^0 \supset R^1 \supset \cdots \supset R^r \supset R^{r+1} = 0$$

of R where each R^i is normal in G and each quotient R^i/R^{i+1} is a vector group with a linear action of G/R.

(i) Suppose that $H^2(G/R, \mathrm{Lie}(R^i/R^{i+1})) = 0$ for each i. Then G has a Levi factor.
(ii) Suppose that G has a Levi factor and that $H^1(G/R, \mathrm{Lie}(R^i/R^{i+1})) = 0$ for each i. Then any two Levi factors of G are conjugate by an element of $R(k)$.

Remark (1.3.1). At least for connected G, Theorem B also simplifies the hypotheses of other results in [Mc 10]; see e.g. the Theorems in §5.4 and §5.5 of that paper.

We conclude this introduction by mentioning some questions which we have left unanswered.

Question E. *Does the conclusion of Theorem A hold when either:*

(i) ***(R)*** *fails to hold, or*
(ii) when $\mathrm{Lie}(U)$ is a simple G-module of dimension > 1, but the identity component of G acts trivially on Lie(U)?

The author is unaware of examples demonstrating a negative answer in either case (i) or (ii).

Finally, we point out that during the initial preparation of this manuscript, the author learned that David Stewart recently obtained a result similar to Theorem B; see [St 13, Theorem 3.3.5]. Stewart assumes that G is a connected linear algebraic group G over an *algebraically closed* field k; he shows (in the language used above) that if G acts on a connected unipotent group U, then the action of G on U is linearly filtered.

1.4. **Notations, assumptions, and conventions.** We write k_{sep} for a separable closure of k, and k_{alg} for an algebraic closure of k_{sep} and hence of k. If $\ell \supset k$ is a field extension, we write $G_{/\ell}$ for the linear algebraic group over ℓ obtained by extension of scalars.

When we speak of a closed subgroup of an algebraic group G, we mean a closed subgroup scheme over k (unless said otherwise, we only consider smooth group schemes); thus the subgroup is required to be "defined over k" in the language of [Sp 98] or [Bo 91]. Similar remarks apply to homomorphisms between linear algebraic groups. We occasionally use the terminology "k-subgroup" or "k-homomorphism" for emphasis.

An action of the algebraic group G on an algebraic group H is always understood to be given by a morphism

$$\alpha : G \times H \to H$$

of varieties; since G and H are smooth group schemes, α will determine an action on H by algebraic group automorphisms provided that the morphism $(h \mapsto \alpha(g, h)) : H_{/k_{\mathrm{alg}}} \to H_{/k_{\mathrm{alg}}}$ is a homomorphism of algebraic groups for each $g \in G(k_{\mathrm{alg}})$.

By a G-module V, we mean a co-module V for the Hopf algebra $k[G]$, where $k[G]$ is the coordinate algebra of G. When V is finite dimensional as a k-vector space, it follows that the action of G on V is determined by a homomorphism of algebraic groups $G \to \mathrm{GL}(V)$ defined over k. In general, since the action of G is determined by a *co-module map* $\Delta : V \to k[G] \otimes V$ [Jan 03, I.2.8], one knows that the action of G on V is *locally finite*: any $v \in V$ is contained in a finite dimensional G-submodule of V.

For a G-module V and a field extension $\ell \supset k$, write $V_{/\ell} = V \otimes_k \ell$ for the $G_{/\ell}$-module obtained by extension of scalars.

We write $V^\vee$ for the dual vector space of the finite dimensional vector space V; if V is a G-module, $V^\vee$ is again a G-module in a natural way.

Suppose that V_1 and V_2 are algebraic groups on which G acts by group automorphisms, and write $\alpha_i : G \times V_i \to V_i$ for the morphisms defining the actions. A homomorphism of algebraic groups $f : V_1 \to V_2$ is G-equivariant provided that

$$(*) \quad f \circ \alpha_1 = \alpha_2 \circ (1_G \times f).$$

Since the group scheme G and the V_i are smooth, it suffices to check that $(*)$ holds on k_{alg}-points of $G \times V_1$, where k_{alg} is an algebraic closure of k. If the V_i are G-modules and f is a linear mapping, we recover the notion of a G-module homomorphism.

2. Representations and p-linear maps

In this section, we collect some results about the representations of a linear algebraic group G to be used in the sequel. As pointed out in the introduction, our applications here involve questions which are mainly of interest only in positive characteristic. Thus, we suppose throughout that the characteristic of k is $p > 0$, though of course with proper reformulation many of the results remain true in characteristic zero.

We collect in §2.1 some results on extension of scalars and complete reducibility of modules for a linear algebraic group. We then discuss in §2.2 the notion of p-linear mappings between vector groups with a chosen linear structure. The main result of §2.2 is Theorem (2.2.7) concerning the behavior of irreducibility of a G-module under p-linear isogenies. Finally, in §2.3 we show by example when k is imperfect that a G-equivariant p-linear isogeny between G-modules need not preserve irreducibility.

The reader mainly interested in the case of algebraically closed k can skip the results in §2.1 and §2.3. The proof of Theorem (2.2.7) is also much simpler in case k is perfect (in particular, when $k = k_{\mathrm{alg}}$).

2.1. **Extension of scalars and completely reducible representations.**

Proposition (2.1.1). *Let V and W be finite dimensional non-zero G-modules.*

(a) If ℓ is any field extension of k and if $\mathrm{Hom}_{G_{/\ell}}(V_{/\ell}, W_{/\ell}) \neq 0$, then $\mathrm{Hom}_G(V, W) \neq 0$.
(b) If $V_{/k_{\mathrm{alg}}}$ and $W_{/k_{\mathrm{alg}}}$ are isomorphic $G_{/k_{\mathrm{alg}}}$-modules, then V and W are isomorphic G-modules.

Proof. Assertion (a) follows from the observation $\mathrm{Hom}_G(V, W) \otimes_k \ell = \mathrm{Hom}_{G_{/\ell}}(V_{/\ell}, W_{/\ell})$ found in [Jan 03, I.2.10(7)] (since ℓ is a flat k-algebra).

For (b), we view the vector space $H = \mathrm{Hom}_G(V, W)$ as an algebraic variety over k isomorphic to affine space $\mathbf{A}^n$ for some n. The subvariety of G-isomorphisms $I = \mathrm{Iso}_G(V, W) \subset H$ is open; this variety has positive dimension by our assumption. Assertion (b) will follow if we argue that the set $I(k)$ of k-points is non-empty. When k is infinite, this follows from the density of the k-points $H(k)$ in H. Now suppose that k is finite, and consider the group scheme $A = \mathrm{Aut}_G(V) = \mathrm{End}_G(V)^\times$. Since A is an open subscheme of the affine space $\mathrm{End}_G(V)$, it is a smooth and connected group scheme over k and hence "is" a connected linear algebraic group over k.

Evidently, I is a torsor over A. According to the Lang-Steinberg theorem, I has a k-rational point, and the assertion follows. □

Proposition (2.1.2). *Let V be a finite dimensional G-module. There is a k-subalgebra $A \subset \mathrm{End}_k(V)$ with the following properties:*

(a) For every field extension ℓ of k and every ℓ-subspace $W \subset V_{/\ell}$, W is a $G_{/\ell}$-submodule of $V_{/\ell}$ if and only if W is an $A_{/\ell}$-submodule of $V_{/\ell}$, where $A_{/\ell} = A \otimes_k \ell$.
(b) The simple A-modules are precisely the composition factors of V (as an A-module, or equivalently as a G-module).
(c) If S and T are composition factors of V, then $\mathrm{Hom}_A(S, T) = \mathrm{Hom}_G(S, T)$.
(d) If **(R)** *holds for G and if S is a simple A-module, then the division algebra $\mathrm{End}_A(S) = \mathrm{End}_G(S)$ has a splitting field which is a finite separable extension of k. In particular, $A/\operatorname{rad}(A)$ is a separable semisimple k-algebra, where $\operatorname{rad}(A)$ denotes the Jacobson radical of A.*

Proof. Write $\rho : G \to \mathrm{GL}(V)$ for the homomorphism of algebraic groups which determines the action of G on V. Let k_{sep} be a separable closure of k, and write $A_1(V) \subset \mathrm{End}_{k_{\mathrm{sep}}}(V_{/k_{\mathrm{sep}}})$ for the k_{sep}-subalgebra generated by the image $\rho(G(k_{\mathrm{sep}}))$.

Write $\Gamma = \mathrm{Gal}(k_{\mathrm{sep}}/k)$ for the absolute Galois group of k. Since ρ and G are defined over k, the subalgebra A_1 is stable under the natural Γ action on $\mathrm{End}_{k_{\mathrm{sep}}}(V_{/k_{\mathrm{sep}}})$. Put $A(V) = A_1(V)^\Gamma$. It follows from Speiser's Lemma [GS 06, Lemma 2.3.8] that the natural mapping $A(V) \otimes_k k_{\mathrm{sep}} \to A_1(V)$ is an isomorphism. We now argue that $A = A(V) \subset \mathrm{End}_k(V) = \mathrm{End}_{k_{\mathrm{sep}}}(V_{/k_{\mathrm{sep}}})^\Gamma$ has the required properties.

(a) follows from the fact – see [Sp 98, Theorem 11.2.7] – that $G(k_{\mathrm{sep}})$ is dense in G. . Indeed, let $\ell \supset k$ be a field extension, let $W \subset V_{/\ell}$ be a subspace, and choose an algebraically closed field L containing ℓ and k_{sep}. Then W is a $G_{/\ell}$-submodule of $V_{/\ell}$ if and only if $W_{/L}$ is stable under the action of $\rho(g)$ for $g \in G(L)$. Since $G(k_{\mathrm{sep}})$ is dense in G, this last condition is equivalent to the stability of $W_{/L}$ under the action of each element of A_1. Since A contains a k_{sep}-basis of A_1, conclude that W is a $G_{/\ell}$-submodule if and only if W is invariant under the action of each element of A, and (a) follows.

Since V is a faithful A-module, (b) follows e.g. from [CR 81, 3.30].

For (c), choose a composition series $0 = C_0 \subset C_1 \subset C_r = V$ for V as G-module, and write $W = \bigoplus_{i=1}^r C_i/C_{i+1}$ for the semisimplification of the G-module V. The morphism $G \to \mathrm{GL}(V)$ has image in the parabolic subgroup $P \subset \mathrm{GL}(V)$ which is the stabilizer of the flag $C_\bullet$, and $\mathrm{GL}(W)$ identifies naturally with the reductive quotient of P. This leads to a natural surjective algebra homomorphism $A(V) \to A(W)$ whose kernel is the Jacobson radical of $A(V)$. Thus, the proof of (c) is reduced to the case of a semisimple G-module V. After replacing V by a suitable quotient, we may further suppose that each simple factor of V occurs with multiplicity one. Now, the density of $G(k_{\mathrm{sep}})$ shows that

$$\mathrm{End}_G(V) = \{X \in \mathrm{End}_K(V) \mid (X \otimes 1_L)g = g(X \otimes 1_L)\ \forall g \in G(k_{\mathrm{sep}})\} = \mathrm{End}_A(V)$$

and since the isotypic components of the G-module V are simple, assertion (c) is immediate.

For (d), write R for the unipotent radical of G. Then the fixed points S^R form a G-submodule of S. Since R is unipotent, $S^R \neq 0$. Since S is simple, conclude that $S = S^R$; i.e. R acts trivially on S. Replacing G by G/R, we may suppose that G is reductive. A result of Tits [Ti 71, Theorem 7.2] now shows that $S_{/k_{\mathrm{sep}}}$ is a semisimple $G_{/k_{\mathrm{sep}}}$-module. On the other hand, it follows from [Sp 98, 13.1.1] that $G_{/k_{\mathrm{sep}}}$ is a split reductive group. According to [Jan 03, II.2], the endomorphism algebra of each simple $G_{/k_{\mathrm{sep}}}$-modules is k_{sep}; thus $\mathrm{End}_{k_{\mathrm{sep}}}(S)$ is a *split* semisimple algebra, and (d) follows. □

Proposition (2.1.3). *Assume that* **(R)** *holds for G, and let V be a G-module. The following are equivalent:*

(a) V is a semisimple G-module
(b) $V_{/\ell} = V \otimes_k \ell$ is a semisimple $G_{/\ell}$-module for some field extension $\ell \supset k$
(c) $V_{/\ell}$ is a semisimple $G_{/\ell}$-module for every field extension $\ell \supset k$

Proof. According to [Jan 03, I.2.13(3)], all G-modules are locally finite, so V is the sum of its finite dimensional G-submodules. In particular, V is semisimple if and only if each finite dimensional G-submodule is semisimple; thus we suppose V to be finite dimensional.

Let A be the algebra determined by the G-module V as in Proposition (2.1.2). For any field extension ℓ of k, that proposition shows that $V_{/\ell}$ is a semisimple $G_{/\ell}$-module if and only if $V_{/\ell}$ is a semisimple $A_{/\ell}$-module; in turn, [CR 81, Prop 3.31] shows that $V_{/\ell}$ is a semisimple $A_{/\ell}$-module if and only if $A_{/\ell}$ is a semisimple ℓ-algebra.

Now the result follows from [Re 03, Theorem 7.18]. □

For a G-module V, write $\mathrm{soc}_G(V)$ for the *socle* of V; namely, $\mathrm{soc}_G(V)$ is the largest semisimple submodule of V. Equivalently, $\mathrm{soc}_G(V)$ is the sum of all simple G-submodules of V.

Proposition (2.1.4). *(a) Let A be a finite dimensional k-algebra, and suppose for each simple A-module L that the division algebra $\mathrm{End}_A(L)$ is split by a finite separable field extension of k. For any field extension ℓ of k and any A-module W, the image of $\mathrm{soc}_A(W)_{/\ell}$ in $W_{/\ell}$ coincides with $\mathrm{soc}_{A_{/\ell}}(W_{/\ell})$.*
(b) Assume that **(R)** *holds for G, and let V be a G-module. For any field extension $k \subset \ell$, the image of $\mathrm{soc}_G(V)_{/\ell}$ in $V_{/\ell}$ coincides with $\mathrm{soc}_{G_{/\ell}}(V_{/\ell})$.*

Proof. Note first that (b) is a consequence of (a). Indeed, since the G-module V is locally finite, it is evidently sufficient to prove the result for finite dimensional V. In that case, let A be the finite dimensional algebra of Proposition (2.1.2) associated to G and V. Since **(R)** holds for G, the endomorphism algebra of each simple A-module has a finite separable splitting field. Moreover, Proposition (2.1.2) shows that the G-socle $\mathrm{soc}_G(V)$ coincides with the A-socle $\mathrm{soc}_A(V)$, and the analogous statement holds after scalar extension. It is now clear that (a) implies (b).

For the proof of (a), write $\mathrm{rad}(A)$ for the Jacobson radical of A. The assumptions show that $A/\mathrm{rad}(A)$ is a separable semisimple algebra. Arguing as in [Re 03, Cor. 7.17], one sees that the image of $\mathrm{rad}(A)_{/\ell}$ in $A_{/\ell}$ coincides with $\mathrm{rad}(A_{/\ell})$.

Write $S = \mathrm{soc}_A(W)$ and $T = \mathrm{soc}_{A_{/\ell}}(W_{/\ell})$. The socle S is equal to the sum of the images of all A-homomorphisms

$$\phi : A/\mathrm{rad}(A) \to W,$$

and the socle T is equal to the sum of the images of all $A_{/\ell}$-homomorphisms

$$\psi : (A_{/\ell})/\mathrm{rad}(A_{/\ell}) \to W_{/\ell},$$

Since $(A_{/\ell})/\mathrm{rad}(A_{/\ell}) \simeq (A/\mathrm{rad}(A))_{/\ell}$, the equality $S_{/\ell} = T$ follows at once. □

2.2. **p-linear maps.** Let V and W be finite dimensional k-vector spaces. We write V_a and W_a for the linear algebraic groups (vector groups) determined by V and W; thus V_a and W_a are vector groups.

If $r \geq 0$ is an integer, an additive homomorphism $f : V \to W$ will be said to be p^r-linear if $f(tv) = t^{p^r} f(v)$ holds for every $t \in k$ and $v \in V$. If $f : V_a \to W_a$ is a homomorphism of algebraic groups and $r > 0$ an integer, we say that f is p^r-linear if $f_\Lambda(tv) = t^{p^r} f_\Lambda(v)$ for all commutative k-algebras Λ, all $t \in \Lambda$, and all $v \in V_a(\Lambda) = V \otimes_k \Lambda$.

Lemma (2.2.1). *The assignment $f \mapsto f_k$ determines a bijection from the collection of all p^r-linear homomorphisms $f : V_a \to W_a$ to the collection of all p^r-linear maps $g : V \to W$.*

Proof. Indeed, a p^r-linear homomorphism $f : V_a \to W_a$ of algebraic groups determines a mapping on k-points $\tilde{f} = f_k : V_a(k) = V \to W_a(k) = W$ with the required property. On the other hand, given a p^r-linear mapping $g : V \to W$ and a commutative k-algebra Λ, define $f_\Lambda : V(\Lambda) = V \otimes_k \Lambda \to W(\Lambda) = W \otimes_k \Lambda$ by the rule

$$f_\Lambda(v \otimes \beta) = g(v) \otimes \beta^{p^r} \quad \text{for } v \in V \text{ and } \beta \in \Lambda;$$

the collection of mappings f_Λ determines the unique p^r-linear homomorphism $f : V_a \to W_a$ with $f_k = g$. □

Proposition (2.2.2). *Let $f : V_a \to W_a$ be a p^r-linear homomorphism of vector groups. Write $\widetilde{f} = f_k$ for the map*

$$\widetilde{f} = f_k : V_a(k) = V \to W_a(k) = W$$

induced by f on k-points.

(a) There is a unique k-vector subspace $X \subset W$ such that the image of f lies in the subgroup $X_a \subset W_a$ and for which $f : V_a \to X_a$ is a surjective p^r-linear morphism of vector groups. Moreover, $f_\ell : V_a(\ell) \to X_a(\ell)$ is surjective for any perfect field ℓ containing k.

(b) $\ker f$ *is a connected group scheme over k, and $\widetilde{f}$ is injective if and only if* $\ker f$ *is finite and infinitesimal.*

Proof. For (a), let $I \subset W$ be the image $\widetilde{f}(V) = \widetilde{f}(V_a(k))$. Since $\widetilde{f}$ is p^r-linear, I is a k^{p^r}-linear subspace of W. Set $X = kI \subset W$; it is clear by construction that f factors through the inclusion $X_a \subset W_a$.

For a field extension $\ell \supset k$, the p^r-linearity of f shows that the image $f_\ell(V_a(\ell))$ coincides with $\ell^{p^r} I$. If ℓ is perfect, $\ell^{p^r} I = \ell I = \ell X = X \otimes_k \ell = X_a(\ell)$ so $f_\ell : V_a(\ell) \to X_a(\ell)$ is surjective when ℓ is perfect.

Since k_{alg} is perfect,

$$f_{k_{\mathrm{alg}}} : V_a(k_{\mathrm{alg}}) \to X_a(k_{\mathrm{alg}})$$

is surjective. Hence the subset $f(V_a)$ inside X_a contains all k_{alg}-points. But $f(V_a)$ is a closed subset of X_a, so by containment of all k_{alg}-points, it must coincide with X_a.

The uniqueness of X follows at once from the fact that $X_a(k_{\mathrm{alg}})$ is uniquely determined as the image of $f_{k_{\mathrm{alg}}}$.

For (b), let $K \subset V = V_a(k)$ be the kernel of $\widetilde{f}$. The p^r-linearity of $\widetilde{f}$ shows that K is a k-linear subspace of V. Let ℓ be any field containing k. The kernel of the mapping f_ℓ induced by f on ℓ-points evidently identifies with $K \otimes_k \ell \subset V \otimes_k \ell = V_a(\ell)$. It follows that the group of ℓ-points $(\ker f)(\ell)$ is equal to $K \otimes_k \ell$. Now [Jan 03, I.8.2] shows that $K = 0$ if and only if $\ker f$ is a finite, infinitesimal group scheme.

In general, K_a is a subgroup scheme of $\ker f$, so f induces a p^r-linear mapping $f_1 : V_a/K_a \to X_a$. Now $\widetilde{f_1}$ is injective, so the group scheme $\ker f_1$ is finite and infinitesimal. In particular, $\ker f_1$ is connected. The connectedness of $\ker f$ now follows since $(\ker f)/K_a \simeq \ker f_1$. □

Proposition (2.2.3). *For a p^r-linear homomorphism of vector groups $f : V_a \to W_a$, the following are equivalent:*

(i) f is an isogeny.
(ii) $f_\ell : V_a(\ell) \to W_a(\ell)$ is a bijective group homomorphism for some perfect field ℓ containing k.
(iii) $f_\ell : V_a(\ell) \to W_a(\ell)$ is a bijective group homomorphism for every perfect field ℓ containing k.

Proof. Suppose (i) holds and let ℓ be a perfect field containing k. By hypothesis, $\ker f$ is a finite infinitesimal group scheme, and f is surjective. The condition on $\ker f$ together with (2.2.2)(b) shows that f_ℓ is injective. Since f is surjective, (2.2.2)(a) shows (in the notation of that Proposition) that $W = X$ and hence that f_ℓ is surjective; thus, (iii) holds.

The implication "(iii) implies (ii)" is clear. Now suppose (ii) to hold for the perfect field extension ℓ of k. Since f_ℓ is a injective, (2.2.2)(b) shows that $\ker f$ is a finite infinitesimal group scheme. It only remains to see that f is surjective. In the notation of (2.2.2)(a), it is sufficient to prove that $W = X$. By hypothesis, f_ℓ is surjective; thus (2.2.2)(a) shows that $W \otimes_k \ell = X \otimes_k \ell$. It then follows for dimension reasons that $W = X$ as required. Thus (i) holds. □

Example (2.2.4). There are (non p^r-linear) homomorphisms $f : V_a \to W_a$ for which the map f_k on k-points has finite kernel, even though f is not an isogeny. For example, let $k = \ell(t)$ be the field of rational functions over a field ℓ of characteristic p, let $V = k^2$, and consider the homomorphism $f : V_a \to \mathbf{G}_a$ given by $f(y,z) = y^p - y - tz^p$. Then the kernel of f_k consists in the elements $(a, 0) \in V$ with $a \in \mathbf{F}_p \subset k$, hence contains p elements. But the kernel of $f : V_a \to \mathbf{G}_a$ is a 1-dimensional *wound* unipotent group – i.e. a non-split k-form of the additive group $\mathbf{G}_a$ – see e.g. [Mc 04, Remark 32] for more details.

Proposition (2.2.5). *Let V, W_1 and W_2 be G-modules, and let $f_i : V \to W_i$ be G-equivariant p^r-linear isogenies for $i = 1, 2$.*

(a) If k is perfect, there is a unique isomorphism of G-modules $\phi : W_1 \to W_2$ with the property that $\phi \circ f_1 = f_2$.
(b) For any k, $W_1 \simeq W_2$ as G-modules.

Proof. When k is perfect, Proposition (2.2.3) implies that the f_i are bijective (on k-points), and in that case ϕ is given by $f_2 \circ {f_1}^{-1}$; this proves (a).

For general k, (a) implies that W_1 and W_2 are isomorphic after extending scalars to some perfect field containing k. Then (b) follows from Proposition (2.1.1). □

Remark (2.2.6). Let V and W are finite dimensional G-modules and let $f : V_a \to W_a$ be a G-equivariant and p^r-linear morphism of vector groups.

(a) The vector subspace $X \subset W$ of (2.2.2)(a) is a G-submodule of W.

Now suppose that k is a *perfect* field.

(b) It follows from Proposition (2.2.3) that the assignment $f \mapsto f_k$ is a bijection from the collection of p^r-linear isogenies $f : V_a \to W_a$ to the collection of bijective p^r-linear homomorphisms $g : V \to W$; we say that $f_k : V \to W$ is G-equivariant provided that $f : V_a \to W_a$ is G-equivariant.
(c) The mapping $X \mapsto f(X)$ is an inclusion-preserving bijection from the set of G-submodules of V to the set of G-submodules of W. In particular, V is a simple, respectively semisimple, G-module if and only if W is a simple, respectively semisimple, G-module.
(d) Suppose that k is a perfect field. For each $r \geq 0$, the Frobenius twist $V^{(r)}$ of V is constructed in [Jan 03, I.9.10]. From the construction of $V^{(r)}$, one has a p^r-linear G-equivariant bijection $F : V \to V^{(r)}$. If the G-equivariant p^r-linear homomorphism $f_k : V \to W$ is bijective, then Proposition (2.2.5) yields a unique isomorphism of G-modules $\phi : V^{(r)} \xrightarrow{\sim} W$ such that $\phi \circ F = f_k$.

Recall that a G-module V is *absolutely simple* if $V_{/\ell}$ is a simple $G_{/\ell}$-module for every field extension ℓ of k; in view of Proposition (2.1.2), V is absolutely simple if and only if $V_{/k_{\mathrm{alg}}}$ is a simple $G_{/k_{\mathrm{alg}}}$-module. We can now give the following crucial result:

Theorem (2.2.7). *Let G be a linear algebraic group for which **(R)** holds, let V and W be finite dimensional G-modules, and let $f : V_a \to W_a$ be a G-equivariant, p^r-linear isogeny. If V is a simple G-module, then W is a semisimple, isotypic G-module. If k is perfect or if V is absolutely simple, then W is a simple G-module.*

Proof. We first argue that W is a semisimple G module. Since **(R)** holds for G, Proposition (2.1.3) shows that the simple module V is semisimple after extending scalars to an algebraic closure. When k is algebraically closed, Remark (2.2.6)(c) shows that f_k determines a bijection between G-submodules of V and those of W; thus W is semisimple after extending scalars to an algebraic closure. Now the semisimplicity of W as a G module follows for any k from another application of Proposition (2.1.3).

If k is perfect, Remark (2.2.6)(c) shows that W is a simple G-module. Now suppose that V is absolutely simple, so that V remains simple after extending scalars to a perfect field ℓ. Then W is simple after scalar extension to ℓ, hence W is already a simple G module (over k).

It remains to establish that the semisimple G-module W is isotypic when k is arbitrary. We may find a finite Galois extension $\ell \supset k$ for which $V_{/\ell}$ is isomorphic to a direct sum of absolutely irreducible $G_{/\ell}$-modules. Now write

$$V_{/\ell} \simeq \bigoplus_{i=1}^{e} V_i \qquad \text{and} \qquad W = \bigoplus_{j=1}^{f} W_j$$

where the V_i are the isotypic components of $V_{/\ell}$ as a $G_{/\ell}$-module, and the W_j are the isotypic components of W as a G-module.

For a simple $G_{/\ell}$-submodule $S \subset V_{/\ell}$, it follows from Remark (2.2.6)(a) and (c) that there is a G-submodule $S' \subset W_{/\ell}$ such that the restriction of $f_{/\ell}$ to S is a $G_{/\ell}$-equivariant p^r-linear isogeny $S_a \to S'_a$. Since S is an absolutely simple $G_{/\ell}$-module, we have already established that S' is also an absolutely simple $G_{/\ell}$-module. In particular, $S' \subset W_{j_0/\ell}$ for some j_0. Since $\mathrm{Hom}_G(W_s, W_t) = 0$ if $s \neq t$, it follows from Proposition (2.1.1) that $W_{s/\ell}$ and $W_{t/\ell}$ have no isomorphic composition factors for any field extension ℓ. Thus j_0 is uniquely determined by S.

This reasoning shows that each isotypic component V_i of $V_{/\ell}$ is mapped to the $G_{/\ell}$-submodule $W_{j(i)/\ell}$ for some $1 \leq j(i) \leq f$. Consider the $G_{/\ell}$-submodule $V' \subset V_{/\ell}$ given by the sum of all those V_i for which $j(i) = 1$. Since W_1 is invariant under the action of Γ, also V' is invariant under the action of Γ. By Speiser's Lemma [GS 06, Lemma 2.3.8] the natural mapping $(V')^\Gamma \otimes_k \ell \xrightarrow{\sim} V'$ is an isomorphism. Thus $(V')^\Gamma$ is a G-submodule of V. Since V is a simple G-module, conclude that $V = (V')^\Gamma$ and hence $V_{/\ell} = V'$. Since f is surjective, we find that $W = W_1$, so indeed W is an isotypic G-module. □

2.3. **Reducible images of simple modules.** In this section, we are going to show by example that the conclusion of Theorem (2.2.7) can't be improved. More precisely, we are going to prove the following result:

(2.3.1). *There exists an imperfect field k, a linear algebraic group G over k, G-modules L, M, and a G-equivariant p-linear isogeny $f : L_a \to M_a$ with L a simple G-module, and M a (semisimple and isotypic but) reducible G-module.*

Remark (2.3.2). The referee pointed out the following example. Suppose that k is algebraically closed, that G is connected and reductive, and write G_1 for the *first Frobenius kernel* of G. Let L be a non-trivial *restricted* simple G-module, so that L is also simple as a module for G_1 [Jan 03, II.3.15]. Let $L^{(1)}$ be the first Frobenius twist of L, and let $F : L \to L^{(1)}$ be the G-equivariant p-linear bijection as in Remark (2.2.6)(d). Then G_1 acts trivially on $L^{(1)}$ so $L^{(1)}$ is a (semisimple and isotypic but) reducible G_1-module; thus the conclusion of (2.3.1) holds for the (non-smooth) group scheme G_1.

Proof of (2.3.1) for a linear algebraic group over k: Let D be a central division algebra over k, and suppose that the order of the class of D in the Brauer group $\mathrm{Br}(k)$ is p. Such division algebras exists. For example, let $k = \mathbf{F}_p((t))$ be the field of formal Laurent series over the finite field $\mathbf{F}_p$ with p elements, and let D be the central k-division algebra having Hasse invariant $1/p + \mathbf{Z} \in \mathbf{Q}/\mathbf{Z}$; see e.g. [GS 06, 6.3.9 and 6.3.10].

In general, it follows from [Jac 96, Theorem 4.1.2] together with the assumption on the order of the class of D in $\mathrm{Br}(k)$ that

$$E = k \otimes_{\sigma,k} D \simeq \mathrm{Mat}_{p\times p}(k)$$

is a split simple k algebra, where the tensor product is taken with respect to the Frobenius mapping $\sigma : k \to k$ given by $\sigma(x) = x^p$.

Lemma (2.3.3). *Suppose that L is a simple G-module, and that $\mathrm{End}_G(L) \simeq D$. There is a G-module M and a G-equivariant p-linear isogeny $\phi : L \to M$ for which M is not simple.*

Proof. Consider the G-module $M = k \otimes_{\sigma,k} L$ together with the mapping $\phi : L \to M$ given by $\phi(x) = 1 \otimes x$. Then ϕ determines a G-equivariant p-linear isogeny $\phi : L_a \to M_a$ by Proposition (2.2.3).

We may view M as an $E = k \otimes_{\sigma,k} D$-module in a natural way, and the action of E commutes with the action of G. Since E contains zero divisors, Schur's Lemma shows that M is a reducible G-module. □

In view of Lemma (2.3.3), the proof of (2.3.1) will be completed by the following lemma.

Lemma (2.3.4). *Let D be a finite dimensional central division algebra over k. Then there is a connected and reductive algebraic group G over k and a simple G-module L with $\mathrm{End}_G(L) \simeq D$.*

Proof. Let $E = D^{\mathrm{opp}}$ be the opposite algebra to the division algebra D, and let $G = \mathrm{GL}_{1,E}$ be the unit group scheme of E; thus G is a reductive algebraic group over k which is a k-form of the group GL_p. Moreover, the action of E on itself by left multiplication makes $L = E$ into a simple G-module for which $\mathrm{End}_G(L) \simeq D$. □

Remark (2.3.5). If the Brauer group $\mathrm{Br}(k)$ contains an element of order p, then k is imperfect; this follows from [Jac 96, Theorem 4.1.8].

3. Additive functions on a vector group

Throughout this section, G denotes a linear algebraic group over k for which condition **(R)** holds, and U denotes a vector group over k on which G acts by automorphisms of algebraic groups.

3.1. The action of an algebraic group on a vector group.

The action of the linear algebraic group G on the vector group U determines a linear representation of G on the Lie algebra $\mathrm{Lie}(U)$.

Definition (3.1.1). The action of G on U is said to be *linear* if there is a G-equivariant isomorphism of algebraic groups $U \simeq \mathrm{Lie}(U)_a$.

An important example is given by a finite dimensional G-module V, then G acts on the vector group V_a, and that action is linear.

Remark (3.1.2). (a) Suppose that the action of G on the vector group U is linear. One might prefer to say instead that the *choice* of G-equivariant isomorphism $U \simeq \mathrm{Lie}(U)_a$ determines a *linear structure* on U fixed by G; in general, there are many such choices.
(b) There are vector groups with non-linear G actions – i.e. for which there is no linear structure on U fixed by G; we give examples in §5 below.

Consider the ring of k-endomorphisms of $\mathbf{G}_a$; as in [Sp 98, 3.3.1], this ring may be canonically identified with the "twisted polynomial ring"

$$\mathscr{R} = \mathscr{R}_k = k\langle\tau\rangle$$

where $\tau : k \to k$ is the Frobenius endomorphism $\tau(x) = x^p$. The ring $\mathscr{R}$ has a left k-basis consisting of the elements $\tau^i, i \geq 0$, and we have the commutation formula $\tau a = a^p \tau$ for $a \in k$.

Following [Sp 98, §3.3], for any linear algebraic group U, we write $\mathscr{A}(U)$ for the collection of all homomorphisms of algebraic groups $f : U \to \mathbf{G}_a$ over k. We view $\mathscr{A}(U)$ as a vector subspace of $k[U]$. Composition of additive functions defines a natural action of the ring $\mathscr{R}$ on $\mathscr{A}(U)$, making $\mathscr{A}(U)$ a (left) $\mathscr{R}$-module.

For $j \geq 0$ we write $\mathscr{A}^j(U)$ for the $\mathscr{R}$-module $\mathscr{R}\tau^j\mathscr{A}(U)$. We have the following description of the R-modules $\mathscr{A}^j(U)$.

Lemma (3.1.3). *Let d be the dimension of the vector group U. For $j \geq 0$, $\mathscr{A}^j(U)$ is a free $\mathscr{R}$-module of rank d for each $j \geq 0$. If $T_1, \dots, T_d \in \mathscr{A}(U)$ is an $\mathscr{R}$-basis of $\mathscr{A}(U)$, then $T_1^{p^j}, \dots, T_d^{p^j}$ is an $\mathscr{R}$-basis of $\mathscr{A}^j(U)$.*

Proof. Since $U \simeq \mathbf{G}_a^d$, the description of $\mathscr{A}(U)$ follows from [Sp 98, 3.3.5]. Since $\mathscr{A}^j(U) = \mathscr{R}\tau^j\mathscr{A}(U)$, it is readily verified that the elements $\tau^j T_i = {T_i}^{p^j}$ for $1 \leq i \leq d$ form an $\mathscr{R}$-basis for $\mathscr{A}^j(U)$. ☐

According to (3.1.3), for $j \geq 0$, $\mathscr{A}^j(U)$ is spanned as k-vector space by elements f^{p^j} for $f \in \mathscr{A}(U)$. Since G acts by algebra automorphisms on $k[U]$, it follows that $\mathscr{A}^j(U)$ is a G-submodule of $\mathscr{A}(U)$.

Write Ω_U for the module of k-differentials on U; thus the $k[U]$-module Ω_U is equipped with a derivation $(f \mapsto df) : k[U] \to \Omega_U$ as in [Sp 98, §4.2]. Write $\Omega_U(0) = \Omega_U/\mathfrak{m}_0\Omega_U$ for the fiber at the k-point $0 \in U(k)$ (our notation follows that of [Sp 98, §4.3.1]). Here, $\mathfrak{m}_0$ denotes the maximal ideal of the ring $k[U]$ corresponding to the k-point 0 of U.

Lemma (3.1.4). *(a) The mapping $f \mapsto df$ induces an isomorphism*

$$\mathscr{A}(U)/\mathscr{A}^1(U) \xrightarrow{\sim} \Omega_U(0) = \Omega_U/\mathfrak{m}_0\Omega_U.$$

In particular, $\mathscr{A}(U)/\mathscr{A}^1(U) \simeq \mathrm{Lie}(U)^\vee$.
(b) If the group G acts on U, the isomorphism of (a) is G equivariant.

Proof. The first assertion of (a) follows at once from the description of $\mathscr{A}(U)$ and $\mathscr{A}^1(U)$ found in Lemma (3.1.3). Now use that $\Omega_U(0) \simeq \mathfrak{m}_0/\mathfrak{m}_0^2 \simeq \mathrm{Lie}(U)^\vee$; cf. [Sp 98, 4.1.4]. Assertion (b) is immediate from the definitions. ☐

Proposition (3.1.5). *Let U and V be vector groups and let $\phi : U \to V$ be a homomorphism of algebraic groups. Upon restriction to $\mathscr{A}(V)$, the comorphism $\phi^* : k[V] \to k[U]$ induces a homomorphism of $\mathscr{R}$-modules*

$$\phi^*{}_{|\mathscr{A}(V)} : \mathscr{A}(V) \to \mathscr{A}(U).$$

Moreover, ϕ is an isomorphism of algebraic groups if and only if $\phi^*{}_{|\mathscr{A}(V)}$ is an isomorphism of $\mathscr{R}$-modules.

Proof. A straightforward verification shows that ${\phi^*}_{|\mathscr{A}(V)}$ maps $\mathscr{A}(V)$ to $\mathscr{A}(U)$ and commutes with the action of the ring $\mathscr{R}$.

We now prove the remaining assertion. If $\phi : U \to V$ is an isomorphism, then $\phi^* : k[V] \to k[U]$ is an isomorphism of algebras; thus ${\phi^*}_{|\mathscr{A}(V)}$ is an isomorphism of $\mathscr{R}$-modules.

Suppose on the other hand that ${\phi^*}_{|\mathscr{A}(V)}$ is an isomorphism of $\mathscr{R}$-modules. We argue that the comorphism $\phi^* : k[V] \to k[U]$ is an isomorphism of algebras. Well, since $\mathscr{A}(U)$ generates $k[U]$ as k-algebra, evidently $\phi^*(k[V]) = k[U]$ so that ϕ^* is onto. By its $\mathscr{R}$-linearity, ${\phi^*}_{|\mathscr{A}(V)}$ induces an isomorphism $\mathscr{A}(V)/\mathscr{A}^1(V) \to \mathscr{A}(U)/\mathscr{A}^1(U)$; in view of Lemma (3.1.4)(a), it follows that the mapping $(\Omega_V)(0) \to (\Omega_U)(0)$ induced by ϕ is a linear isomorphism; taking duals, this means that the tangent mapping to ϕ at 0 is an isomorphism. Thus the morphism ϕ is dominant [Sp 98, 4.3.6] so that its comorphism ϕ^* is injective. This shows that ϕ^* – and hence ϕ – is an isomorphism, as required. $\square$

Proposition (3.1.6). *Let V be a G-module, so that V_a is a vector group with an action of G. Given a G-equivariant homomorphism $f : U \to V_a$, the comorphism f^* determines by restriction a homomorphism of G-modules*

$$f^*_{|V^\vee} : V^\vee \to \mathscr{A}(U).$$

This assignment determines a bijection between G-equivariant homomorphisms of algebraic groups $U \to V_a$ over k and homomorphisms of G-modules $V^\vee \to \mathscr{A}(U)$.

Proof. The assignment $f \mapsto f^*$ determines a bijection between the collection of all morphisms $U \to V_a$ of k-varieties and the set $\mathrm{Hom}_{k-\mathrm{alg}}(k[V_a], k[U])$. By the universal property of the symmetric algebra $k[V_a] = \mathrm{Sym}(V^\vee)$, the restriction mapping $g \mapsto g_{|V^\vee}$ determines a bijection $\mathrm{Hom}_{k-\mathrm{alg}}(k[V_a], k[U]) \xrightarrow{\sim} \mathrm{Hom}_k(V^\vee, k[U])$.

Let $\lambda : V^\vee \to k[U]$ be k-linear, and write $f_\lambda : U \to V_a$ for the morphism to which it corresponds under the above bijections. It is straightforward to see that f_λ is a group homomorphism if and only if the image of λ is contained in $\mathscr{A}(U)$, and that f_λ is G-equivariant if and only if λ is a G-module homomorphism. The result follows at once. $\square$

3.2. **A condition for linearity.** We keep the notation and assumptions of § 3.1. We begin by studying the filtration

$$\mathscr{A}(U) \supset \mathscr{A}^1(U) \supset \mathscr{A}^2(U) \supset \cdots$$

of $\mathscr{A}(U)$ by G-submodules.

Proposition (3.2.1). *(a)* $\bigcap_{i\geq 1} \mathscr{A}^i(U) = \{0\}$.
(b) For each $i, r \geq 1$, multiplication by τ^r defines a G-equivariant p^r-linear isogeny

$$(\mathscr{A}(U)/\mathscr{A}^i(U))_a \to (\mathscr{A}^r(U)/\mathscr{A}^{i+r}(U))_a.$$

Proof. Assertion (a) follows from the evident observation that $\bigcap_{i\geq 1} \mathscr{R}\tau^i = \{0\}$ in $\mathscr{R}$. .

Recall that $\mathscr{A}(U)$ is a G- and $\mathscr{R}$- submodule of the coordinate ring $k[U]$. Left multiplication by τ^r on $k[U]$ coincides with the p^r-th power mapping $f \mapsto \tau^r \cdot f = f^{p^r}$. Since $k[U]$ is an integral domain, this mapping is injective. Since the action of G on $k[U]$ preserves the algebra structure, multiplication by τ^r is G-equivariant. Now (b) follows from Proposition (2.2.3). $\square$

Proposition (3.2.2). *Suppose that $V \subset \mathscr{A}(U)$ is a G-submodule for which $V \cap \mathscr{A}^1(U) = 0$. Consider the G-equivariant homomorphism of algebraic groups $\phi : U \to (V^\vee)_a$ corresponding via Proposition (3.1.6) to the inclusion $V \to \mathscr{A}(U)$.*
(a) ϕ is a separable surjection; i.e. ϕ is surjective and $d\phi : \mathrm{Lie}(U) \to \mathrm{Lie}((V^\vee)_a) = V^\vee$ is surjective.
(b) If $\mathscr{A}(U) = V + \mathscr{A}^1(U)$, then $V^\vee \simeq \mathrm{Lie}(U)$ and $d\phi$ is an isomorphism.

Proof. For (a), it suffices to argue that $d\phi$ is surjective. Dualizing, it is the same to argue that the mapping

$$\phi^* : \Omega_{(V^\vee)_a}(0) \to \Omega_U(0)$$

induced by the comorphism ϕ^* on the fibers at zero of the respective modules of differentials is injective.

But Lemma (3.1.4) gives natural identifications

$$\Omega_{(V^\vee)_a}(0) = \mathscr{A}((V^\vee)_a)/\mathscr{A}^1((V^\vee)_a) = \mathrm{Lie}((V^\vee)_a)^\vee = V \quad \text{and} \quad \Omega_U(0) = \mathscr{A}(U)/\mathscr{A}^1(U);$$

under these identifications, the induced mapping $\phi^* : \Omega_{(V^\vee)_a}(0) \to \Omega_U(0)$ identifies with the composite

$$V \to \mathscr{A}(U) \to \mathscr{A}(U)/\mathscr{A}^1(U),$$

which is injective by hypothesis.

For (b), we have by hypothesis an isomorphism $V \simeq \mathscr{A}(U)/\mathscr{A}^1(U)$ so indeed $V^\vee \simeq \mathrm{Lie}(U)$ by (3.1.4). Thus $\dim U = \dim V_a$ so that the surjective linear mapping $d\phi$ is an isomorphism for dimension reasons. □

Proposition (3.2.3). *Suppose that* $\dim U = 1$. *Then the action of G on U is linear.*

Proof. According to (3.1.3), $\mathscr{A}(U)$ is an $\mathscr{R}$-module of rank 1. Fix an R-basis element $f \in \mathscr{A}(U)$. Since the units $\mathscr{R}^\times$ of $\mathscr{R}$ coincide with the scalars $k^\times$, any other $\mathscr{R}$-basis element of $\mathscr{A}(U)$ is a $k^\times$-multiple of f. Since G acts by automorphisms on U, it follows that $V = kf$ is a (one dimensional) G-submodule of $\mathscr{A}(U)$, and is a complement to $\mathscr{A}^1(U)$ in $\mathscr{A}(U)$. It is now easy to see that the mapping $\phi : U \to V^\vee$ of Proposition (3.2.2) is an isomorphism (indeed, $k[U]$ is the polynomial ring $k[f]$, the comorphism ϕ^* is injective, and the image of ϕ^* evidently contains f). □

Theorem (3.2.4). *Assume **(R)** holds for the linear algebraic group G, and suppose that G acts by group automorphisms on the vector group U. If $\mathrm{Lie}(U)$ is a simple module for G, then $\mathscr{A}(U)$ is a semisimple G-module. In particular, there is a G-submodule $V \subset \mathscr{A}(U)$ for which $\mathscr{A}(U) = V + \mathscr{A}^1(U)$ and $\mathscr{A}^1(U) \cap V = \{0\}$.*

Proof. In the proof, we are going to abbreviate $\mathscr{A}^i = \mathscr{A}^i(U)$ for $i \geq 0$. In view of Proposition (3.2.1)(a), the Theorem will follow if show that the quotient G-module $\mathscr{A}/\mathscr{A}^i$ is semisimple for each $i \geq 1$. We proceed by induction on i; when $i = 1$, $\mathscr{A}/\mathscr{A}^1$ is isomorphic to the simple G-module $\mathrm{Lie}(U)^\vee$. Now let $i > 1$ and suppose that the G-module $\mathscr{A}/\mathscr{A}^{i-1}$ is known to be semisimple. There is a short exact sequence of G-modules

$$(\flat) \quad 0 \to \mathscr{A}^1/\mathscr{A}^i \to \mathscr{A}/\mathscr{A}^i \xrightarrow{\psi} \mathscr{A}/\mathscr{A}^1 \to 0.$$

According to Proposition (3.2.1)(b), multiplication by τ determines a G-equivariant p-linear isogeny

$$\mathscr{A}/\mathscr{A}^{i-1} \to \mathscr{A}^1/\mathscr{A}^i.$$

The G-module $\mathscr{A}/\mathscr{A}^{i-1}$ is semisimple by the induction hypothesis, and it now follows from Theorem (2.2.7) that $\mathscr{A}^1/\mathscr{A}^i$ is semisimple. Since $\mathscr{A}/\mathscr{A}^1$ is a simple G-module, the semisimplicity of $\mathscr{A}/\mathscr{A}^i$ will follow if we argue that $(\flat)$ is split exact.

Since G-modules are locally finite, we may choose a simple G-submodule $L \subset \mathscr{A}(U)$. In view of Proposition (3.2.1)(a) and the simplicity of L, there is $r \geq 0$ for which $L \subset \mathscr{A}^r$ and $L \cap \mathscr{A}^{r+1} = 0$.

The image of L is contained in the G-socle of $\mathscr{A}^r/\mathscr{A}^{r+i}$. It follows that

$$\mathscr{A}^r/\mathscr{A}^{r+i} = \mathscr{A}^{r+1}/\mathscr{A}^{r+i} + \mathrm{soc}(\mathscr{A}^r/\mathscr{A}^{r+i}).$$

By Proposition (3.2.1)(b), multiplication by τ^r defines a G-equivariant p^r-linear isogeny

$$\mathscr{A}/\mathscr{A}^i \to \mathscr{A}^r/\mathscr{A}^{r+i}.$$

It now follows from Remark (2.2.6)(c) that

$$\mathscr{A}/\mathscr{A}^i = \mathscr{A}^1/\mathscr{A}^i + \mathrm{soc}(\mathscr{A}/\mathscr{A}^i).$$

Since $\mathscr{A}/\mathscr{A}^1$ is simple, it follows that the restriction of ψ to $\mathrm{soc}(\mathscr{A}/\mathscr{A}^i)$ is surjective; thus there is an exact sequence of G-modules

$$(\sharp) \quad \mathrm{soc}(\mathscr{A}/\mathscr{A}^i) \to \mathscr{A}/\mathscr{A}^1 \to 0.$$

Since $\mathrm{soc}(\mathscr{A}/\mathscr{A}^i)$ is semisimple, $(\sharp)$ is split; moreover, a choice of a splitting for the sequence $(\sharp)$ splits the sequence $(\flat)$, as required. □

Suppose that H is a linear algebraic group over k and that V is a semisimple H-module. Given a simple H-module L, write $V_{(L)}$ for the L-isotypic component of V; since V is semisimple, we have

$$V = \bigoplus_L V_{(L)},$$

the sum being taken over a system of isomorphism classes of simple H-modules L.

Lemma (3.2.5). *Let H be a linear algebraic group, let V and W be semisimple H-modules for which $V \simeq W$ as H-modules. Suppose that $\dim V_{(L)} = \dim W_{(L)} < \infty$ for each simple H-module L. If $\Phi : V \to W$ is an H-module that is injective, then Φ is an isomorphism.*

Proof. Indeed, Φ gives by restriction an H-module homomorphism $\Phi_{(L)} : V_{(L)} \to W_{(L)}$ for each simple module L; since Φ is injective and since $V_{(L)} \simeq W_{(L)}$, the finite dimensionality shows that $\Phi_{(L)}$ is an isomorphism for each L, hence Φ is an isomorphism. □

Theorem (3.2.6). *Assume **(R)** holds for the linear algebraic group G, that G acts by group automorphisms on the vector group U, and that $\operatorname{Lie}(U)$ – and hence also $\mathscr{A}(U)/\mathscr{A}^1(U)$ – is a simple module for the identity component G^0 of G. Then the action of G on U is linear.*

Proof. Apply Theorem (3.2.4) to learn that $\mathscr{A}(U)$ is a semisimple G-module. In particular, choose a G-submodule $V \subset \mathscr{A}(U)$ for which $\mathscr{A}(U) = V + \mathscr{A}^1(U)$ and $V \cap \mathscr{A}^1(U) = 0$. Write $\phi : U \to (V^\vee)_a$ for the G-equivariant homomorphism of algebraic groups corresponding as in Proposition (3.2.2) to the inclusion mapping $V \to \mathscr{A}(U)$.

We will argue that $\phi : U \to (V^\vee)_a$ is an isomorphism of algebraic groups. We first observe that the unipotent radical R of G acts trivially on U. Indeed, since $\mathscr{A}(U)$ is a semisimple G-module, R acts trivially on $\mathscr{A}(U)$. Since $\mathscr{A}(U)$ generates the k-algebra $k[U]$, R acts trivially on $k[U]$ and hence on U. Replacing G by G/R, we now suppose G to be connected and reductive.

According to Proposition (3.2.3), we may suppose that $\dim U > 1$. According to Proposition (3.1.5) it suffices to prove that

$$\Phi = (\phi^*)_{|\mathscr{A}(V^\vee)} : \mathscr{A}(V^\vee) \to \mathscr{A}(U)$$

is an isomorphism of G-modules. Proposition (3.2.2) show that the tangent mapping $d\phi$ is an isomorphism. In particular, ϕ is dominant - hence surjective - so the comorphism $\phi^* : k[V^\vee] \to k[U]$ is injective. We conclude that $\Phi = (\phi^*)_{|\mathscr{A}(V^\vee)}$ is an injective homomorphism of G-modules.

Now, $\mathscr{A}(U)/\mathscr{A}^1(U) \simeq \mathscr{A}(V^\vee)/\mathscr{A}^1(V^\vee)$ as G-modules. For each $i \geq 0$ it follows from Proposition (3.2.1)(b) and Remark (2.2.6)(d) that the $G_{/k_{\text{alg}}}$-modules

$$(\mathscr{A}^i(U)/\mathscr{A}^{i+1}(U))_{/k_{\text{alg}}} \quad \text{and} \quad (\mathscr{A}^i(V^\vee)/\mathscr{A}^{i+1}(V^\vee))_{/k_{\text{alg}}}$$

are both isomorphic to the i-th Frobenius twist $((\mathscr{A}(U)/\mathscr{A}^1(U))_{/k_{\text{alg}}})^{(i)}$. It then follows from (2.1.1)(b) that

$$\mathscr{A}^i(U)/\mathscr{A}^{i+1}(U)) \simeq \mathscr{A}^i(V^\vee)/\mathscr{A}^{i+1}(V^\vee)$$

as G-modules.

According to Theorem (2.2.7), both $\mathscr{A}(U)$ and $\mathscr{A}(V^\vee)$ are semisimple G-modules. In view of that semisimplicity, Proposition (3.2.1)(a) implies that $\mathscr{A}(U)$ is G-isomorphic to the direct sum of all $\mathscr{A}^i(U)/\mathscr{A}^{i+1}(U)$ and that a similar statement holds for $\mathscr{A}(V^\vee)$. In particular, $\mathscr{A}(U) \simeq \mathscr{A}((V^\vee)_a)$ as G-modules.

To show that Φ is an isomorphism, we are going to apply Lemma (3.2.5) to the linear algebraic group $H = G^0$ with $V = \mathscr{A}(U)$ and $W = \mathscr{A}((V^\vee)_a)$. Since Φ is injective, that Lemma will show Φ to be an isomorphism provided we argue that

($\sharp$) each isotypic component of V is a finite dimensional G^0-module.

Fix a simple G^0-module L. Then ($\sharp$) will follow if we show that

($\sharp\sharp$) there is $N = N(L) \geq 0$ such that $\operatorname{Hom}_{G^0}(L, \mathscr{A}^i(U)/\mathscr{A}^{i+1}(U)) = 0$ for $i \geq N$.

According to Proposition (2.1.3), both $L_{/k_{\text{alg}}}$ and $\operatorname{Lie}(U)_{/k_{\text{alg}}}$ are semisimple $G^0_{/k_{\text{alg}}}$-modules. Write

$$\operatorname{Lie}(U)^\vee_{/k_{\text{alg}}} \simeq L_1 \oplus \cdots \oplus L_d \quad \text{and} \quad L_{/k_{\text{alg}}} \simeq S_1 \oplus \cdots \oplus S_e$$

for non-trivial simple $G^0_{/k_{\text{alg}}}$-modules L_i and S_j. Using Proposition (2.1.1), we see that ($\sharp\sharp$) will follow if we show for each $1 \leq j \leq e$ that

($\sharp\sharp\sharp$) there is $M = M(j) \geq 0$ such that $\operatorname{Hom}_{G_{/k_{\text{alg}}}}(S_j, (\mathscr{A}^i(U)/\mathscr{A}^{i+1}(U))_{/k_{\text{alg}}}) = 0$ for $i \geq M$.

Now

$$(\mathscr{A}^i(U)/\mathscr{A}^{i+1}(U))_{/k_{\text{alg}}} \simeq ((\mathscr{A}(U)/\mathscr{A}^1(U))_{/k_{\text{alg}}})^{(i)} \simeq L_1^{(i)} \oplus \cdots \oplus L_d^{(i)}.$$

Since $\dim U > 1$, $\mathrm{Lie}(U)$ is a non-trivial simple module for G^0. Thus $\mathrm{Lie}(U)^\vee \simeq \mathscr{A}(U)/\mathscr{A}^1(U)$ has no G^0-fixed points. In particular, the simple $G_{/k_{\mathrm{alg}}}$-modules L_i are non-trivial. Since G^0 is connected and reductive, [Jan 03, Prop. II.3.16 and II.2.7] shows for $1 \le \ell \le d$ that the G^0-modules $\{L_\ell^{(i)} \mid i \ge 0\}$ are simple and pairwise non-isomorphic. Now ($\sharp\sharp\sharp$) follows at once. Thus Φ is an isomorphism and the proof of the Theorem is complete. □

4. Linear filtrations for connected, split unipotent groups

Throughout this section, G denotes a *connected* linear algebraic group for which condition **(R)** holds. In this section, we consider a split unipotent group U over k on which G acts by group automorphisms.

Recall from § 1.3 that the action of G on U is said to be linearly filtered if there is a filtration of U

$$U = U^0 \supset U^1 \supset \cdots \supset U^r \supset U^{r+1} = 0$$

by G-invariant closed subgroups U^i such that for each $0 \le i \le r$, U^i/U^{i+1} is a vector group with a linear action of G.

4.1. ***G*-complete reducibility and linear filtrations.** We begin the discussion of linear filtrations with an example illustrating the importance of this notion [2].

Recall that a linear algebraic group M over a field k is said to be *linearly reductive* provided that each of its linear representations is completely reducible. It follows from [DG 70, IV §3.3.6] that this notion is geometric; i.e. that M is linearly reductive if and only if $M_{/K}$ is linearly reductive for each field extension $k \subset K$.

Recall that Serre has defined the notion of G-complete reducibility: a subgroup H of a reductive group G is G-completely reducible over k provided that for each k-parabolic subgroup P of G which contains M, there is a Levi factor L of P (defined over k) which contains M.

Proposition (4.1.1). *If the linearly reductive k-group M is a subgroup of a reductive group G over k, then M is G-completely reducible over k.*

Proof. Fix a parabolic subgroup P of G containing M. It follows from [SGA 3, XXVI §2 Prop. 2.1] that the action of P on the unipotent radical R_uP is linearly filtered in the sense above. Thus, the action of M on R_uP is linearly filtered. If $\pi : P \to P/R_uP$ is the quotient mapping, put $\widetilde{M} = \pi^{-1}\pi(M)$. Then R_uP is the unipotent radical of $\widetilde{M}$. The group $\widetilde{M}$ is evidently the semidirect product of M and R_uP – i.e. M is a Levi factor of $\widetilde{M}$.

Choose a Levi factor L of P, let $\tau : L \to P/R_u$ be an isomorphism, and let $M' = \tau^{-1}\pi(M)$. Then M' is a second Levi factor of $\widetilde{M}$. The complete reducibility of linear H-modules implies that $H^1(M, V) = 0$ for every linear M-representation V. It now follows from [Mc 10, Thm 5.1] that M and M' are conjugate by a k-rational element $x \in R_u(P)(k)$ – i.e. $M' = xMx^{-1}$. But then M is contained in the k-Levi subgroup $x^{-1}Lx$ of P, which shows that indeed M is G-completely reducible. □

When k is algebraically closed, the preceding proposition is Lemma 2.6 of [BMR 05].

4.2. **Construction of a linear filtration of a vector group with *G*-action.** Suppose that U is a positive dimensional vector group on which G acts by group automorphisms.

Lemma (4.2.1). *There is a simple G-module L and a separable and surjective G-equivariant homomorphism $U \to L_a$ of algebraic groups.*

Proof. Write $\mathscr{A} = \mathscr{A}(U)$. We first claim that $\mathrm{soc}\,\mathscr{A} \not\subset \mathscr{A}^1$. In view of Proposition (2.1.4), it suffices to prove this claim after extending scalars to an algebraic closure; thus for the time being we suppose $k = k_{\mathrm{alg}}$. In that case, multiplication by $\tau^r \in \mathscr{R}$ defines a p^r-linear bijective mapping $\sigma^r : \mathscr{A} \to \mathscr{A}^r$; since G acts on $k[U]$ by algebra automorphisms, evidently the restriction of σ^r to any G-submodule $V \subset \mathscr{A}$ defines a G-equivariant p^r-linear bijection $\sigma^r_{|V} : V \to \sigma^r V$. Since k is perfect, it follows from (2.2.6) that σ^r induces an inclusion preserving bijection between G-submodules of $\mathscr{A}$ and G-submodules of $\mathscr{A}^r$.

[2] I thank Brian Conrad, Cyril DeMarche, Sebastian Herpel, and David Stewart for some discussions concerning this example. Especially, I thank DeMarche for pointing out the reference to SGA3 treating the case where G is not split over k

Since G-modules are locally finite, any non-zero G-module has a non-zero socle. In particular, $\operatorname{soc}\mathscr{A} \neq 0$ so we may choose a simple submodule $L \subset \mathscr{A}$. In view of Proposition (3.2.1)(a) and the simplicity of L, evidently $L \subset \mathscr{A}^r$ and $L \cap \mathscr{A}^{r+1} = 0$ for a suitable integer $r \geq 0$. Thus, $\operatorname{soc}(\mathscr{A}^r) \not\subset \mathscr{A}^{r+1}$. Since σ^r determines an inclusion preserving bijection between the G-submodules of $\mathscr{A}$ and of $\mathscr{A}^r$, it indeed follows that $\operatorname{soc}(\mathscr{A}) \not\subset \mathscr{A}^1$.

Having proved the claim, we return to the original setting; in particular, k is now arbitrary. Since $\operatorname{soc}\mathscr{A} \not\subset \mathscr{A}^1$, we may choose a simple G-submodule $L \subset \mathscr{A}$ such that $L \cap \mathscr{A}^1 = 0$. It then follows from (3.2.2)(a) that there is a G-equivariant, separable, surjective homomorphism of algebraic groups $\phi : U \to (L^\vee)_a$, as required. □

Lemma (4.2.2). *(a) There is a positive dimensional, closed, G-invariant subgroup $W \subset U$ and a G-equivariant isomorphism $W \simeq \operatorname{Lie}(W)_a$ of algebraic groups.*
(b) The action of G on U is linearly filtered.

Proof. To prove (a), we proceed by induction on the composition length n of $\operatorname{Lie}(U)$ as a G-module. First suppose that $n = 1$, so that $\operatorname{Lie}(U)$ is a simple G-module. Since G is connected, Theorem (3.2.6) implies that indeed $U \simeq \operatorname{Lie}(U)_a$.

Now suppose $n > 1$ and that the result is known for vector groups V with G-action for which the composition length of the G-module $\operatorname{Lie}(V)$ is strictly less than n. Use Lemma (4.2.1) to find a separable and surjective G-equivariant homomorphism of algebraic groups $\phi : U \to L_a$ for a simple G-module L. Since $d\phi$ is surjective, the kernel of ϕ is a *smooth* group scheme over K; thus the identity component V of $\ker\phi$ is a vector group on which G acts. There is a short exact sequence of G-modules

$$0 \to \operatorname{Lie}(V) \to \operatorname{Lie}(U) \to \operatorname{Lie}(L_a) = L \to 0.$$

In particular, the composition length of the G-module $\operatorname{Lie}(V)$ is $n-1$, so by induction V – and hence U – contains a closed G-invariant positive dimensional subgroup W for which there is a G-equivariant isomorphism $W \simeq \operatorname{Lie}(W)_a$ of algebraic groups. Now (b) follows from (a) by induction on the dimension of U. □

4.3. **Linear filtrations of split unipotent groups.** We apply the results of the previous section to study connected, split unipotent groups over k. We obtain the following Theorem:

Theorem (4.3.1). *Let G be a connected linear algebraic group over k for which* **(R)** *holds. Let U be a connected, split, unipotent group over k and suppose that G acts by group automorphisms on U. Then the action of G on U is* linearly filtered.

Proof. According to [Sp 98, Exerc. 14.3.12(2) and (3)] the derived subgroup (U, U) is a connected, split unipotent group over k, and the quotient $U/(U, U)$ is a connected, commutative, split, unipotent group over k. The subgroup (U, U) is characteristic – in particular, it is invariant under the the action of G. If the conclusion of the Theorem holds for (U, U) and for $U/(U, U)$, it clearly holds for U.

By induction on the dimension of U, we are thus reduced to the case where U is commutative. In that case, let $U^{(p)}$ be the subgroup generated by p-th powers. $U^{(p)}$ is again characteristic, hence invariant under the action of G. Moreover, [Sp 98, Exerc. 14.3.12(2) and (3)] again implies that $U^{(p)}$ and $U/U^{(p)}$ are connected, split, commutative unipotent groups. If the conclusion of the Theorem holds for $U^{(p)}$ and $U/U^{(p)}$, it holds for U. Observe that the $U/U^{(p)}$ has exponent p, so that according to [CGP 10, Theorem B.2.5] the group $U/U^{(p)}$ is a vector group.

By another induction on the dimension of U, it is enough to give the proof when U is a vector group. The result in that case has already been established in Lemma (4.2.2)(b). □

5. Examples of non-linear actions

Let G be a linear algebraic group over k. In this section, we give examples of vector groups U having an action of G by group automorphisms which is not linear – i.e. for which there is no G-equivariant isomorphism $\operatorname{Lie}(U)_a \to U$. For simplicity, *we assume that k is algebraically closed*. In particular, in this section we often write G for the group of k-points $G(k)$. And if V is a vector space, we often identify the corresponding vector group V_a with V.

5.1. **Linear extensions of G-modules.** Suppose that the G-module E is an extension of the G-module V by the G-module W; i.e. there is a short exact sequence of G-module

$$(*) \quad 0 \to W \to E \to V \to 0.$$

Extensions of the form $(*)$ are parametrized by a cohomology group, namely

$$\mathrm{Ext}^1_G(V,W) \simeq H^1(G, \mathrm{Hom}_k(V,W))$$

Recall that the action $g \star \phi$ of an element $g \in G$ on an element $\phi \in \mathrm{Hom}_k(V,W)$ is given by rule

$$g \star \phi = g \circ \phi \circ g^{-1} (= \rho_V(g) \circ \phi \circ \rho_W(g^{-1}))$$

where in the parenthetical formulation $\rho_V : G \to \mathrm{GL}(V)$ and $\rho_W : G \to \mathrm{GL}(W)$ are the homomorphisms defining the G-modules. Using the above identification, elements in $\mathrm{Ext}^1_G(V,W)$ may be viewed as equivalence classes of 1-cocycles $G \to \mathrm{Hom}_k(V,W)$.

For convenience, we are going to reformulate this cocycle description of $\mathrm{Ext}^1_G(V,W)$. Let us write

$$Z^1 = Z^1(G, \mathrm{Hom}_k(V,W)) = \{\sigma \mid \sigma_{gh} = g \star \sigma_h + \sigma_g \quad \text{for } g,h \in G\}.$$

where $\sigma : G \to \mathrm{Hom}_k(V,W)$ denotes a *regular mapping* denoted by $(g \mapsto \sigma_g)$.

Similarly, write

$$\begin{aligned} Z^1_0 = Z^1_0(G, \mathrm{Hom}_k(V,W)) &= \{\tau \mid \tau_{gh} = g \circ \tau_h + \tau_g \circ h \quad \text{for } g,h \in G\} \\ &= \{\tau \mid \tau_{gh} = \rho_W(g) \circ \tau_h + \tau_g \circ \rho_V(h) \quad \text{for } g,h \in G\}. \end{aligned}$$

where again $\tau : G \to \mathrm{Hom}_k(V,W)$ denotes a regular mapping denoted by $(g \mapsto \tau_g)$.

Since $\mathrm{Hom}_k(V,W)$ is a k-vector space, also Z^1 and Z^1_0 both have a natural structure of k-vector space. Given $\phi \in \mathrm{Hom}_k(V,W)$, define two regular functions $G \to \mathrm{Hom}_k(V,W)$ by the rules

$$\partial(\phi)_g = g \star \phi - \phi \qquad \text{and} \qquad \partial_0(\phi)_g = g \circ \phi - \phi \circ g = \rho_W(g) \circ \phi - \phi \circ \rho_V(g)$$

for $g \in G$. Of course, ∂ is the usual cohomology boundary mapping, so $\partial\phi \in Z^1$. One checks that $\partial_0\phi \in Z^1_0$, so these rules determine linear mappings

$$\partial : \mathrm{Hom}_k(V,W) \to Z^1 \qquad \text{and} \qquad \partial_0 : \mathrm{Hom}_k(V,W) \to Z^1_0.$$

(5.1.1). *The mapping $\iota : Z^1 \to Z^1_0$ defined by $\sigma \mapsto (g \mapsto \sigma_g \circ g = \sigma_g \circ \rho_W(g))$ determines an isomorphism*

$$H^1(G, \mathrm{Hom}_k(V,W)) = Z^1 / \operatorname{im} \partial \xrightarrow{\sim} Z^1_0 / \operatorname{im} \partial_0.$$

Sketch. Of course, $H^1(G, \mathrm{Hom}_k(V,W)) = Z^1 / \operatorname{im}\partial$ is the *definition* of Hochschild cohomology. Now one checks for $\sigma \in Z^1$ that the rule $\iota(\sigma)$ defined by $g \mapsto \sigma_g \circ g$ determines an element of Z^1_0. Since ι is evidently invertible, the result follows upon observing that $\iota \circ \partial = \partial_0$. □

Given a short exact sequence $(*)$, choose a linear section $s : V \to E$ to the projection $\pi : E \to V$. Using s, one forms the regular function

$$\tau : G \to \mathrm{Hom}_k(V,W) \quad \text{via} \quad g \mapsto \tau_g = g \circ s - s \circ g = \rho_W(g) \circ s - s \circ \rho_V(g),$$

and one readily checks that $\tau \in Z^1_0$. Conversely, given $\tau \in Z^1_0(G, \mathrm{Hom}_k(V,W))$, one constructs a corresponding extension

$$0 \to W \to E_\tau \to V \to 0$$

where $E_\tau = V \times W$ as varieties, with G-action given by

$$g. \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} gv \\ gw + \tau_g(v) \end{pmatrix} \quad \text{for } g \in G(k), v \in V \text{ and } w \in W.$$

Another linear section $s' : V \to E$ has the form $s + \phi$ for $\phi \in \mathrm{Hom}_k(V,W)$. And the extension $(*)$ is split if and only if $s + \phi \in \mathrm{Hom}_G(V,W)$ for some ϕ. If $\tau \in Z^1_0$ is constructed from the section s, the section $s + \phi$ is G-linear if and only if $\tau = \partial_0\phi$.

This correspondence describes the bijection between elements of $H^1(G, \mathrm{Hom}_k(V,W)) \simeq \mathrm{Ext}^1_G(V,W)$ and isomorphism classes of extensions $(*)$.

5.2. **Non-linear extensions.** Now, consider G-modules V and W and fix a cocycle

$$\tau \in Z^1_0(G, \mathrm{Hom}_k(V,W))$$

as in §5.1.

Write $F : W \to W^{(1)}$ for the G-equivariant p-linear bijection of Remark (2.2.6)(d) and write $\mathrm{Hom}_p(V, W^{(1)})$ for the group of all p-linear mappings $V \to W^{(1)}$.

There is a regular function

$$\tilde{\tau} = F_* \circ \tau : G \xrightarrow{\tau} \mathrm{Hom}_k(V,W) \xrightarrow{F_*} \mathrm{Hom}_p(V, W^{(1)})$$

where $F_*(f) = F \circ f$. Evidently $\tilde{\tau}$ satisfies the cocycle condition

$$(**) \quad \tilde{\tau}_{gh} = \rho_{W^{(1)}}(g) \circ \tilde{\tau}_h + \tilde{\tau}_g \circ \rho_V(h).$$

We use $\tilde{\tau}$ to define a vector group $\widetilde{E} = E_{\tilde{\tau}}$ with an action of G as follows. The underlying vector group is $\widetilde{E} = V \times (W^{(1)})$ and the action of an element $g \in G = G(k)$ on an element of $\widetilde{E}(k) = V \times W^{(1)}$ is given by the rule

$$g. \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} gv \\ gw + \tilde{\tau}_g(v) \end{pmatrix} = \begin{pmatrix} gv \\ gw + F(\tau_g(v)) \end{pmatrix} \quad \text{for } v \in V \text{ and } w \in W^{(1)};$$

using the cocycle condition $(**)$, one now checks that this rule defines an action of G on $\tilde{E}$ by group automorphisms.

Lemma (5.2.1). *There is an isomorphism of G-modules* $\mathrm{Lie}(\widetilde{E}) \simeq V \oplus W^{(1)}$.

Proof. Indeed, since V and $W^{(1)}$ are linear representations of G, there are G-equivariant isomorphisms $V \simeq \mathrm{Lie}(V)$ and $W^{(1)} \simeq \mathrm{Lie}(W^{(1)})$. Thus $\mathrm{Lie}(\widetilde{E})$ is a (linear) extension of the G-module V by the G-module $W^{(1)}$.

Write $s : V \to \widetilde{E}$ for the section $v \mapsto \begin{pmatrix} v \\ 0 \end{pmatrix}$ so that $\tilde{\tau}_g = g \circ s - s \circ g$. Then $ds : V = \mathrm{Lie}(V) \to \mathrm{Lie}(\widetilde{E})$ is a linear section, so by the rule described in §5.1, the extension $\mathrm{Lie}(\widetilde{E})$ of V by $W^{(1)}$ is determined by the cocycle σ for which $\sigma_g = g \circ ds - ds \circ g = d\tilde{\tau}_g = d(F \circ \tau_g)$.

But the tangent mapping of $F : W \to W^{(1)}$ is identically zero; thus $\sigma(g)$ is zero for each $g \in G(k)$. It follows that $\mathrm{Lie}(\widetilde{E})$ is the trivial extension of V by $W^{(1)}$. $\square$

We are going to argue that in general the action of G on $\widetilde{E}$ is not linear. If V is a finite dimensional k-vector space viewed as a vector group, write $\mathscr{A}(V) \subset k[V]$ for the space of additive functions on V as in §3. Recall that $k[V] = \mathrm{Sym}(V^\vee)$ is the symmetric algebra on the vector space dual to V. If $k[V] = \bigoplus_{m \geq 0} k[V]_m$ is the grading for which $V^\vee = k[V]_1$, write for each $m \geq 1$

$$\mathscr{A}(V)_m = \mathscr{A}(V) \cap k[V]_m.$$

We require the following result.

Lemma (5.2.2). *Let V and W be finite dimensional k-vector spaces, and let $\phi : V \to W$ be a homomorphism of vector groups.*

(a) $\mathscr{A}(V)_m \neq 0$ if and only if $m = p^r$ for some r. Moreover, $\mathscr{A}(V)_1 = V^\vee$ and $\mathscr{A}(V)_{p^r} = (V^\vee)^{p^r}$ where the p^r-th powers are taken in the algebra $k[V]$.

(b) $\mathscr{A}(V) = \bigoplus_{r \geq 0} \mathscr{A}(V)_{p^r}$.

(c) If $\lambda : W^\vee \to \mathscr{A}(V)$ is the linear map determined by ϕ as in Proposition (3.1.6), then ϕ is p-linear if and only if the image of λ lies in $\mathscr{A}(V)_p$.

Proof. (a) and (b) follow from the description found in Lemma (3.1.3). Note that in (a), the equality $\mathscr{A}(V)_{p^r} = (V^\vee)^{p^r}$ holds since k is algebraically closed.

For (c), first use (b) to write $\lambda = \sum_{r \geq 0} \lambda_r$ where $\lambda_r : W^\vee \to \mathscr{A}(V)_{p^r}$ is a linear map for $r \geq 0$. Write $\phi^* : k[W] \to k[V]$ for the comorphism of ϕ. The grading in (a), respectively the analogous grading for $k[W]$, is determined by the action of $\mathbf{G}_m$ on $k[V]$, respectively on $k[W]$, obtained from the scalar action of $\mathbf{G}_m$ on V, respectively W. Thus, ϕ is p-linear if and only if for each $i \geq 0$, ϕ^* maps $k[W]_i$ to $k[V]_{pi}$ if and only if $\lambda = \lambda_1$, and (c) follows. $\square$

We now prove:

Proposition (5.2.3). *Let $\tau \in Z^1_0 = Z^1_0(G, \mathrm{Hom}_k(V, W))$ and construct the vector group $\tilde{E}$ with G-action using the regular function $\tilde{\tau}$ as above. If the cohomology class $[\tau] \in Z^1_0/\operatorname{im}\partial_0 \simeq H^1(G, \mathrm{Hom}_k(V, W))$ is non-zero, there is no G-equivariant isomorphism between* $\mathrm{Lie}(\tilde{E})$ *and* $\tilde{E}$.

Proof. In view of Lemma (5.2.1), it suffices to prove that if the class $[\tau]$ is non-zero, there is no G-equivariant homomorphism $V \to \tilde{E}$ which is a section to the quotient mapping $\pi : \tilde{E} \to V$. In fact, we are going to prove the equivalent assertion that if there is a G-equivariant section to π, then $[\tau] = 0$.

In the explicit description of $\tilde{E}$ given above, let us fix the section $s : V \to \tilde{E}$ given by

$$v \mapsto \begin{pmatrix} v \\ 0 \end{pmatrix} \quad \text{for } v \in V.$$

Then any homomorphism $V \to \tilde{E}$ of algebraic groups which is a section to π has the form

$$F_\phi : v \mapsto \begin{pmatrix} v \\ \phi(v) \end{pmatrix}$$

for some homomorphism of algebraic groups $\phi : V \to W^{(1)}$.

A calculation shows that the homomorphism $F_\phi : V \to \tilde{E}$ is G-equivariant if and only if

$$\tilde{\tau}_g = \rho_{W^{(1)}}(g) \circ \phi - \phi \circ \rho_V(g) = g \circ \phi - \phi \circ g$$

for each $g \in G$. So let us fix ϕ such that F_ϕ is G-equivariant. Note for $g \in G$, $t \in k^\times$ and $v \in V$ that

$$\tilde{\tau}_g(tv) = t^p \tilde{\tau}_g(v),$$

and therefore

$$g\phi(tv) - \phi(gtv) = t^p(g\phi(v) - \phi(gv));$$

in other words, $g \circ \phi - \phi \circ g$ is p-linear for each $g \in G$.

According to Proposition (3.1.6) ϕ is uniquely determined by a linear map $\lambda : (W^{(1)})^\vee \to \mathscr{A}(V)$. In view of Lemma (5.2.2)(a), we may write $\lambda = \sum_{r \geq 0} \lambda_r$ where λ_r is a linear mapping $(W^{(1)})^\vee \to \mathscr{A}(V)_{p^r}$ (and $\lambda_r = 0$ for all but finitely many r).

Applying Lemma (5.2.2)(b) shows that the p-linear homomorphism $g \circ \phi - \phi \circ g : V \to W^{(1)}$ is uniquely determined by a linear mapping $\mu : (W^{(1)})^\vee \to \mathscr{A}(V)_p = (V^\vee)^p$.

We thus deduce that

$$\mu = \sum_{r \geq 0} (g \circ \lambda_r - \lambda_r \circ g)$$

so that

$$\mu = g \circ \lambda_1 - \lambda_1 \circ g \quad \text{and} \quad g \circ \lambda_r - \lambda_r \circ g = 0 \quad \text{for } r \neq 1.$$

According to Lemma (5.2.2)(b), the linear map λ_1 determines a p-linear map $\Lambda : V \to W^{(1)}$. It follows at once that there is a linear map $\Gamma : V \to W$ such that $\Lambda = F \circ \Gamma$.

Now observe for $g \in G$ that the group homomorphism $F \circ \tau_g = \tilde{\tau}_g : V \to W^{(1)}$ satisfies

$$F \circ \tau_g = g \circ \Lambda - \Lambda \circ g = g \circ F \circ \Gamma - F \circ \Gamma \circ g = F \circ (g \circ \Gamma - \Gamma \circ g),$$

where we have invoked the G-equivariance of the mapping F. Since F is bijective, deduce that $\tau_g = g \circ \Gamma - \Gamma \circ g$ for each $g \in G$. Since k is algebraically closed, this proves that τ is the trivial co-cycle, as required. □

Remark (5.2.4). Note that for G-modules V and W, the cohomology group $H^1(G, \mathrm{Hom}_k(V, W))$ may be identified with $\mathrm{Ext}^1_G(V, W)$. Thus, according to Proposition (5.2.3), any non-split extension of G-modules gives rise to a vector group U with G-action for which there is no G-equivariant isomorphism between U and $\mathrm{Lie}(U)$.

There are *many* non-split extensions of G-modules. If G is a reductive group with maximal torus T contained in the Borel group B, let $\lambda \in X^*(T)$ be a dominant weight. Let $H^0(\lambda)$ be the standard module determined by λ as in [Jan 03, §II.2]. Then $H^0(\lambda)$ has a unique simple submodule $L(\lambda)$ [Jan 03, §II.2] and hence there is a short exact sequence of G-modules

$$(\clubsuit) \quad 0 \to L(\lambda) \to H^0(\lambda) \to C \to 0;$$

this short exact sequence is non-split if and only if $C \neq 0$, i.e. if and only if $H^0(\lambda)$ is not simple.

To give an explicit example, let $G = \mathrm{SL}(V)$. In that case, [Jan 03, II.2.16] shows that $V = H^0(\omega) = L(\omega)$ for a fundamental dominant weight ω, and that moreover $H^0(p\omega) \simeq \mathrm{Sym}^p(V)$ is the p-th symmetric power of V. On the other hand, $L(p\omega) \simeq V^{(1)}$ is the first Frobenius twist of V. Since $\dim V^{(1)} = \dim V < \dim \mathrm{Sym}^p(V)$, one sees that $H^0(p\omega)$ is not simple so that (♣) is not split. In this case, our construction gives a non-linear extension of $C = \mathrm{Sym}^p(V)/V^{(1)}$ by $L(p\omega)^{(1)} = L(p^2\omega) = V^{(2)}$.

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