CORRECTION: REDUCTIVE SUBGROUP SCHEMES OF A PARAHORIC GROUP SCHEME

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In the published version of the manuscript (McNinch 2020), the proof of Theorem 4.5.6 is flawed and should be corrected as described below. I would like to thank Anis Zidani for bringing the flawed proof to my attention, and for critical comments on some preliminary attempts to fix the error.

1. Some reformulations

First of all, Definition 4.5.3 should be replaced with:

Definition A. *Assume that G splits over an unramified extension of* K. *In this setting, a smooth affine $\mathscr{A}$-group scheme $\mathscr{P}$ with $\mathscr{P}_{\mathrm{K}} = G$ is a parahoric group scheme if there is an unramified extension* L *of* K *with ring of integers $\mathscr{B}$ such that $\mathscr{P}_{\mathscr{B}}$ identifies with a parahoric $\mathscr{B}$-group scheme constructed as in §4.3 for the split group G_{L}.*

Before proceeding further, we pause to give a useful characterization of parahoric group schemes under our assumptions. Choose an unramified splitting field L for G which is finite and Galois over K, let $\mathscr{B}$ be the ring of integers of L, and let $\Gamma = \mathrm{Gal}(\mathrm{L}/\mathrm{K})$ be the Galois group.

Proposition B. *Let $\mathscr{P}$ be a parahoric group scheme. Then there is a maximal* K*-torus T and an finite, unramified galois extension* L *of* K *with the following properties:*

i) T_{L} is split,
ii) there is a Chevalley system for G_{L} and a Γ-invariant facet F in $V = X_*(T) \otimes \mathbf{Q}$ such that $\mathscr{Q} = \mathscr{P}_{\mathscr{B}}$ is the parahoric group scheme constructed from $x \in F$ as in §4.3

Proof. It follows from (Bruhat and Tits 1984), 5.2.6 that the parahoric group scheme $\mathscr{Q} = \mathscr{P}_{/\mathscr{B}}$ is determined by a Γ-stable facet F in the Bruhat-Tits building of G_{L} for any apartment containing F, as described by the construction of §4.3

According to (Kaletha and Prasad 2023) Prop 4.2.10 F is a facet of a *special* K-apartment A of the building of G_{L}. This means that – after possibly replacing L by a larger unramified galois extension of K – there is a maximal K-torus T of G such that T_{L} is split and A is the apartment of the building of G_{L} determined by T_{L}. Choose a Chevalley system for G_{L} and use it to identify the apartment A with $V = X_*(T_{\mathrm{L}}) \otimes \mathbf{Q}$; thus F identifies with a facet in V. Now the parahoric subgroup scheme $\mathscr{Q}$ is determined as in §4.3 by a point x in the facet F of V, and the parahoric group scheme $\mathscr{P}$ arises via étale descent from $\mathscr{Q}$. $\square$

Remark 4.5.4 should be *deleted*. The third and fourth sentences of this remark are erroneous (and play no role in the remainder of the argumentation).

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2. Invariance of the construction of §4.3

Recall that $\mathscr{A}$ is a complete discrete valuate ring with field of fractions K and residue field k. Suppose that G is a split reductive group over K. Let T be a K-split maximal torus and let $\mathscr{P}$ be a parahoric group scheme for G constructed as in §4.3 from a choice of Chevalley system and the corresponding schematic root datum $\mathscr{D}_x$ determined by a point $x \in V = X_*(T) \otimes \mathbf{Q}$. Write $\mathscr{T} \subset \mathscr{P}$ for the split $\mathscr{A}$-torus with $\mathscr{T}_{\mathrm{K}} = T$; recall that we may identify $X_*(\mathscr{T})$ with $X_*(T)$ and with $X_*(\mathscr{T}_{\mathrm{k}})$.

In §4.4 we have constructed a subgroup scheme $\mathscr{M}$ of $\mathscr{P}$ with the following properties:

R1. $\mathscr{M}$ is locally closed in $\mathscr{P}$ and is smooth over $\mathscr{A}$.
R2. $\mathscr{M}$ is a split reductive $\mathscr{A}$-group scheme.
R3. $\mathscr{M}_{\mathrm{k}}$ is a Levi factor of $\mathscr{P}_{\mathrm{k}}$.

Now, the point x determines a μ-homomorphism $\phi_x : \mu \to \mathscr{T}$ and we have

R4. $\mathscr{M}$ is the identity component of the centralizer $C_x = C_{\mathscr{P}}(\phi_x)$.

It follows from (Bruhat and Tits 1984), I.2.6 that

R5. $\mathscr{C}_x$ is the schematic closure in $\mathscr{P}$ of $C_x = C_G(\phi_{x/\mathrm{K}})$.

Write $\Phi \subset X^*(T)$ for the root system of G and for $\alpha \in \Phi$ write U_α for the corresponding root subgroup of G. It follows from Theorem 3.4.6 that

R6. $C_x = \langle T, U_\alpha \mid \alpha \in \Phi_x \rangle$ where $\Phi_x = \{\alpha \in \Phi \mid \langle \alpha, x \rangle \in \mathbf{Z}\}$.

Lemma C. *The subset Φ_x depends only on $\mathscr{P}$ and $\mathscr{T}$; it is independent of the choice of Chevalley system and point $x \in V = X_*(T) \otimes \mathbf{Q}$ used to construct $\mathscr{P}$.*

Proof. Indeed, $\Phi_x \subset X_*(T) = X_*(\mathscr{T}_{\mathrm{k}})$ is precisely the set of non-zero weights for the adjoint action of the torus $\mathscr{T}_{\mathrm{k}}$ on $\mathrm{Lie}(\mathscr{P}_{\mathrm{k}}/R_u(\mathscr{P}_{\mathrm{k}}))$ where $R_u(\mathscr{P}_{\mathrm{k}})$ is the unipotent radical of $\mathscr{P}_{\mathrm{k}}$. □

3. Reformulation and proof of Theorem 4.5.6

Finally, Theorem 4.5.6 and its proof should be replaced by the following.

Theorem D. *Assume that G splits over an unramified extension of* K *and that $\mathscr{P}$ is a parahoric group scheme attached to G. There is a reductive $\mathscr{A}$-subgroup scheme $\mathscr{M}$ of $\mathscr{P}$ containing $\mathscr{T}$ such that*

(a) the special fiber $\mathscr{P}_{\mathrm{k}}$ has a Levi decomposition with Levi factor $\mathscr{M}_{\mathrm{k}}$, and
(b) the generic fiber $\mathscr{M}_{\mathrm{K}}$ is a connected reductive subgroup of G containing T, and $\mathscr{M}_{\mathrm{L}}$ is a subgroup of G_{L} of type $C(\mu)$ for some unramified extension L *of* K.

Proof. According to Proposition B, there is an unramified finite Galois extension L of K for G, a maximal K-torus T of G for which T_{L} is split, and a Γ-invariant facet F of $V = X_*(T_{\mathrm{L}}) \otimes \mathbf{Q}$ such that $\mathscr{Q} = \mathscr{P}_{\mathscr{B}}$ is the parahoric group scheme constructed from any point $x \in F$ as in §4.3, where $\mathscr{B}$ is the ring of integers of L.

Using Theorem 4.4.2 we find a locally closed, reductive $\mathscr{B}$-subgroup scheme $\mathscr{N}$ of $\mathscr{Q}_F$ such that $\mathscr{N}_{\mathrm{L}}$ is a subgroup of G_{L} of type $C(\mu)$ and $\mathscr{N}_\ell$ is a Levi factor of $\mathscr{Q}_\ell$.

We are going to argue that $\mathscr{N}$ descends to an $\mathscr{A}$-group scheme. Using étale decent – (Bruhat and Tits 1984), 5.1.8 – we must argue: (♣) $\mathscr{N}$ is Γ-invariant.

Let $\mathscr{T}$ be the split $\mathscr{B}$-torus with $\mathscr{T}_{\mathrm{L}} = T_{\mathrm{L}}$ and let $\phi_x : \mu \to \mathscr{T}$ be the μ-homomorphism determined by $x \in V = X_*(T_{\mathrm{L}})$. Write $\mathscr{C}_x$ for the centralizer in $\mathscr{Q}$ of ϕ_x. In view of R4 of Section 2, (♣) will follow if we show that $\mathscr{C}_x$ is Γ-invariant.

Now according to R5 of Section 2, $\mathscr{C}_x$ is the schematic closure in $\mathscr{Q}$ of the centralizer C_x of $\phi_{x/\mathrm{L}}$ in G_{L}. Thus (♣) will follow provided we show that the subgroup C_x of G_{L} is Γ-invariant.

Finally, according to R6 of Section 2,

$$C_x = \langle T_{\mathrm{L}}, U_\alpha \mid \alpha \in \Phi_x \rangle$$

where $\Phi_x = \{\alpha \in \Phi \mid \langle x, \alpha \rangle \in \mathbf{Z}\}$ and for $\alpha \in \Phi$, U_α is the corresponding root subgroup of G_{L}. Thus, (♣) will follow provided that we show that the subset $\Phi_x \subset X^*(T_{\mathrm{L}})$ is Γ-invariant.

Now the Γ-invariance of Φ_x follows from Lemma C. This confirms (♣) and shows that $\mathscr{N}$ descends to an $\mathscr{A}$-subgroup scheme $\mathscr{M}$ of $\mathscr{P}$ with $\mathscr{M}_{\mathscr{B}} = \mathscr{N}$.

Now, $\mathscr{M}$ is smooth over $\mathscr{A}$ with reductive fibers; thus $\mathscr{M}$ is a reductive $\mathscr{A}$-subgroup scheme of $\mathscr{P}$. Moreover, since $\mathscr{M}_{\mathrm{L}} = \mathscr{N}_{\mathrm{L}}$ is a subgroup of G_{L} of type $C(\mu)$ containing T_{L}, we see that $\mathscr{M}_{\mathrm{K}}$ is a connected and reductive subgroup of G containing T . Finally, since $\mathscr{M}_\ell = \mathscr{N}_\ell$ is a Levi factor of $\mathscr{P}_\ell = \mathscr{Q}_\ell$, we see that $\mathscr{M}_k$ is a Levi factor of $\mathscr{P}_k$. Thus $\mathscr{M}$ has the required properties, and the proof is concluded. □

Bibliography

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