ERRATA FOR "THE SECOND COHOMOLOGY FOR SMALL IRREDUCIBLE MODULES FOR SIMPLE ALGEBRAIC GROUPS"

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Let G be a quasisimple, simply connected algebraic group over the algebraically closed field of characterisctic $p > 0$.

The main result of (McNinch 2002) – namely, Theorem B from the introduction of that paper (which is Theorem 5.4 in the body of the paper) – is incorrect when $p \le 3$. Here is a correct formulation of that result

Theorem 1 (Corrected version of Theorem B/Theorem 5.4)**.** *Let $p > 2$. If L is an irreducible G-module with* $\dim L \le p$ *and if $H^2(G, L) \neq 0$, then either*

(a) $p = 3$ and $G = \mathrm{SL}_3$, or

(b) $L \simeq \mathfrak{g}^{[d]}$ for some $d \ge 1$.

Here is a more precise formulation of the error(s) that led to the incorrect formulation of this main theorem:

- Prop. 4.2(2) isn't correct when $p = 2, 3$; see Proposition 2 below.
- Lemma 5.3(2) is not correct. In fact, the Lemma contains statements (1) and (2); the given proof correctly confirms (1) and claims – incorrectly – that the proof of (2) is "the same". See Lemma 3 below.
- The errors in Prop. 4.2(2) and Lemma 5.3(2) require a reformulation of Theorem 5.3; see Theorem 5 below.

We are going to describe corrections for these errors, and then we will sketch the proof of Theorem 1.

Before doing so, let me acknowledge that in a sequence of emails dated February-April 2009, David Stewart pointed out the error(s) just mentioned, and he communicated corrections to me. Morever, he provided examples showing that as a result of the two errors just mentioned, the conclusion of the main Theorem of the paper sometimes fails when $p = 2$ or $p = 3$

Here is a correct formulation of Prop. 4.2(2):

Proposition 2. *If $p \ge h \ge 3$ then either $H^2(G_1, k)^{[-1]} \simeq \mathrm{Lie}(G)^\vee$ or $p = 3$ and G has root system of type A_2.*

Sketch. The corresponding result (McNinch 2002, Prop. 4.2(2)) is deduced from (Andersen and Jantzen 1984, Cor. 6.3). In fact, the result in *loc. cit.* implies the result for $p > 3$, and indeed yields a more complicated description of $H^2(G_1, k)$ when $p = 3$ and G has type A_2 which I originally overlooked. For a more explicit argument confirming the claim of this Proposition, see (Bendel, Nakano, and Pillen 2007, Theorem 6.2) $\square$

Part (2) of lemma 5.3 should be replaced by the following:

Lemma 3. *Let $E_2^{p,q} \implies H^{p+q}$ be a convergent first quadrant spectral sequence. Assume that*

$$E_2^{1,0} = E_2^{1,1} = E_2^{2,0} = E_2^{2,1} = E_2^{3,0} = 0.$$

Then $H^2 \simeq E_2^{0,2}$.

Date: November 2016.

Remark 4. The changes from the original formulation are the stipulations that the terms $E_2^{2,1}$ and $E_2^{3,0}$ vanish.

Proof. The result follows if we argue that $E_\infty^{0,2} \simeq E_2^{0,2}$. Of course, for $a \geq 2$, $E_{a+1}^{0,2}$ is the cohomology of the sequence

$$E_a^{-a,1+a} \to E_a^{0,2} \to E_a^{a,3-a}.$$

Since E_2 is zero outside the first quadrant, it will be enough to argue that $E_a^{a,3-a} = 0$ for all $a \geq 2$. This is immediate for $a \geq 4$. When $a = 2$ we have assumed that $E_2^{2,1} = 0$.

Finally, when $a = 3$, first note that we have assume $E_2^{3,0} = 0$. Now, $E_3^{3,0}$ is the cohomology of the sequence

$$E_2^{1,1} \to E_2^{3,0} \to E_2^{5,-1};$$

since $E_2^{1,1} = 0$ by assumption, it follows that $E_3^{3,0} = E_2^{3,0} = 0$. This completes the proof. □

Now, (McNinch 2002, Prop. 4.2(2) and Lemma 5.3) are used in the proof of Theorem 5.3. We must reformulate the statement of that Theorem as follows:

Theorem 5 (Corrected version of Theorem 5.3)**.** *Let $p > 2$ and suppose also that $p \geq h$. Let V be a G-module for which $H^i(G, V) = 0$ for $i = 1, 2$, and let $d \geq 1$. Then:*

(a) $H^1(G, V^{[d]}) = 0$;

(b) if $p > h$, then $H^2(G, V^{[d]}) = \mathrm{Hom}_G(\mathfrak{g}, V)$;

(c) if $p = h$ and $\dim V \leq p$, then either $H^2(G, V^{[d]}) = 0$ or $p = 3$, G has root system A_2 and $\dim V = 3$.

Remark 6. The changes from (McNinch 2002) are the stipulation that $p > 2$ and the requirement that $p > h$ in (b). Assertion (c) is new.

Sketch of proof of Theorem 5. We are going to sketch the outline of the proof in order to point out where new arguments are required.

Recall that we write $E_2^{p,q} = H^p(G, H^q(G, V^{[d]})^{[-1]})$ for the terms of the "second page" of the Lyndon-Hochschild-Serre spectral sequence for the normal subgroup scheme G_1 – the first Frobenius kernel – of G.

The proof of (a) proceeds precisely as in (McNinch 2002).

We now proceed to prove (b) and (c). As noted in the introduction to (McNinch 2002), the hypothesis "$\dim V \leq p$" implies that V is a semisimple G-representation; see (Jantzen 1997). Thus in the remainder of proof of the Theorem, we may and will suppose V to be irreducible.

As in the original proof, since $p > 2$ we have $H^1(G_1, k) = 0$ by (McNinch 2002, Prop. 4.2(1)); since $d > 0$, it follows that (♣) $E_2^{n,1} = 0$ for all $n \geq 0$.

In particular, $E_\infty^{1,1} = E_2^{1,1} = 0$, so the only possible non-zero E_∞-terms of total degree 2 are precisely $E_\infty^{0,2}$ and $E_\infty^{2,0}$.

If $d > 1$, proceed as in the original proof (McNinch 2002, p.468); one gets

$$H^2(G, V^{[d]}) \simeq E_2^{2,0} \simeq H^2(G, V^{[d-1]}),$$

thus reducing the proof of (b) and (c) to the case $d = 1$.

Now suppose that $d = 1$. Then $E_2^{2,0} = 0$ by our assumption on V. Note that (a) shows $E_2^{1,0} = 0$. Moreover $E_2^{1,1} = E_2^{2,1} = 0$ by (♣).

We now prove (b), so we suppose that $p > h$. This assumption excludes the situation in which $p = 3$ and the root system of G is of type A_2; thus Proposition 2 gives a G-module isomorphism $H^2(G_1, k) \simeq \mathfrak{g}^\vee$.

Now suppose that

$$0 \neq E_2^{0,2} = \mathrm{Hom}_G(\mathfrak{g}, V).$$

Since $p > h$, one knows that $\mathfrak{g}$ is an irreducible G-module; see e.g. the proof of (McNinch 2002, Lemma 4.1.B). Since V is irreducible, there is a G-isomorphism $V \simeq \mathfrak{g} \simeq H^0(\widetilde{\alpha})$. In particular, $H^3(G, V) = 0$

by (McNinch 2002, Prop. 3.4(c)) so that $E_2^{3,0} = 0$. Now the hypotheses of Lemma 3 are verified; that Lemma shows that $H^2(G, V^{[1]}) = E_2^{0,2}$ as required.

Finally, we prove (c). Since $\dim V \leq p$ and $p = h$, we find using (McNinch 2002, Prop. 5.1) that the root system of G is of type A_{p-1} and V is a Frobenius twist of $L(\varpi_1)$ or $L(\varpi_{p-1})$. If $p > 3$, then Proposition 2 gives an isomorphism $H^2(G_1, k) \simeq \mathfrak{g}^\vee$. Thus $E_2^{0,2} \neq 0 \implies \mathrm{Hom}_G(\mathfrak{g}, V) \neq 0$. But the adjoint representation $\mathfrak{g}$ has length two; it has a trivial composition factor and a composition factor $L(\widetilde{\alpha})$ of dimension $p^2 - 2$. Since V is irreducible of dimension p, deduce that $\mathrm{Hom}_G(\mathfrak{g}, V) = 0$. This shows that $E_2^{0,2} = 0$ and hence $H^2(G, V^{[1]}) = E_\infty^{0,2} = 0$, as required. □

Sketch of proof of Theorem 1. As in the given proof of (McNinch 2002, Theorem 5.4), write L' for a simple G-module for which $(L')^{[d]} \simeq L$ for $d \geq 0$ and such that $\mathrm{Lie}(G)$ acts non-trivially on L'. Recall that $p \geq h$ by (McNinch 2002, Prop. 5.1), and that $H^i(G, L') = 0$ for $i \geq 0$ by (McNinch 2002, Prop 5.2). We are finished if $d = 0$. If $d > 1$ then Theorem 5(b) applies. If $p > h$, that result shows that $H^2(G, L) = \mathrm{Hom}_G(\mathfrak{g}, L')$ which proves the required result in this case.

If now $p = h$, Theorem 5(c) applies. It shows that $H^2(G, L) = 0$ unless $p = 3$ and G has root system of type A_2, which completes the proof. □

Remark 7. David Stewart observed that there are actually counter-examples to the original formulation of the main Theorem of (McNinch 2002) when $p = 2$ and when $p = 3$. For the $p = 2$ examples, the interested reader is referred to Stewarts paper (Stewart 2010).

Let $G = \mathrm{SL}_3 = \mathrm{SL}(V)$ and suppose that $p = 3$. Stewart pointed out to me that $H^2(G, L^{[d]}) \neq 0$ for $d \geq 1$ where L is either the "natural" 3-dimensional G-module V or its dual $V^\vee$.

References

Andersen, Henning Haahr and Jens Carsten Jantzen (1984). "Cohomology of induced representations for algebraic groups". In: *Math. Ann.* 269.4, pp. 487–525.

Bendel, Christopher P., Daniel K. Nakano, and Cornelius Pillen (2007). "Second cohomology groups for Frobenius kernels and related structures". In: *Adv. Math.* 209.1, pp. 162–197.

Jantzen, Jens Carsten (1997). "Low-dimensional representations of reductive groups are semisimple". In: *Algebraic groups and Lie groups*. Vol. 9. Austral. Math. Soc. Lect. Ser. Cambridge Univ. Press, Cambridge, pp. 255–266.

McNinch, George (2002). "The second cohomology of small irreducible modules for simple algebraic groups". In: *Pacific J. Math.* 204.2, pp. 459–472.

Stewart, David I. (2010). "The second cohomology of simple SL_2-modules". In: *Proc. Amer. Math. Soc.* 138.2, pp. 427–434.

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