

# DIMENSIONAL CRITERIA FOR SEMISIMPLICITY OF REPRESENTATIONS

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## 1. INTRODUCTION AND PRELIMINARIES

**1.1. Overview.** Let  $k$  denote a field. Let  $G$  be a group and suppose that the  $k$  vector space  $V$  is a module for  $G$ . When the characteristic of  $k$  is positive, several recent results have considered the semisimplicity of  $V$  by relating the dimension of  $V$  to the characteristic  $p$ .

For example, J.-P. Serre proved in [22] the following theorem: suppose that  $V$  and  $W$  are two semisimple representations of  $G$  over a field  $k$  of characteristic  $p > 0$ . If

$$\dim_k V + \dim_k W < p + 2,$$

then the representation of  $G$  on the tensor product  $V \otimes_k W$  is again semisimple.

Suppose now that  $k$  is algebraically closed, and that  $G$  is a reductive, connected algebraic  $k$  group. In this setting, we shall usually consider rational modules  $V$ . In the spirit of the above theorem, M. Larsen proved in [14] that for a rational  $G$  module  $V$ , a condition involving  $\dim_k V$  and  $p$  implies the semisimplicity of  $V$ . His result was subsequently improved by J. Jantzen in [12]. Jantzen proved that every rational  $G$ -module of dimension  $\leq p$  is semisimple (see Theorem II of [12]).

In this paper, we add another parameter to Jantzen's semisimplicity condition. Assume that  $G$  is almost simple and let  $\ell$  be the rank of  $G$ . In (3.1.a), we will specify a constant  $\mathcal{C}$  depending on  $\ell$  (and the type of the group); for classical type groups,  $\mathcal{C}$  is roughly  $\ell^2$ . We obtain the following extension of Jantzen's result:

**Theorem 1.** *(The Main Theorem) Let  $G$  be an almost simple algebraic group, and let  $V$  be a rational  $G$  module. If  $\dim_k V \leq \mathcal{C} \cdot p$ , then either  $V$  is semisimple, or  $V$  has a sub-quotient isomorphic to a Frobenius twist of one of the indecomposable modules described in Proposition 5.1.1.*

The bulk of this paper is occupied with the proof of this theorem; the actual text of the proof is located in section 5. The main theorem has the following immediate corollaries:

**Corollary 1.1.1.** *(The  $\ell$  Corollary) Suppose that  $G$  is an almost simple group and that  $V$  is a rational  $G$  module. If  $\dim_k V \leq \ell p$ , then  $V$  is semisimple.*

*Proof.* Since in all cases  $\mathcal{C} \geq \ell$ , the corollary follows from Theorem 1 together with the observation that each module  $E$  listed in Proposition 5.1.1 satisfies  $\dim_k E > \ell p$ .  $\square$

**Corollary 1.1.2.** *Suppose that  $G$  is a reductive group with root system  $\Phi$ . Let  $\Phi = \bigcup_{i=1}^r \Phi_i$  be the decomposition of  $\Phi$  into irreducible components, and let  $\ell_i$  denote the rank of  $\Phi_i$ . Put*

$$\ell_{\min} = \min\{\ell_i \mid i = 1, 2, \dots, r\}.$$

*If  $V$  is a rational  $G$  module and  $\dim_k V \leq \ell_{\min} p$ , then  $V$  is semisimple.*

*Proof.* This result follows from corollary 1.1.1 together with Lemma 3.1 from [12].  $\square$

**Corollary 1.1.3.** *Suppose that  $G$  is almost simple of rank  $\ell$ , and that  $V$  is an abstract (i.e. not necessarily rational) representation of  $G$ . Suppose that  $\dim_k V \leq \ell p$ . Then  $V$  is completely reducible.*

*Proof.* According to Theorem 1 of [20], the map  $G \rightarrow \mathrm{GL}(V)$  factors as a map

$$G \xrightarrow{\phi} H = G \times G \times \cdots \times G \xrightarrow{\psi} \mathrm{GL}(V)$$

where  $\phi$  is a “twisted diagonal embedding” with dense image and  $\psi$  is a rational homomorphism. It follows that  $V$  is a semisimple module for  $G$  if and only if it is a semisimple module for  $H$ ; since the quantity  $\ell_{\min}$  for the group  $H$  is  $\ell$ , we get the result by applying Corollary 1.1.2.  $\square$

**1.2. Algebraic Group Notions and Notations.** We fix here some notations which will be in force throughout this paper. Let  $k$  be an algebraically closed field of characteristic  $p \geq 0$ . Denote by  $G$  a connected, reductive algebraic  $k$  group. There are numerous standard notations associated with such a  $G$ , which we summarize here:

$T$	a maximal torus of $G$
$X = \mathrm{Hom}(T, \mathbb{G}_m)$	the character group of $T$
$Y = \mathrm{Hom}(\mathbb{G}_m, T)$	all 1-parameter subgroups of $T$
$\langle ?, ? \rangle : X \times Y \rightarrow \mathbb{Z}$	the canonical duality
$\Phi \subset X$	the root system
$\Phi^\vee \subset Y$	the co-roots
$T \subseteq B, B^-$	opposite Borel subgroups

The choice of Borel subgroup  $B$  determines the following:

$\Phi_+, \Phi_+^\vee$	systems of positive roots in $\Phi, \Phi^\vee$
$\Delta, \Delta^\vee$	bases for $\Phi, \Phi^\vee$

The quantity  $\ell$  will always denote the semisimple rank of  $G$ . All representations considered in this paper are *rational*, i.e. those for which the action of  $G$  is described by matrices whose coefficients are regular functions of the algebraic variety  $G$ .

A few remarks concerning root systems are now in order. First of all,  $\mathcal{W}$  will denote the Weyl group of the root system  $\Phi$ . This group acts on both  $X$  and  $Y$  in a well-known manner. Furthermore, the pairing  $\langle ?, ? \rangle$  is  $\mathcal{W}$  equivariant. A reductive group is classified by its so-called *root datum*, namely the quadruple  $(X, \Phi, Y, \Phi^\vee)$  together with the pairing  $\langle ?, ? \rangle : X \times Y \rightarrow \mathbb{Z}$ . For a treatment of this classification, one may refer to [10] II.1, especially II.1.13, 14, 15.

In the consideration of the proof of Theorem 1, we may restrict our attention to almost simple groups, i.e. those having irreducible root systems. Any semisimple group has a central extension by a simply connected group; it is sufficient to prove Theorem 1 in the case where

$G$  is simply connected. To prove our main theorem, we can therefore restrict our attention to simply connected groups of types  $A_\ell, B_\ell, C_\ell, D_\ell, E_\ell, F_4,$  and  $G_2$ .

Let us assume now that  $G$  is simply connected; one then knows that  $Y$  is generated by the  $\alpha_i^\vee$  ( $1 \leq i \leq \ell$ ). Since  $\langle \cdot, \cdot \rangle$  is a perfect pairing,  $X \simeq Y^*$ . Let  $\{\omega_i \mid 1 \leq i \leq \ell\} \subset X$  be the dual basis (under the pairing  $\langle \cdot, \cdot \rangle$ ) to  $\{\alpha_i \mid 1 \leq i \leq \ell\} \subset Y$ . The weights  $\omega_i$  are the fundamental dominant weights; they form a  $\mathbb{Z}$  basis of  $X$ . Fix a  $\mathcal{W}$  invariant quadratic form  $\eta(\cdot)$  on  $X$  and let  $\psi(\cdot, \cdot)$  be the corresponding bilinear form characterized by  $\psi(x, x) = \frac{1}{2}\eta(x)$  for  $x \in X$ . One may now identify  $Y$  with the lattice  $\{y \in X_{\mathbb{Q}} \mid \psi(y, X) \subseteq \mathbb{Z}\} \subseteq X_{\mathbb{Q}} = X \otimes_{\mathbb{Z}} \mathbb{Q}$ ; this being done, the correspondence between roots and co-roots is given by  $\alpha = \frac{1}{\eta(\alpha^\vee)}\alpha^\vee$  ( $\alpha \in \Phi$ ).

**1.3. Bourbaki notation.** In [3] Planche I through Planche IX, the irreducible root systems are constructed and information concerning them is recorded. For each type, the fundamental dominant weights and the roots are described in terms of a basis  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d$  for a Euclidean space  $E$ . To tie in the construction there with the set-up described above, one should take for  $\eta(\cdot)$  the usual quadratic form with respect to the  $\varepsilon_i$ . One then takes for  $X$  the  $\mathbb{Z}$  span inside  $E$  of the fundamental dominant weights  $\omega_1, \dots, \omega_\ell$ , and for  $Y$  the  $\mathbb{Z}$  span inside  $E$  of the co-roots  $\alpha^\vee = \frac{1}{\eta(\alpha)}\alpha$  ( $\alpha \in \Phi$ ). We shall use this notation and the information recorded in this source; especially we will freely express roots and weights in terms of the  $\varepsilon_i$ , and we will use the descriptions of the action of  $\mathcal{W}$  on  $X$  given there.

We list here the descriptions of the simple roots and fundamental weights in terms of the  $\varepsilon_i$ , as well as the description of the action of  $\mathcal{W}$ , for the classical root system types.

- For  $\Phi = A_\ell$ ,  $\dim E = \ell + 1$ . We have  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  and  $\omega_i = (\sum_{s=1}^i \varepsilon_s) - \frac{i}{\ell+1} (\sum_{s=1}^\ell \varepsilon_s)$  for  $1 \leq i \leq \ell$ . The Weyl group  $\mathcal{W} \simeq \text{Sym}_{\ell+1}$  acts by permuting the indices of the  $\varepsilon_i$ .
- For  $\Phi = B_\ell$ ,  $\dim E = \ell$ . We have  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  for  $i = 1, 2, \dots, \ell - 1$  and  $\alpha_\ell = \varepsilon_\ell$ . We have also  $\omega_i = \varepsilon_1 + \dots + \varepsilon_i$  for  $i = 1, 2, \dots, \ell - 1$  and  $\omega_\ell = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_\ell)$ . The Weyl group  $\mathcal{W} \simeq \text{Sym}_\ell \cdot (\mathbb{Z}/2\mathbb{Z})^\ell$  acts as the group of all permutations and sign changes of the  $\varepsilon_i$ .
- For  $\Phi = C_\ell$ ,  $\dim E = \ell$ . We have  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  for  $i = 1, 2, \dots, \ell - 1$  and  $\alpha_\ell = 2\varepsilon_\ell$ . We have  $\omega_i = \varepsilon_1 + \dots + \varepsilon_i$  for  $i = 1, 2, \dots, \ell$ . The Weyl group acts just as for  $B_\ell$ .
- For  $\Phi = D_\ell$ ,  $\dim E = \ell$ . We have  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  for  $i = 1, 2, \dots, \ell - 2$ ,  $\alpha_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_\ell$ , and  $\alpha_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell$ . We have  $\omega_i = \varepsilon_1 + \dots + \varepsilon_i$  for  $i = 1, 2, \dots, \ell - 2$ ,  $\omega_{\ell-1} = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{\ell-1} - \varepsilon_\ell)$ , and  $\omega_\ell = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{\ell-1} + \varepsilon_\ell)$ . The Weyl group  $\mathcal{W} \simeq \text{Sym}_\ell \cdot (\mathbb{Z}/2\mathbb{Z})^\ell$  acts as the group of all permutations and even numbers of sign changes of the  $\varepsilon_i$ .

**1.4. Parabolic subgroups and subsystems.** Fix a subset  $I \subset \Delta$ . One can form the *parabolic subgroup*  $P = P_I \supseteq B^-$  ( $P$  is sometimes called an opposite parabolic subgroup as it contains the opposite Borel subgroup). Such a subgroup is not reductive in general. We denote its unipotent radical by  $Q$ . There is a subgroup of  $P$  which is complementary to  $Q$

called a *Levi factor*; it is denoted  $L$ . Thus  $P \simeq QL$  is a semidirect product, and  $L \simeq P/Q$  is a reductive group with root system determined by  $I$ .

Observe that  $L$  contains the entire maximal torus  $T$ . Let  $L'$  denote the derived group of  $L$ , and let  $T' = T \cap L'$ ;  $T'$  is then a maximal torus of  $L'$ . The inclusion  $T' \subset T$  induces a surjection  $X(T) \rightarrow X(T')$  which corresponds to the restriction of a character to the smaller torus.

The group  $L'$  is a semisimple group whose Dynkin diagram is determined (in an obvious manner) by  $I$ .

**1.5. Classical Group Realizations.** Let  $V$  be an  $\ell + 1$ -dimensional vector space over  $k$ , and let  $G$  be the group  $\mathrm{SL}(V)$ . A fixed basis  $\vec{e} = (e_1, e_2, \dots, e_{\ell+1})$  of  $V$  determines a subgroup  $T < G$  consisting of those elements of  $G$  which act diagonally with respect to the  $e_i$  ( $i = 1, \dots, \ell + 1$ ). Fix such a basis; according to [2] V.23.2,  $G$  is an almost simple algebraic  $k$  group of type  $A_\ell$ , and  $T$  is a maximal torus. Let  $\chi_i \in X$  denote the character of  $T$  describing the action of  $T$  on  $e_i$  ( $i = 1, 2, \dots, \ell + 1$ ); in Bourbaki notation,  $\chi_i$  is identified with  $\varepsilon_i - \frac{1}{(\ell + 1)} \sum_{j=1}^{\ell+1} \varepsilon_j \in X \subseteq Y_{\mathbb{Q}}$ . Note that the so-called ‘natural’ module  $V$  for  $G$  is the simple (or Weyl) module whose highest weight is  $\omega_1$ .

Let  $W$  be a  $d$  dimensional vector space over  $k$  and let  $\beta$  be a non degenerate symmetric or skew symmetric bilinear form on  $W$ ; if  $\beta$  is symmetric, assume that  $p \neq 2$ . We write  $\Omega = \Omega(W, \beta)$  for the group of determinant 1 isometries. Let  $\ell = \left\lfloor \frac{d}{2} \right\rfloor$ . According to [2] V.23.3 and V.23.4,  $\Omega$  is an almost simple algebraic  $k$  group of rank  $\ell$ ; the type of  $\Omega$  is  $C_\ell$  if  $\beta$  is skew symmetric,  $B_\ell$  if  $\beta$  is symmetric and  $d$  is odd, and  $D_\ell$  if  $\beta$  is symmetric and  $d$  is even. Since  $\beta$  is non-degenerate, we can find complementary maximal isotropic subspaces  $E$  and  $F$  of  $W$ , with bases  $\vec{e} = (e_1, \dots, e_\ell)$  and  $\vec{f} = (e_{-\ell}, \dots, e_{-1})$  satisfying  $\beta(e_i, e_{-j}) = \delta_{i,j}$  for  $i, j \geq 1$ . If  $\dim_k W = 2\ell + 1$ , find  $u \in W$  so that  $W = E \oplus F \oplus ku$ . Denote by  $S$  the subgroup of  $\Omega$  which acts diagonally with respect to the basis  $\vec{b} = (\vec{e}, u, \vec{f})$ , where the notation refers to concatenation of the tuples and  $u$  is deleted if  $\dim_k W$  is even. Again according to [2] V.23.3 and V.23.4,  $S$  is a maximal torus. Let me describe how one realizes the characters  $\varepsilon_i$  for these groups of isometries. Let  $T$  be the maximal torus of  $\mathrm{SL}(W)$  constructed as above with respect to the basis  $\vec{b}$ , and let  $X(T)$  denote its character group. Let  $X(S)$  denote the character group for  $\Omega$ . The inclusion  $\Omega \subset \mathrm{SL}(W)$  induces a surjection  $\psi : X(T) \rightarrow X(S)$ . Let  $\chi_i \in X(T)$  ( $-\ell \leq i \leq \ell$ ) be the characters for the basis  $\vec{b}$  as above; we take  $\varepsilon_i = \psi(\chi_i - \chi_{-i}) \in X(S)$  ( $1 \leq i \leq \ell$ ). In particular, the ‘natural’ module  $W$  has weights  $\{\pm\omega_1 = \varepsilon_1, \pm\varepsilon_2, \dots, \pm\varepsilon_\ell, \varepsilon_0\}$  (with  $\varepsilon_0$  dropped in the even dimensional case).

If  $\beta$  is symmetric, the group  $\Omega$  is *not* a simply connected group. There is a simply connected double covering group  $\mathrm{Spin}(V)$ . When we discuss representations of groups of types  $B_\ell$  or  $D_\ell$ , then generically we are referring to  $\mathrm{Spin}(V)$ . Whenever possible, we will choose to factor representations through the group  $\Omega$ .

*Remark 1.5.1.* To avoid the restriction  $p \neq 2$ , one should define a group of type  $B_\ell$  or  $D_\ell$  as the stabilizer of a non-degenerate quadratic form  $\phi$ ; see [2] V.23.5, and 23.6. In the case of  $p > 2$ , this quadratic form satisfies  $\phi(v) = \frac{1}{2}\beta(v, v)$  for  $v \in V$ .

*Remark 1.5.2.* The embeddings  $\Omega(V, \beta) < \mathrm{SL}(V)$  and  $S < T$ , as well as the surjection  $X(T) \rightarrow X(S) \rightarrow 0$ , correspond to the situations described in Theorem 8.1 (a),(b),(c) of [21], as well as the remarks preceding that theorem. This theorem describes circumstances under which certain irreducible  $\mathrm{SL}(V)$  modules restrict to irreducible modules for  $\Omega$ ; we shall apply this result later.

## 2. BACKGROUND MATERIAL ON REPRESENTATIONS

**2.1. Induced modules, Weyl modules, and simple modules.** Each weight  $\lambda \in X$  determines a one dimensional module  $k_\lambda$  for  $T$ ; an element  $t \in T$  acts on this module as multiplication with  $\lambda(t)$ . Since  $T$  is a quotient of  $B^-$ , we get also a  $B^-$  module structure on  $k_\lambda$ . There is a corresponding induced module  $\mathrm{ind}_{B^-}^G(k_\lambda)$  for  $G$ ; refer to [10] I.3.3 for a complete definition. The functor  $\mathrm{ind}_{B^-}^G(?)$  is left exact; we denote by  $H^i(?) = H^i(G/B^-, ?)$  ( $i \geq 0$ ) its derived functors. We use the abbreviation  $H^i(\lambda) = H^i(k_\lambda)$ .

For a rational  $G$ -module  $M$ , the formal character of  $M$  is given by

$$\mathrm{ch} M = \sum_{\lambda \in X} \dim_k M_\lambda e^\lambda \in \mathbb{Z}[X]^{\mathcal{W}}.$$

For  $\lambda \in X$ , we put  $\chi(\lambda) = \sum_{i \geq 0} (-1)^i \mathrm{ch} H^i(\lambda)$ . If  $\lambda \in X_+$ , Kempf's vanishing theorem (see [10] II.4.5) shows that  $H^i(\lambda) = 0$  for  $i > 0$ . It follows that  $\chi(\lambda) = \mathrm{ch} H^0(\lambda)$  ( $\lambda \in X_+$ ).

According to [10] Proposition II.5.10,  $\chi(\lambda)$  is given by the Weyl character formula whenever  $\lambda \in X_+$ . In particular, if one writes  $\Pi(H^0(\lambda)) = \Pi(\lambda)$  for the set of weights  $\xi \in X$  such that  $H^0(\lambda)_\xi \neq 0$ , then  $\Pi(\lambda)$  is the saturated set of weights with the highest weight  $\lambda$ ; see [8] 13.4 and Proposition 21.3. In particular,  $\Pi(\lambda)$  may be characterized as the union of the orbits  $\mathcal{W}\mu$  of all dominant weights  $\mu \leq \lambda$ .

Let  $w_0$  be the longest word in  $\mathcal{W}$ . For a dominant weight  $\lambda$ , we define  $\lambda^* = -w_0\lambda \in X_+$ . We put  $V(\lambda) = H^0(\lambda^*)^*$  (the dual or contragredient module),  $V(\lambda)$  is the so-called ‘‘Weyl module.’’ According to [10] Corollary II.5.11, the character of  $V(\lambda)$  is nicely behaved; indeed  $\mathrm{ch} V(\lambda) = \chi(\lambda)$ .

Fix  $\lambda \in X_+$ . Since  $\chi(\lambda)$  is given by Weyl's character formula, [8] Corollary 24.3 shows that the Weyl degree formula may be used to compute  $\dim_k H^0(\lambda) = \dim_k V(\lambda)$ . This formula says that

$$(2.1.a) \quad \dim_k V(\lambda) = \prod_{\alpha > 0} \frac{\langle \lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} \quad (\lambda \in X_+).$$

The Weyl module enjoys certain nice properties; it is a highest weight module in the sense that its  $\lambda$  weight space is 1 dimensional, is  $B^+$  stable, and generates  $V(\lambda)$  as a  $G$  module. In fact, the Weyl module is a *universal* highest weight module: for any  $G$  module  $M$ , we have by [10] Lemma II.2.13:

$$(2.1.b) \quad \mathrm{Hom}_G(V(\lambda), M) \simeq \mathrm{Hom}_{B^+}(k_\lambda, M).$$

Observe that by duality  $\mathrm{soc} H^0(\lambda) \simeq V(\lambda)/\mathrm{rad} V(\lambda)$  ( $\lambda \in X_+$ ). It is known that the modules  $\mathrm{soc} H^0(\lambda)$  are simple; furthermore

$$(2.1.c) \quad \{L(\lambda) = \mathrm{soc} H^0(\lambda) \mid \lambda \in X_+\}$$

is a complete set of representatives for the isomorphism classes of simple rational  $G$ -modules. For proof of this fact, see [10] Corollary II.2.7. Observe that  $L(\lambda)^* \simeq L(\lambda^*)$ .

**2.2. Restricted representations.** Assume that  $p > 0$ , and let  $F$  be the Frobenius endomorphism of  $G$ . For a rational  $G$ -module  $M$ , and  $r \in \mathbb{N}$ , denote by  $M^{[r]}$  the rational  $G$ -module which is the same as  $M$  as a vector space with  $G$  module structure twisted by  $F^r$ . For each  $r \in \mathbb{N}$ , write

$$(2.2.a) \quad X_r = \{\mu \in X : 0 \leq \langle \mu, \alpha_i \rangle < p^r \quad (1 \leq i \leq \ell)\}.$$

A *restricted weight* is a weight in  $X_1$ . Each weight  $\mu \in X_+$  has a  $p$ -adic expansion  $\mu = \sum_{j=0}^t p^j \mu_j$  where  $\mu_j \in X_1$  for each  $0 \leq j \leq t$ . The  $p$ -adic expansion of a weight yields a corresponding tensor product decomposition of the simple module according to the following:

**Proposition 2.2.1.** (*Steinberg's Tensor Product Theorem*) *With  $\mu$  as above, one has:*

$$(2.2.b) \quad L(\mu) \simeq L(\mu_0) \otimes L(\mu_1)^{[1]} \otimes L(\mu_2)^{[2]} \otimes \cdots \otimes L(\mu_t)^{[t]}$$

*Proof.* See [10] Corollary II.3.17. □

For an arbitrary dominant weight  $\lambda$ , there is no general formula which describes the dimension of the various weight spaces for  $L(\lambda)$ . However, for restricted  $\lambda$ , the following theorem due to A. Premet (see [17]), and I. Suprenenko (see [25]) for the case of a group of type  $A_\ell$ , gives some useful information concerning these weight spaces.

*Definition 2.2.2.* Let  $G$  be a reductive algebraic  $k$  group. If  $G$  has a component of type  $B_\ell$ ,  $C_\ell$ , or  $F_4$ , the prime  $p = 2$  will be called special. If  $G$  has a component of type  $G_2$ , the primes  $p = 2, 3$  will be called special. No other prime is special.

**Proposition 2.2.3.** *Assume that the prime  $p$  is not special, and let  $\mu \in X_1$ . If  $\gamma \in X$  is such that  $V(\mu)_\gamma \neq 0$ , then  $L(\mu)_\gamma \neq 0$ .*

*Proof.* The result is Theorem 1 of [17]. □

**2.3. Extensions.** Let  $\mathcal{R}_G$  denote the category of rational  $G$  modules. Let  $N$  be a rational  $G$ -module. We denote by  $\text{Ext}_G^i(?, N)$  the  $i^{\text{th}}$  derived functor of  $\text{Hom}(?, N) : \mathcal{R}_G \rightarrow \mathcal{R}_G$ . As usual, one may identify elements  $\sigma = \sigma_E \in \text{Ext}_G^1(M, N)$  with certain equivalence classes of short exact sequences  $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$  of rational  $G$ -modules; a sequence in the equivalence class is split precisely when  $\sigma_E = 0$ .

We record some facts concerning extensions of simple rational  $G$ -modules.

**Lemma 2.3.1.** *Let  $\lambda, \mu \in X_+$ .*

- (a)  $\text{Ext}_G^1(L(\lambda), L(\mu)) \simeq \text{Ext}_G^1(L(\mu), L(\lambda))$ .
- (b)  $\text{Ext}_G^1(L(\lambda), L(\lambda)) = 0$ .
- (c) *If  $\mu \not\prec \lambda$ ,  $\text{Ext}_G^1(L(\lambda), L(\mu)) \simeq \text{Hom}_G(\text{rad } V(\lambda), L(\mu))$ .*

*Proof.* (a) may be found in [10] II.12 (4). (b) is [10] II.2.12 (1). (c) is [10] Proposition II.14. □

The central investigation of this paper is the study of extensions between two simple modules whose dimensions are suitably constrained. We show in the next section that “most of the time” an appropriate dimensional constraint placed on a simple module  $L(\lambda)$  gives information about the quantity  $\langle \lambda + \rho, \alpha_0 \rangle$ . The following result shows that this quantity is significant to extension theory.

**Lemma 2.3.2.** *Suppose that  $\Phi$  is not of type  $A_1$ . Let  $\lambda, \lambda' \in X_+$  satisfy  $\langle \lambda + \rho, \alpha_0 \check{\cdot} \rangle < p$  and  $\langle \lambda' + \rho, \alpha_0 \check{\cdot} \rangle < p$ . If  $\lambda \neq \lambda'$ , then  $\text{Ext}_G^1(L(\lambda + p\mu), L(\lambda' + p\mu')) = 0$  for every choice of  $\mu, \mu' \in X_+$ .*

*Proof.* This is Lemma 1.7 of [12]. □

For  $r \geq 1$ , the  $r$ th Frobenius kernel of  $G$ , written  $G_r$ , is the group scheme theoretic kernel of  $F^r$ . For  $r = 1$ , the representation theory of  $G_1$  is identical to that of the restricted enveloping algebra  $\mathcal{U}^{[p]}(\mathfrak{g})$  of the Lie algebra  $\mathfrak{g}$  of  $G$ ; see [10] I.9.6.

**Lemma 2.3.3.** *Let  $\lambda = \tau + p^r\mu, \lambda' = \tau' + p^r\mu'$  ( $\tau, \tau' \in X_r, \mu, \mu' \in X_+, r \in \mathbb{N}$ ).*

- (a) *If  $\tau \neq \tau'$  then  $\text{Ext}_G^1(L(\lambda), L(\lambda')) \simeq \text{Hom}_G(L(\mu), \text{Ext}_{G_r}^1(L(\tau), L(\tau'))^{[-r]} \otimes L(\mu'))$ .*
- (b) *Assume that  $\tau = \tau'$ . If  $p \neq 2$ , or if the root system  $\Phi$  does not have an irreducible component of type  $C_\ell$ , then*

$$\text{Ext}_G^1(L(\lambda), L(\lambda')) \simeq \text{Ext}_G^1(L(\mu), L(\mu')).$$

*Proof.* For (a), see [10] II.10.17 (3). Note that the symbol  $\oplus$  should be replaced with  $\otimes$  in this citation. (b) follows from the result II.12.9 together with II.10.17 (2) from [10]. □

### 3. ALLOWABLE WEIGHTS.

In this section, we develop a technique which will permit us to relate the dimension of a simple module  $L(\lambda)$  to the quantity  $\langle \lambda + \rho, \alpha_0 \check{\cdot} \rangle$ . The key point is that for restricted  $\lambda$  and primes which are not special, Premet's Theorem (Proposition 2.2.3) guarantees that  $|\Pi(\lambda)|$  is a lower bound for  $\dim_k L(\lambda)$ . Working with the quantity  $|\Pi(\lambda)|$  has several advantages; in particular, this quantity is independent of  $p$ .

**3.1. Allowable weights defined.** Throughout this section,  $\Phi$  denotes an *irreducible* root system of rank  $\ell$ . Let  $\tilde{\alpha}$  (respectively  $\alpha_0$ )  $\in \Phi_+$  be the long (respectively short) root of maximal height, and let

$$\mathcal{C} = \mathcal{C}(\Phi) = \max \left\{ \frac{|\mathcal{W}\tilde{\alpha}|}{2}, \frac{|\mathcal{W}\alpha_0|}{2} \right\}.$$

Computation of the quantity  $\mathcal{C}$  is straightforward. One uses the tables in [3] to determine the long and short root of maximal height; they are simply the long and short root which are dominant weights. It is then a simple matter to apply the definition of  $\mathcal{C}$  to obtain the following

$$(3.1.a) \quad \begin{array}{ll} \text{For type } A_\ell, & \mathcal{C} = \binom{\ell+1}{2}. \\ \text{For type } B_\ell, & \mathcal{C} = \ell(\ell-1). \\ \text{For type } E_6, & \mathcal{C} = 36. \\ \text{For type } E_8, & \mathcal{C} = 120. \\ \text{For type } G_2, & \mathcal{C} = 3. \end{array} \quad \begin{array}{ll} \text{For type } C_\ell, & \mathcal{C} = 2 \binom{\ell}{2} = \ell(\ell-1). \\ \text{For type } D_\ell, & \mathcal{C} = \ell(\ell-1). \\ \text{For type } E_7, & \mathcal{C} = 63. \\ \text{For type } F_4, & \mathcal{C} = 12. \end{array}$$

*Definition 3.1.1.* A weight  $\lambda \in X_+$  will be called *allowable* provided that  $|\Pi(\lambda)| > \mathcal{C} \cdot \langle \lambda + \rho, \alpha_0 \check{\cdot} \rangle$ . A set of weights  $S \subseteq X_+$  will be called allowable provided every weight in it is allowable.

We remark that for a root system of type  $A_1$ ,  $\mathcal{C} = 1$  and

$$|\Pi(n\omega_1)| = (n+1) = \langle n\omega_1 + \rho, \alpha_0^\vee \rangle \quad (n \in \mathbb{N}).$$

Hence there are *no* allowable weights in the rank 1 case.

For a root system of rank  $\ell > 1$ , we shall verify that there are only finitely many weights which are not allowable. For each irreducible root system, a finite set of weights  $\mathcal{I}$  containing the non-allowable weights is listed in table 3.1.1. Section 3.2 is devoted to the proof that  $\mathcal{I}$  contains the non-allowable weights.

Table 3.1.1. The set  $\mathcal{I}$ .

For the classical types of root systems,  $\mathcal{I}$  consists of the diagram automorphism conjugates of the following weights.

The classical types.			
Type $A_\ell$ , $\ell \geq 2$	Type $B_\ell$ , $\ell \geq 3$	Type $C_\ell$ , $\ell \geq 2$	Type $D_\ell$ , $\ell \geq 4$
$\omega_i$ , $\ell > 2i$ ( $i = 1, 2, 3$ )	$\omega_i$ , $\ell > i$ ( $i = 1, 2, 3$ )	$\omega_i$ , $\ell > i$ ( $i = 1, 2, 3$ )	$\omega_i$ , $\ell > i + 1$ ( $i = 1, 2, 3$ )
$\omega_4$ , $7 \leq \ell \leq 15$	$\omega_\ell$ , $3 \leq \ell \leq 11$	$\omega_4$ , $4 \leq \ell \leq 6$	$\omega_4$ , $\ell = 6$
$\omega_5$ , $9 \leq \ell \leq 11$		$\omega_5$ , $\ell = 5, 6$	$\omega_\ell$ , $4 \leq \ell \leq 12$
		$\omega_6$ , $\ell = 6$	
$r\omega_1$ ( $r = 2, 3$ )	$r\omega_1$ ( $r = 2, 3$ )	$r\omega_1$ ( $r = 2, 3$ )	$2\omega_1$
$4\omega_1$ , $\ell = 4, 5$			
$\omega_1 + \omega_2$ , $\ell \geq 3$	$\omega_1 + \omega_2$	$\omega_1 + \omega_2$	$\omega_1 + \omega_2$ , $\ell \geq 5$
$2\omega_1 + \omega_\ell$ , $\ell \geq 3$	$\omega_2 + \omega_\ell$ , $\ell = 3, 4$	$2\omega_2$ , $\ell = 2, 3$	$2\omega_4$ , $\ell = 5$
$\omega_i + \omega_\ell$ , $\ell > i$ ( $i = 1, 2, 3$ )	$2\omega_\ell$ , $\ell = 3, 4$	$\omega_1 + \omega_3$ , $\ell = 3$	$\omega_1 + \omega_\ell$ , $4 \leq \ell \leq 7$
$2\omega_2$ , $3 \leq \ell \leq 6$	$\omega_1 + \omega_\ell$ , $3 \leq \ell \leq 5$		$\omega_4 + \omega_5$ , $\ell = 5$
$2\omega_1 + \omega_2$ , $2 \leq \ell \leq 5$			
$3\omega_1 + \omega_2$ , $\ell = 2$			
$\omega_4 + \omega_6$ , $\ell = 6$			
$\omega_2 + \omega_3$ , $\ell = 4$			

#### Exceptional Types

- Type  $E_6$ .  $\mathcal{I} = \{\omega_2, \omega_1 + \omega_6, \omega_1, \omega_6, \omega_5, \omega_3, 2\omega_1, 2\omega_6\}$ .
- Type  $E_7$ .  $\mathcal{I} = \{\omega_1, \omega_6, 2\omega_7, \omega_7, \omega_2\}$ .
- Type  $E_8$ .  $\mathcal{I} = \{\omega_1, \omega_8\}$ .
- Type  $F_4$ .  $\mathcal{I} = \{\omega_1, \omega_3, \omega_4, 2\omega_4\}$ .
- Type  $G_2$ .  $\mathcal{I} = \{\omega_1, \omega_2, 2\omega_2, 3\omega_2\}$ .



**3.2. The main result on allowable weights.** Throughout the remainder of this section, let  $\Phi$  be an irreducible root system of rank  $\ell \geq 2$ , and let  $\mathcal{I}$  be the set specified in table 3.1.1.

**Proposition 3.2.1.** *Suppose that  $\lambda$  is a non-0 weight. If  $\lambda \in X_+ \setminus \mathcal{I}$ , then  $\lambda$  is allowable.*

We develop some intermediate results before presenting the proof of this Proposition. First note the following lemma:

**Lemma 3.2.2.** *(Minimal weights) Let  $\Phi$  be irreducible. The non-0 minimal weights for the usual partial order  $\leq$  on  $X_+$  are as follows:*

$$\begin{array}{ll} A_\ell & : \ \varpi_i, \ i = 1, \dots, \ell \quad B_\ell & : \ \varpi_\ell \\ C_\ell & : \ \varpi_1 \quad D_\ell & : \ \varpi_1, \ \varpi_\ell, \ \varpi_{\ell-1} \\ E_6 & : \ \varpi_1, \ \varpi_6 \quad E_7 & : \ \varpi_7 \end{array}$$

*Proof.* See [8], §13.2, exercise 13 (p. 72). □

We shall often work not with the usual partial order  $\geq$  on weights, but instead with a somewhat different partial order that we denote by  $\succeq$ . It will be defined only on  $X_+$ . Its definition involves the choice of a particular set  $\mathcal{R}$ ; we take  $\mathcal{R}$  as described in table 3.2.1. In each case note that  $\mathcal{R} \subset \mathbb{Z}_{\geq 0}\Delta$ . In most situations, the elements of  $\mathcal{R}$  listed above are actually positive roots. The representation of roots and weights described in section 1.3 make this verification straightforward. We remark that despite the fact that the root systems  $B_2$  and  $C_2$  are the same, the sets  $\mathcal{R}$  do not coincide; we will later choose to work with the set  $\mathcal{R}$  defined for  $C_2$  in this case.

Table 3.2.1. The set  $\mathcal{R}$  for the irreducible root systems.

- $\Phi = A_\ell$ ,  $\ell \geq 2$ ,  $\mathcal{R} = \Phi_+$ .
- $\Phi = B_\ell$ ,  $\mathcal{R} = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_\ell\}$ .
- $\Phi = C_\ell$ ,  $\mathcal{R} = \{\varepsilon_i + \varepsilon_{i+1} \mid 1 \leq i \leq \ell - 1\} \cup \{\varepsilon_1 - \varepsilon_2\}$ .
- $\Phi = D_\ell$ ,  $\mathcal{R} = \{\varepsilon_i + \varepsilon_{i+1} \mid 1 \leq i \leq \ell - 2\} \cup \{\varepsilon_1 - \varepsilon_2, \varepsilon_{\ell-1} + \varepsilon_\ell, \varepsilon_{\ell-1} - \varepsilon_\ell, \varepsilon_1 - \varepsilon_\ell, \varepsilon_1 + \varepsilon_\ell\}$ .
- $\Phi = E_6$ ,  $\mathcal{R} = \{\varpi_2, \varpi_4, \varpi_3 + \varpi_5, \varpi_1 + \varpi_6, \varpi_3 - \varpi_6, \varpi_5 - \varpi_1, 2\varpi_1 - \varpi_3, 2\varpi_6 - \varpi_5\}$ .
- $\Phi = E_7$ ,  $\mathcal{R} = \{\varpi_1, \varpi_3, \varpi_4, \varpi_6, \varpi_5 - \varpi_2, \varpi_2 - \varpi_7, 2\varpi_7 - \varpi_6\}$ .
- $\Phi = E_8$ ,  $\mathcal{R} = \{\varpi_i \mid i = 1, 2, \dots, 8\}$ .
- $\Phi = F_4$ ,  $\mathcal{R} = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$ .
- $\Phi = G_2$ ,  $\mathcal{R} = \{\varpi_1 - \varpi_2, \varpi_2\}$ .

**Definition 3.2.3.** Let  $\succeq$  be the minimal partial ordering on  $X_+$  with the property that for any pair of weights  $\lambda, \tau \in X_+$ ,  $\lambda \succeq \tau$  whenever  $\lambda = \tau$  or their difference  $\lambda - \tau$  lies in  $\mathcal{R}$ .

In general, two dominant weights  $\lambda, \tau$  will satisfy  $\lambda \succeq \tau$  if there is a sequence of elements  $\gamma_1, \gamma_2, \dots, \gamma_t \in \mathcal{R}$  such that

$$(3.2.a) \quad \lambda = \tau + \sum_{i=1}^t \gamma_i \quad \text{and} \quad \lambda + \sum_{i=1}^m \gamma_i \in X_+ \text{ for all } m, 1 \leq m \leq t.$$

The partial order  $\succeq$  does not in general coincide with the usual partial order  $\geq$  on  $X_+$ . However, since  $\mathcal{R} \subset \mathbb{Z}_{\geq 0}\Delta$ , it is true that  $\lambda \geq \mu$  whenever  $\lambda \succeq \mu$  ( $\lambda, \mu \in X_+$ ). We stress that each partial sum  $\tau + \sum_{i=1}^m \gamma_i$  is required to be *dominant*; in particular, one does not necessarily recover the usual partial order even for type  $A_\ell$  (where  $\mathcal{R} = \Phi_+$ ).

**Lemma 3.2.4.** *Let  $\succeq$  be the partial order on  $X_+$  determined by the set  $\mathcal{R}$  specified in table 3.2.1. The minimal weights for the partial order  $\succeq$  coincide with the minimal weights for the usual partial order (see Lemma 3.2.2).*

*Proof.* Suppose that  $\lambda$  is not a  $\succeq$ -minimal weight. We exhibit a dominant weight  $\mu$  with  $\mu \prec \lambda$ .

Let  $\Phi = A_\ell$ . If there are indices  $1 \leq i < j \leq \ell$  with  $\langle \lambda, \alpha_i^\vee \rangle > 0$  and  $\langle \lambda, \alpha_j^\vee \rangle > 0$ , take  $\mu = \lambda - (\alpha_i + \cdots + \alpha_j) \in X_+$ . Otherwise,  $\lambda = n\varpi_i$  ( $1 \leq i \leq \ell$ , and  $n \in \mathbb{N}$ ); furthermore,  $n > 1$  since  $\lambda$  is not  $\succeq$ -minimal. Take  $\mu = \lambda - \alpha_i$ .

Let  $\Phi = B_\ell$ . If there is some  $1 \leq i \leq \ell - 1$  such that  $\langle \lambda, \alpha_i^\vee \rangle > 0$ , take  $\mu = \lambda - \varepsilon_i$ . Otherwise,  $\langle \lambda, \alpha_i \rangle = 0$  for  $i < \ell$ , so that  $\lambda = n\varpi_\ell$  for some  $n \in \mathbb{N}$ . Since  $\lambda$  is not  $\succeq$ -minimal,  $n > 1$ . Take  $\mu = \lambda - \varepsilon_\ell$ .

Let  $\Phi = C_\ell$ . If there is an index  $1 < i \leq \ell$  such that  $\langle \lambda, \alpha_i^\vee \rangle > 0$ , take  $\mu = \lambda - (\varepsilon_{i-1} + \varepsilon_i)$ . Otherwise,  $\lambda = n\varpi_1$  ( $n \in \mathbb{N}$ ). According to Lemma 3.2.2,  $n > 1$ . Take  $\mu = \lambda - (\varepsilon_1 - \varepsilon_2)$ .

Let  $\Phi = D_\ell$ . If there is an index  $1 < i < \ell - 1$  with  $\langle \lambda, \alpha_i^\vee \rangle > 0$  take  $\mu = \lambda - (\varepsilon_{i-1} + \varepsilon_i)$ . Otherwise,  $\lambda = n_1\varpi_1 + n_{\ell-1}\varpi_{\ell-1} + n_\ell\varpi_\ell$  ( $n_1, n_{\ell-1}, n_\ell \in \mathbb{N}$ ). If  $n_1, n_\ell > 0$  take  $\mu = \lambda - (\varepsilon_1 + \varepsilon_\ell)$ ; if  $n_1, n_{\ell-1} > 0$ , take  $\mu = \lambda - (\varepsilon_1 - \varepsilon_\ell)$ ; finally if  $n_\ell, n_{\ell-1} > 0$  take  $\mu = \lambda - (\varepsilon_{\ell-2} + \varepsilon_{\ell-1})$ . We are thus reduced to consideration of  $\lambda = n\varpi_r$  ( $r = 1, \ell - 1, \ell$ , and  $n \in \mathbb{N}$ ). According to Lemma 3.2.2, one has  $n > 1$ . Take  $\mu = \lambda - (\varepsilon_1 - \varepsilon_2)$  if  $r = 1$ ,  $\mu = \lambda - (\varepsilon_{\ell-1} + \varepsilon_\ell)$  if  $r = \ell$ , and  $\mu = \lambda - (\varepsilon_{\ell-1} - \varepsilon_\ell)$  if  $r = \ell - 1$ .

Let  $\Phi = E_\ell$ ,  $\ell = 6, 7, 8$ . When  $\ell = 8$  the assertion is trivial. Write  $n_i = \langle \lambda, \alpha_i^\vee \rangle$  for  $i = 1, 2, \dots, \ell$ . Suppose that  $\ell = 6$ . If  $n_2 > 0$  (respectively  $n_4 > 0$ ,  $n_3 > 0$ ,  $n_5 > 0$ ), take  $\mu = \lambda - \varpi_2$  (respectively  $\mu = \lambda - \varpi_4$ ,  $\mu = \lambda - (\varpi_3 - \varpi_6)$ ,  $\mu = \lambda - (\varpi_5 - \varpi_1)$ ). If both  $n_1 > 0$  and  $n_6 > 0$ , take  $\mu = \lambda - (\varpi_1 + \varpi_6)$ . Replacing  $\lambda$  with its conjugate under the diagram automorphism if necessary, we can therefore assume that  $\lambda = n_1\varpi_1$ . Since  $\lambda$  is not minimal, one has  $n_1 > 1$ . Now take  $\mu = \lambda - (2\varpi_1 - \varpi_3)$ .

Now assume  $\ell = 7$ . If  $n_1 > 0$  (respectively  $n_3 > 0$ ,  $n_4 > 0$ ,  $n_6 > 0$ ,  $n_5 > 0$ ,  $n_2 > 0$ ), take  $\mu = \lambda - \varpi_1$  (respectively  $\mu = \lambda - \varpi_3$ ,  $\mu = \lambda - \varpi_4$ ,  $\lambda - \varpi_6$ ,  $\mu = \lambda - (\varpi_5 - \varpi_2)$ ,  $\mu = \lambda - (\varpi_2 - \varpi_7)$ ). Thus, we can assume that  $\lambda = n_7\varpi_7$ . Since  $\lambda$  is not  $\succeq$ -minimal,  $n_7 > 1$ . Now take  $\mu = \lambda - (2\varpi_7 - \varpi_6)$ .

When  $\Phi = F_4$ , the result is trivial. For type  $G_2$ , write  $\lambda = n_1\varpi_1 + n_2\varpi_2$ . If  $n_2 > 0$ , take  $\mu = \lambda - \varpi_2$ . If  $n_2 = 0$ , then  $n_1 \neq 0$ ; we may therefore take  $\mu = \lambda - (\varpi_1 - \varpi_2) = (n_1 - 1)\varpi_1 + \varpi_2$ .  $\square$

The partial order  $\succeq$  provides an inductive tool for studying allowable weights; the key property is that  $\Pi(\mu) \subset \Pi(\lambda) \setminus \mathcal{W}\lambda$  whenever  $\lambda \succeq \mu$  and  $\mu \neq \lambda$ . The following two results show how to exploit this property.

For an arbitrary weight  $\sigma \in X$ , write  $\text{pos}(\sigma) = \sum \varpi_j$  where the sum is taken over those  $j$  such that  $\langle \sigma, \alpha_j^\vee \rangle > 0$ .

**Lemma 3.2.5.** *Let  $\lambda - \mu = \sigma \in \mathcal{R}$ . If  $\mu$  is allowable, and  $|\mathcal{W}\text{pos}(\sigma)| \geq \mathcal{C}\langle \sigma, \alpha_0^\vee \rangle$ , then  $\lambda$  is allowable.*

*Proof.* Since  $0 \notin \mathcal{R}$ , we know  $\lambda \neq \mu$ . For each  $i$  with  $\langle \sigma, \alpha_i^\vee \rangle > 0$ , the fact that  $\mu = \lambda - \sigma$  is dominant yields  $\langle \lambda, \alpha_i^\vee \rangle > 0$ . It follows that  $\text{Stab}_{\mathcal{W}}(\lambda) \subset \text{Stab}_{\mathcal{W}}(\text{pos}(\sigma))$  so that  $|\mathcal{W}\lambda| \geq |\mathcal{W}\text{pos}(\sigma)|$ . Using the fact that  $\mu$  is allowable together with the hypothesis on  $\sigma$ , one obtains  $|\Pi(\lambda)| \geq |\Pi(\mu)| + |\mathcal{W}\lambda| > \mathcal{C}\langle \mu + \rho, \alpha_0^\vee \rangle + |\mathcal{W}\lambda| \geq \mathcal{C}\langle \mu + \rho, \alpha_0^\vee \rangle + \mathcal{C}\langle \sigma, \alpha_0^\vee \rangle = \mathcal{C}\langle \lambda + \rho, \alpha_0^\vee \rangle$ .  $\square$

**Lemma 3.2.6.** *Let  $\succeq$  be the partial order determined by the set  $\mathcal{R}$  as above. Fix  $\lambda \succeq \mu$ , and assume that  $\mu$  is allowable.*

- (a) *If  $\Phi \neq A_\ell, B_\ell$  for  $\ell \geq 3$ , then  $\lambda$  is allowable.*
- (b) *If  $\Phi = A_\ell$  when  $\ell \geq 3$ , and if  $\lambda - \mu \in \mathcal{R} \setminus \{\alpha_1, \alpha_\ell\}$  then  $\lambda$  is allowable.*
- (c) *If  $\Phi = B_\ell$  when  $\ell \geq 3$ , and if  $\lambda - \mu \in \mathcal{R} \setminus \{\varepsilon_1\}$  then  $\lambda$  is allowable.*

*Proof.* Let us write  $n_\sigma = \langle \sigma, \alpha_0^\vee \rangle$ .

For part (a), we show first that  $|\mathcal{W} \text{pos}(\sigma)| \geq \mathcal{C}n_\sigma$  for every  $\sigma \in \mathcal{R}$  for those root systems which have not been excluded. Repeated application of Lemma 3.2.5 will then yield the result.

For  $\Phi = A_2$ , note that  $\mathcal{C} = 3$ . We have  $\mathcal{R} = \{\alpha_1, \alpha_2, \alpha_0 = \alpha_1 + \alpha_2\}$ . Observe that  $n_{\alpha_i} = 2, 1, 1$  and  $|\mathcal{W} \text{pos}(\alpha_i)| = 6, 3, 3$  when  $i = 0, 1, 2$  resp. Thus,  $\mathcal{C} \cdot n_\sigma = |\mathcal{W} \text{pos}(\sigma)|$  for every  $\sigma \in \mathcal{R}$  and the claim follows.

For  $\Phi = C_\ell$ ,  $\ell \geq 2$ , recall that  $\alpha_0 = \varepsilon_1 + \varepsilon_2$ . One checks that  $n_\sigma = 0$  with the exceptions  $n_{\varepsilon_1 + \varepsilon_2} = 2$  and  $n_{\varepsilon_2 + \varepsilon_3} = 1$ . The claim now follows from the observation that

$$|\mathcal{W} \text{pos}(\varepsilon_1 + \varepsilon_2)| = 2^2 \binom{\ell}{2} = 2\mathcal{C}$$

and (when  $\ell \geq 3$ ),  $|\mathcal{W} \text{pos}(\varepsilon_2 + \varepsilon_3)| = 2^3 \binom{\ell}{3} = \frac{4}{3}(\ell - 2)\mathcal{C}$ .

Note that, using the identification  $B_2 = C_2$ , this settles  $B_2$  as well (observe that  $\varepsilon_2 + \varepsilon_3$  does not occur in  $\mathcal{R}$  for this rank).

For  $\Phi = D_\ell$ , recall  $\alpha_0 = \varepsilon_1 + \varepsilon_2$ . One checks that  $n_\sigma = 0$  unless  $\sigma = \varepsilon_1 + \varepsilon_2, \varepsilon_2 + \varepsilon_3, \varepsilon_1 + \varepsilon_\ell$  or  $\varepsilon_1 - \varepsilon_\ell$ , when  $n_\sigma = 2, 1, 1, 1$  resp. The claim now follows by observing that

$$|\mathcal{W} \text{pos}(\varepsilon_1 + \varepsilon_2)| = 2^2 \binom{\ell}{2} = 2\mathcal{C}, \quad |\mathcal{W} \text{pos}(\varepsilon_2 + \varepsilon_3)| = 2^3 \binom{\ell}{3} = \frac{4}{3}(\ell - 2)\mathcal{C},$$

and  $|\mathcal{W} \text{pos}(\varepsilon_1 \pm \varepsilon_\ell)| = 2^{\ell-1} \cdot \ell = \frac{2^{\ell-1}\mathcal{C}}{\ell - 1}$ .

For  $\Phi = E_\ell$ ,  $\ell = 6, 7, 8$ , and  $\Phi = F_4, G_2$ , the verification that  $|\mathcal{W} \text{pos}(\sigma)| \geq n_\sigma$  is a straightforward numerical calculation; we omit the details here. This completes the verification of (a).

Note that (b) will follow from Lemma 3.2.5 provided that we show  $|\mathcal{W} \text{pos}(\sigma)| \geq \mathcal{C}n_\sigma$  for every  $\sigma \in \mathcal{R} \setminus \{\alpha_1, \alpha_\ell\}$ . To see this, recall that an arbitrary  $\sigma \in \Phi_+$  has the form  $\varepsilon_i - \varepsilon_j$ ; one has for such a  $\sigma$  the computations  $\text{pos}(\sigma) = \mathfrak{w}_i + \mathfrak{w}_{j-1}$  (where  $\mathfrak{w}_0$  is interpreted as 0) and  $\langle \sigma, \alpha_0^\vee \rangle = \langle \varepsilon_i - \varepsilon_j, (\varepsilon_1 - \varepsilon_{\ell+1})^\vee \rangle = \delta_{i,1} + \delta_{j,\ell+1}$ .

The inequality is trivial unless  $\sigma = \varepsilon_1 - \varepsilon_j$  ( $1 < j$ ) or  $\varepsilon_j - \varepsilon_{\ell+1}$  ( $j < \ell + 1$ ). For these  $\sigma$ , we observe that whenever  $\sigma$  is different from  $\alpha_1, \alpha_\ell$ , one has first of all  $|\mathcal{W} \text{pos}(\sigma)| \geq 2 \binom{\ell+1}{2}$ , and secondly  $\langle \sigma, \alpha_0^\vee \rangle \leq 2$ . The result follows.

Finally, (c) will follow from Lemma 3.2.5 provided that  $|\mathcal{W} \text{pos}(\sigma)| \geq \mathcal{C}n_\sigma$  for  $\sigma \in \mathcal{R} \setminus \{\varepsilon_1\}$ . We have  $n_{\varepsilon_j} = 0$  for  $j > 1$ , so the inequality is trivial in these cases.  $\square$

*Remark 3.2.7.* To check that a given weight  $\lambda$  is allowable, one typically needs to find a lower bound for  $|\Pi(\lambda)|$ . If one specifies a collection of subdominant weights  $\mu_1, \mu_2, \dots, \mu_r \leq \lambda$ , one

has:

$$(3.2.b) \quad |\Pi(\lambda)| \geq \sum_{i=1}^r |\mathcal{W}\mu_i|.$$

**Proposition 3.2.8.** *Let  $\Phi$  be irreducible, and let  $\succeq$  be the partial order determined by the set  $\mathcal{R}$  as above. There is a set of allowable weights  $\mathcal{A}_\tau$  for each minimal weight  $\tau$  with the following property: for each  $\lambda \in X_+$  either  $\lambda \in \mathcal{I}$  or there is a minimal weight  $\tau$  and an element  $\mu \in \mathcal{A}_\tau$  such that  $\lambda \succeq \mu$ .*

*Outline of Proof:* Let  $\lambda$  be a dominant weight. Find the minimal weight  $\tau$  so that  $\lambda \succeq \tau$ . According to Lemma 3.2.4, we can find a sequence of weights  $\gamma_1, \gamma_2, \dots, \gamma_t \in \mathcal{R}$  satisfying condition (3.2.a). Let us write  $\lambda_k = \tau + \sum_{i=1}^k \gamma_i$  ( $k = 1, 2, \dots, t$ ).

In order to verify the proposition, we will proceed as follows. For each minimal weight  $\tau$ , we write down each possibility for  $\lambda_1$ ; these possibilities are precisely those dominant weights obtained from  $\tau$  by adding a weight in  $\mathcal{R}$ . We label each possibility for  $\lambda_1$  with either a  $(*)$  if it lies in  $\mathcal{I}$  or a  $(\dagger)$  otherwise.

We may now iterate this procedure; for each possibility for  $\lambda_1$  marked with a  $(*)$ , we determine all of the possibilities for  $\lambda_2$ . Again, we label these possibilities either with  $(*)$  or  $(\dagger)$ . We continue this procedure in the obvious manner. Because the set  $\mathcal{I}$  is finite, this process must terminate in a finite number of steps. Once it has terminated, it is clear that the proposition will follow provided that each weight marked with  $(\dagger)$  is allowable; for this verification, one may use the technique described in remark 3.2.7.

If a weight  $\mu$  occurs as a possibility for  $\lambda_i$  which is marked with  $(*)$ , transitivity of the partial order  $\succeq$  implies that we may omit  $\mu$  as a possibility for any  $\lambda_j$  with  $j > i$ .

The results of carrying out the procedure outlined above are recorded in the Appendix to this paper; the verification that the weights marked with  $\dagger$  are allowable is an unpleasant chore – our assertions may be verified by the interested reader. In order to demonstrate this technique, we present the following example.

Let  $\Phi = A_\ell$ ,  $\ell > 4$ , and consider the weight  $\tau = 0$ . The above procedure yields:

$$\begin{aligned} \lambda_1 &= \omega_1 + \omega_\ell (*) \\ \lambda_2 &= 2\omega_1 + 2\omega_\ell (\dagger), 2\omega_1 + \omega_{\ell-1} (\dagger), \omega_2 + 2\omega_\ell (\dagger), \omega_2 + \omega_{\ell-1} (\dagger) \end{aligned}$$

One must now verify that the weights marked with  $\dagger$  are allowable. Observe that each of the weights  $\gamma$  marked with  $\dagger$  satisfies  $\gamma \geq \mu$  where  $\mu = \omega_2 + \omega_{\ell-1}$ . It follows that for any such  $\gamma$ ,  $|\Pi(\gamma)| \geq |\Pi(\mu)|$ . Observe that  $\mu$  has the subdominant weight  $\omega_1 + \omega_\ell$ ; applying 3.2.7 we obtain

$$|\Pi(\gamma)| \geq |\Pi(\mu)| \geq 6 \binom{\ell+1}{4} + 2 \binom{\ell+1}{2}.$$

On the other hand, for any such  $\gamma$  we have  $\mathcal{C}\langle \gamma + \rho, \alpha_0 \rangle \leq (\ell+4) \cdot \binom{\ell+1}{2}$ . From these data, it is easy to verify that each  $\gamma$  is allowable.  $\square$

*Proof of Proposition 3.2.1:* We point out that for  $\Phi \neq A_\ell$  for  $\ell \geq 3$  and  $\Phi \neq B_\ell$  for  $\ell \geq 3$ , the proposition follows from Proposition 3.2.8. Indeed, an arbitrary weight  $\lambda \notin \mathcal{I}$  satisfies  $\lambda \succeq \mu$

for some  $\mu \in \mathcal{A}_\tau$  and some minimal weight  $\tau$ . With the above root systems excluded, we may apply (a) of 3.2.6; this result shows that  $\lambda$  is allowable.

Suppose that  $\ell \geq 3$  and  $\Phi = A_\ell$  or  $\Phi = B_\ell$ . If  $\lambda \notin \mathcal{I}$ , we have again  $\lambda \succeq \mu$  for some  $\mu \in \mathcal{A}_\tau$  and some minimal weight  $\tau$ . We may reduce to the case where  $\lambda - \mu \in \mathcal{R}$ . Let  $\sigma$  be  $\alpha_1$  if  $\Phi = A_\ell$  and  $\varepsilon_1$  if  $\Phi = B_\ell$ . Up to diagram automorphism, parts (b) and (c) of Lemma 3.2.6 show that we may assume  $\lambda - \mu = \sigma$ .

We suppose now that  $\lambda$  is not allowable and deduce a contradiction. Suppose that  $\eta \in \mathcal{R} \setminus \Gamma\sigma$  (where  $\Gamma$  is the group of diagram automorphisms). If  $\lambda - \eta$  is dominant and allowable, then according to Lemma 3.2.6, we would have  $\lambda$  allowable contrary to our assumption. Suppose that  $\Phi = A_\ell$  and both  $\alpha_1$  and  $\alpha_\ell$  may be subtracted from  $\lambda$  to yield dominant weights. Then also subtracting  $\tilde{\alpha}$  yields an allowable weight; again, this is incompatible with our assumption.

We may thus suppose that  $\sigma$  is the *only* element of  $\mathcal{R}$  which may be subtracted from  $\lambda$  to yield a dominant weight. A little thought then shows that  $\lambda = n\varpi_1$  ( $n \in \mathbb{N}$ ) or  $\lambda = n\varpi_1 + \varpi_\ell$  ( $n \in \mathbb{N}$ ), where the latter configuration occurs only for  $B_\ell$ . Since  $\lambda \notin \mathcal{I}$ , we have  $n \geq 4$ .

Suppose for the moment that  $\ell \geq 4$  and  $\lambda = n\varpi_1$ , and let  $\mu_1 = \lambda - \alpha_1 = (n-2)\varpi_1 + \varpi_2$ ,  $\mu_2 = \lambda - 2\alpha_1 - \alpha_2 = (n-3)\varpi_1 + \varpi_3$ . Note that  $\mu_2 \notin \mathcal{I}$ , so by induction  $\mu_2$  is allowable. Furthermore, notice that  $\mu_1 \notin \Pi(\mu_2)$ . We can therefore conclude that when  $\Phi = A_\ell$  we have:

$$\begin{aligned} |\Pi(\lambda)| &> |\Pi(\mu_2)| + |\mathcal{W}\mu_1| > \mathcal{C}\langle \mu_2 + \rho, \alpha_0^\vee \rangle + (\ell+1)\ell \\ &= \mathcal{C}\langle \lambda - 3\varpi_1 + \varpi_3 + \rho, \alpha_0^\vee \rangle + 2\mathcal{C} \\ &= \mathcal{C}\langle \lambda + \rho, \alpha_0^\vee \rangle - 2\mathcal{C} + 2\mathcal{C} \\ &= \mathcal{C}\langle \lambda + \rho, \alpha_0^\vee \rangle. \end{aligned}$$

When  $\Phi = B_\ell$  we have:

$$\begin{aligned} |\Pi(\lambda)| &> |\Pi(\mu_2)| + |\mathcal{W}\mu_1| > \mathcal{C}\langle \mu_2 + \rho, \alpha_0^\vee \rangle + 4\ell(\ell-1) \\ &= \mathcal{C}\langle \lambda - 3\varpi_1 + \varpi_3 + \rho, \alpha_0^\vee \rangle + 4\mathcal{C} \\ &= \mathcal{C}\langle \lambda + \rho, \alpha_0^\vee \rangle - 4\mathcal{C} + 4\mathcal{C} \\ &= \mathcal{C}\langle \lambda + \rho, \alpha_0^\vee \rangle. \end{aligned}$$

One obtains a similar inequality for  $\lambda = n\varpi_1 + \varpi_\ell$  in type  $B_\ell$ ; in that case,  $\mu_1 = (n-2)\varpi_1 + \varpi_2 + \varpi_\ell$  and  $\mu_2 = (n-3)\varpi_1 + \varpi_3 + \varpi_\ell$ . These inequalities prove that  $\lambda$  is allowable; the result now holds for  $\ell \geq 4$ .

Essentially the same argument settles  $\ell = 3$ ; although additional arguments are necessary to show that  $4\varpi_1$  is allowable when  $\Phi = A_3$  or  $\Phi = B_3$ .  $\square$

#### 4. TECHNIQUES FOR UNDERSTANDING WEYL MODULES

**4.1. “Small” simple modules.** It is clear that we need to understand the simple modules with dimension  $\leq \mathcal{C}p$ . Steinberg’s tensor product theorem reduces us to the restricted case. Furthermore, the following lemma provides a characterization of the highest weight of such a simple module.

**Lemma 4.1.1.** *Let  $\ell \geq 2$ , and assume that  $p$  is not special. Let  $\mu$  be a restricted weight, and suppose that  $\dim_k L(\mu) \leq \mathcal{C}p$ . Then either  $\mu$  satisfies  $\langle \mu + \rho, \alpha_p^\vee \rangle < p$ , or  $\mu \in \mathcal{I}$ .*

*Proof.* Since  $p$  is not special and  $\mu$  is restricted, Proposition 2.2.3 applies; this proposition implies that  $\dim_k L(\mu) \geq |\Pi(\mu)|$ . We suppose that  $\mu$  is not in  $\mathcal{I}$ ; thus,  $\mu$  is allowable by Proposition 3.2.1. We now have

$$\mathcal{C} \cdot p \geq \dim_k L(\mu) \geq |\Pi(\mu)| > \mathcal{C}\langle \mu + \rho, \alpha_0^\vee \rangle.$$

Dividing by  $\mathcal{C}$  shows that  $\langle \mu + \rho, \alpha_0^\vee \rangle < p$  as desired.  $\square$

The case where  $p$  is special will be treated specially later on. The condition  $\langle \lambda + \rho, \alpha_p^\vee \rangle < p$  shows that  $\lambda$  lies in the so-called *lowest dominant alcove*. For such  $\lambda$ ,  $L(\lambda)$  is well understood; see Proposition 4.4.3 below. We now wish to investigate the structure of the Weyl modules with highest weights in  $\mathcal{I}$ . When it does not carry us too far afield, we occasionally consider a broader class of modules than is necessitated by the set  $\mathcal{I}$ .

**4.2. Some particularly simple Weyl modules.** When the high weight of a Weyl module is *minuscule*, or minimal in the partial ordering on dominant weights, the Weyl module is particularly easy to understand.

**Proposition 4.2.1.** (*Minuscule weights*) *Suppose that  $\lambda \in X_+$  is minuscule. Then*

$$H^0(\lambda) \simeq V(\lambda) \simeq L(\lambda) \text{ and } \Pi(L(\lambda)) = \mathcal{W}\lambda$$

*Proof.* We know that  $\Pi(\lambda)$  is the union of the  $\mathcal{W}$  orbits of the subdominant weights to  $\lambda$ . Since  $\lambda$  is minuscule, it has no proper subdominant weights; thus  $\Pi(H^0(\lambda)) = \mathcal{W}\lambda$ . Since  $\mathcal{W}$  permutes the weights of any module, we know that all weight spaces are 1 dimensional. It follows that any weight vector generates  $H^0(\lambda)$  as a  $G$  module. In particular,  $H^0(\lambda)$  has no proper non-trivial submodules. Thus  $H^0(\lambda) = \text{soc } H^0(\lambda)$  and the result follows.  $\square$

Using this proposition and some results from [21], we obtain the following:

**Proposition 4.2.2.** *Let  $\Phi = A_\ell, C_\ell, B_\ell,$  or  $D_\ell$  and let the group  $G$  be realized as  $\text{SL}(V)$  in the first case,  $\text{Sp}(V)$  in the second, and  $\text{Spin}(V)$  (where these groups are described in section 1.5). We have the following:*

- (a) *For  $\Phi = A_\ell$  and  $i = 1, 2, \dots, \ell$ ,  $V(\mathfrak{w}_i) = L(\mathfrak{w}_i) \simeq \bigwedge^i V$ .*
- (b) *For  $\Phi = A_\ell$  and  $r < p$ ,  $V(r\mathfrak{w}_1) = L(r\mathfrak{w}_1) \simeq S^r V$ .*
- (c) *For  $\Phi = B_\ell$ ,  $V(\mathfrak{w}_\ell) = L(\mathfrak{w}_\ell)$ . This module has dimension  $2^\ell$ .*
- (d) *For  $\Phi = D_\ell$ ,  $V(\mathfrak{w}_{\ell-1}) = L(\mathfrak{w}_{\ell-1})$  and  $V(\mathfrak{w}_\ell) = L(\mathfrak{w}_\ell)$ . Each has dimension  $2^{\ell-1}$ .*
- (e) *For  $\Phi = B_\ell$ , suppose that  $p \neq 2$  and  $1 \leq i \leq \ell - 1$ . Then  $V(\mathfrak{w}_i) = L(\mathfrak{w}_i) \simeq \bigwedge^i V$ .*
- (f) *For  $\Phi = D_\ell$ ,  $V(\mathfrak{w}_1) = L(\mathfrak{w}_1) = V$ .*
- (g) *For  $\Phi = D_\ell$ , suppose that  $p \neq 2$  and  $2 \leq i \leq \ell - 2$ . Then  $V(\mathfrak{w}_i) = L(\mathfrak{w}_i) \simeq \bigwedge^i V$ .*
- (h) *For  $\Phi = C_\ell$  and  $1 \leq r < p$ ,  $V(r\mathfrak{w}_1) = L(r\mathfrak{w}_1) \simeq S^r V$ .*

*Proof.* For (a), note that  $\mathfrak{w}_i$  is a minuscule weight. According to Proposition 4.2.1,  $L(\mathfrak{w}_i) = V(\mathfrak{w}_i)$ . By taking a basis of weight vectors for  $V$ , it is easy to verify that  $\bigwedge^i V$  has 1 dimensional weight spaces, and that the weights of this module are precisely the  $\mathcal{W}$  conjugates of  $\mathfrak{w}_i$ . It follows that  $L(\mathfrak{w}_i) \simeq \bigwedge^i V$ .

Statement (b) is proved in 1.14 of [21].

For part (c) and (d), let  $\lambda$  be one of the designated weights. One knows in all cases that  $\lambda$  is minuscule; one applies Lemma 4.2.1 and a calculation of  $|\mathcal{W}\lambda|$  to obtain the result. (These representations are the “spin” and “half-spin” modules for  $\text{Spin}(V)$ .)

For parts (e), (f), and (g), note that the group  $\Omega = \mathrm{SO}(V)$  acts on  $V(\mathfrak{w}_i)$ . To see that  $L(\mathfrak{w}_i) \simeq \bigwedge^i V$ , note that according to remark 1.5.2, we may apply [21] (8.1)(b) to learn that  $\bigwedge^i V$  restricts to an irreducible  $\Omega$  module. Since in characteristic 0,  $V(\mathfrak{w}_i) = L(\mathfrak{w}_i)$ , we get immediately  $\chi(\mathfrak{w}_i) = \mathrm{ch}(\bigwedge^i V)$ ; the equality  $V(\mathfrak{w}_i) = L(\mathfrak{w}_i)$  now follows since  $\mathrm{ch}(\bigwedge^i V)$  is independent of characteristic.

For part (g), the isomorphism  $H^0(r\mathfrak{w}_1) \simeq S^r V$  is shown in [10]. According to 1.5.2, we may apply [21] (8.1)(c) to the embedding  $\Omega \subset \mathrm{SL}(V)$ ; when  $r < p$  this result guarantees the simplicity of  $S^r V$ . It follows that  $H^0(r\mathfrak{w}_1) \simeq L(r\mathfrak{w}_1) \simeq V(r\mathfrak{w}_1)$ .  $\square$

**4.3. Characters and the dot action of the Weyl group.** The so-called “dot action” of the ordinary Weyl group  $\mathcal{W}$  on  $X$  is given by the formula:

$$(4.3.a) \quad w.\lambda = w(\lambda + \rho) - \rho, \text{ for } w \in \mathcal{W} \text{ and } \lambda \in X.$$

Let  $\mathcal{D} = \{\lambda \in X \mid \langle \lambda + \rho, \alpha^\vee \rangle \geq 0, \text{ for every } \alpha \in \Phi_+\}$ . Since  $\mathcal{D}$  is simply the affine translation by  $\rho$  of the “fundamental Weyl chamber” for the usual action of  $\mathcal{W}$ ,  $\mathcal{D}$  is a fundamental domain for the dot action of  $\mathcal{W}$  on  $X$ .

The domain  $\mathcal{D}$  is useful in describing the characters  $\chi(\lambda)$  for  $\lambda \in X$ . Here,  $\chi(\lambda)$  is computed in *characteristic 0*. We would like to write  $\chi(\lambda)$  as  $\pm\chi(\mu)$  for some  $\mu \in X_+$ ; the following lemma provides the necessary tools to do so.

**Lemma 4.3.1.** *Let  $\lambda \in X$ .*

- (a) *If  $w.\lambda \in \mathcal{D}$  for some  $w \in \mathcal{W}$ , then  $\chi(\lambda) = (-1)^{l(w)}\chi(w.\lambda)$ , where  $l(w)$  denotes the length of the Weyl group element  $w$ .*
- (b) *If  $\lambda \in \mathcal{D}$  but  $\lambda \notin X_+$ , then  $\chi(\lambda) = 0$ .*

*Proof.* See [10] II, 5.5 Corollary (a).  $\square$

**4.4. Linkage and the affine Weyl group.** Let  $\mathcal{W}_p$  be the affine Weyl group of the root system  $\Phi$ .  $\mathcal{W}_p$  is generated by affine reflections  $s_{\alpha, np}$  for  $\alpha \in \Phi$  and  $n \in \mathbb{Z}$ . We always assume that  $\mathcal{W}_p$  is acting on  $X$  via the dot action. Let  $\vec{m} = (m_\alpha)_{\alpha \in \Phi_+} \in \mathbb{Z}^{|\Phi_+|}$  and set  $\mathcal{C}_{\vec{m}} = \{\mu \in X \mid (m_\alpha - 1)p < \langle \mu + \rho, \alpha^\vee \rangle < m_\alpha p, \alpha \in \Phi_+\}$ .  $\bar{\mathcal{C}}_{\vec{m}}$ , the closure of  $\mathcal{C}_{\vec{m}}$ , is described by replacing the strict inequalities  $<$  with  $\leq$ . Let  $\vec{1} = (1, 1, \dots, 1)$  and  $\mathcal{C} = \mathcal{C}_{\vec{1}}$ . The  $\mathcal{C}_{\vec{m}}$  are called alcoves, and  $\mathcal{C}$  is the lowest dominant alcove. One has

**Proposition 4.4.1.** *For every  $\vec{m} \in \mathbb{Z}^{|\Phi_+|}$ , the closure of  $\mathcal{C}_{\vec{m}}$  is a fundamental domain for the action of  $\mathcal{W}_p$  on  $X$ .*

*Proof.* See [3], ch. V, §3, th. 2.  $\square$

The importance to us of conjugacy under  $\mathcal{W}_p$  (also referred to as “linkage”) is the following proposition:

**Proposition 4.4.2.** *(The Linkage Principle) Let  $\nu, \mu \in X_+$ . If  $\mathrm{Ext}_G^1(L(\nu), L(\mu)) \neq 0$ , then  $\mu \in \mathcal{W}_p.\nu$ .*

For a proof of the proposition, see [10], II, Corollary 6.17.

**Corollary 4.4.3.** *(The Lowest Alcove Condition) Let  $\mu \in X_+$ , and suppose  $\langle \mu + \rho, \alpha_0^\vee \rangle \leq p$ . Then  $V(\mu)$  is simple.*

*Proof.* Suppose that  $V(\mu)$  is not simple. Thus  $\text{rad } V(\mu)$  has a simple quotient  $L(\gamma)$  for some  $\gamma \in X_+$ . Since  $V(\mu)$  is a highest weight module, one has  $\gamma < \mu$ .

Since  $\gamma \leq \mu$ , we have  $\langle \gamma + \rho, \alpha_0^\vee \rangle \leq \langle \mu + \rho, \alpha_0^\vee \rangle$ . Thus  $0 \leq \langle \gamma + \rho, \alpha^\vee \rangle \leq \langle \gamma + \rho, \alpha_0^\vee \rangle \leq \langle \mu + \rho, \alpha_0^\vee \rangle \leq p$  for each  $\alpha \in \Phi_+$ ; we deduce that  $\mu, \gamma \in \bar{\mathbf{C}}$ . Since  $\bar{\mathbf{C}}$  is a fundamental domain for  $\mathcal{W}_p$ , we deduce that  $\gamma \notin \mathcal{W}_p\mu$ ; according to Proposition 4.4.2, this is a contradiction.  $\square$

**4.5. Andersen-Jantzen Sum Formula.** Let  $D$  be the group of divisors of  $\mathbb{Z}$ , i.e the group of all formal sums  $\sum_{p \in \mathfrak{P}} n_p [p]$  with  $n_p \in \mathbb{Z}$  almost all 0, the sum taken over  $\mathfrak{P}$  a complete set of primes in  $\mathbb{Z}$  up to sign. For  $q \in \mathbb{Q}$ , write  $\text{div}(q) = \sum_{p \in \mathfrak{P}} \nu_p(q) [p] \in D$ .

If  $M$  is a finitely generated torsion module over  $\mathbb{Z}$ , write  $\nu_p(M)$  for the length of the  $\mathbb{Z}_{(p)}$  module  $M_{(p)} = M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$  and let  $\nu(M) = \sum_{p \in \mathfrak{P}} \nu_p(M) [p] \in D$ . For example,  $\nu(\mathbb{Z}/n\mathbb{Z}) = \text{div}(n)$ .

There is a group scheme  $G_{\mathbb{Z}}$  over  $\mathbb{Z}$  such that  $G = G_k$  is obtained by extension of scalars from  $G_{\mathbb{Z}}$ . If  $M$  is a rational  $G_{\mathbb{Z}}$  module which is a finitely generated torsion module for  $\mathbb{Z}$ , we write  $\nu^c(M) = \sum_{\lambda \in X} \nu(M_\lambda) e^\lambda \in D[X] = D \otimes_{\mathbb{Z}} \mathbb{Z}[X]$ .

If  $A$  and  $B$  are instead  $G_{\mathbb{Z}}$  modules which are *free* over  $\mathbb{Z}$ , and  $\phi : A \rightarrow B$  is a  $G_{\mathbb{Z}}$  module map which is bijective after tensoring with  $\mathbb{Q}$ , we write  $\nu^c(\phi) = \nu^c(\text{coker } \phi)$ .

Let  $\lambda \in X_+$  and let  $V_{\mathbb{Q}}(\lambda)$  be the Weyl module for  $G_{\mathbb{Q}}$ . One can construct a  $\mathbb{Z}$  lattice  $V_{\mathbb{Z}}(\lambda)$  in  $V_{\mathbb{Q}}(\lambda)$  with the property that for any field  $k$ , one has  $V_k(\lambda) \simeq V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} k$ ; see [10] II.8.3 (3). The induced module  $H_{\mathbb{Z}}^0(\lambda)$  is free over  $\mathbb{Z}$ , and satisfies  $H_k^0(\lambda) \simeq H_{\mathbb{Z}}^0(\lambda) \otimes_{\mathbb{Z}} k$  as well; see [10] II.8.8 (1) (2) and [10] II.8.7 (1). Following the construction in [10] II.8.16, one has an essentially unique homomorphism  $T_\lambda : V_{\mathbb{Z}}(\lambda) \rightarrow H_{\mathbb{Z}}^0(\lambda)$  (written  $\tilde{T}_{w_0}(w_0 \cdot \lambda)$  in [10]) which satisfies several nice conditions:

$$(4.5.a) \quad \text{For any field } k, \text{ Im}(T_\lambda \otimes 1_k) = L_k(\lambda)$$

$$(4.5.b) \quad \nu^c(T_\lambda) = - \sum_{\alpha > 0} \sum_{i=1}^{\langle \lambda + \rho, \alpha^\vee \rangle - 1} \text{div}(i) \chi(\lambda - i\alpha)$$

For proof of (4.5.a) see the remark in [10] II after Proposition 6.16; for (4.5.b), see [10] II, proposition 8.16. We shall refer to (4.5.b) as the ‘‘general sum formula.’’

Fix the field  $k$  and assume  $\text{char}(k) = p > 0$ . Define filtrations of  $V_{\mathbb{Z}}(\lambda)$  and  $V_k(\lambda)$  by the following: for  $i \geq 0$ , put

$$V_{\mathbb{Z}}^i = \{x \in V_{\mathbb{Z}}(\lambda) \mid T_\lambda(x) \in p^i H_{\mathbb{Z}}^0(\lambda)\}$$

and  $V_k^i = V_{\mathbb{Z}}^i \otimes_{\mathbb{Z}} k$ . One can restate (4.5.a) and (4.5.b) in these terms:

**Corollary 4.5.1.** *(The  $p$ -Sum Formula) One has  $V_k^0/V_k^1 \simeq L(\lambda)$ . The remaining terms of the filtration satisfy:*

$$(4.5.c) \quad \sum_{i > 0} \text{ch } V_k^i = \sum_{\alpha > 0} \sum_{0 < mp < \langle \lambda + \rho, \alpha^\vee \rangle} \nu_p(mp) \chi(s_{\alpha, mp} \cdot \lambda).$$

For a proof and more discussion of this result, see [10] II, Proposition 8.19.

*Remark 4.5.2.* If  $\sum_{i > 0} \text{ch } V_k^i = \text{ch } L(\mu)$  for some  $\mu \in X_+$ , then it is immediate that  $V_k^1 \simeq L(\mu)$  and  $V_k^i = 0$  for  $i \geq 2$ .



One situation where the general sum formula has been evaluated for *unbounded* rank is detailed in the following:

**Lemma 4.5.3.** *Let  $\Phi = C_\ell$  and let  $\lambda = \mathfrak{w}_s$  ( $1 \leq s \leq \ell$ ). One has:*

$$(4.5.d) \quad \nu^c(T_\lambda) = \sum_{j=0}^{i-1} \operatorname{div} \left( \frac{\ell + 1 - j - i}{i - j} \right) \chi(\mathfrak{w}_{2j}) \quad \text{if } s = 2i, \text{ and}$$

$$(4.5.e) \quad \nu^c(T_\lambda) = \sum_{j=0}^{i-1} \operatorname{div} \left( \frac{\ell - j - i}{i - j} \right) \chi(\mathfrak{w}_{2j+1}) \quad \text{if } s = 2i + 1.$$

*Proof.* These calculations are given in [9]; see especially pages 92–94 where the calculations are worked out in detail (for type  $D_\ell$  rather than  $C_\ell$ ). Also, there is a typographic error in the formula for type  $C_{\text{odd}}$ .  $\square$

*Remark 4.5.4.* We shall use these calculations later (Proposition 4.8.2) to fully describe the Weyl modules  $V(\mathfrak{w}_2)$  and  $V(\mathfrak{w}_3)$  for a group of type  $C_\ell$ . We may immediately apply Lemma 4.5.3 to deduce the simplicity of  $V(\mathfrak{w}_s)$  under the following conditions: when  $s = 4$  and  $p > \ell - 1$ ; when  $s = 5$ ,  $\ell = 5, 6$ , and  $p > 3$ ; when  $s = 6$ ,  $\ell = 6$ , and  $p > 3$ .

We shall now evaluate the general sum formula for a few other weights for unbounded rank. We utilize the following technical result. Recall that the *support* of an element  $z \in \mathbb{Z}\Phi$  is the set of simple roots  $\alpha_i$  so that  $k_i$  is non-zero when we write  $z = \sum_i k_i \alpha_i$  with  $k_i \in \mathbb{Z}$ .

**Lemma 4.5.5.** *Let  $\gamma$  be a regular weight and let  $w \in \mathcal{W}$ . Let  $I$  be the support  $\gamma - w\gamma$ . Then  $w \in \mathcal{W}_I$ .*

*Proof.* We argue by induction on  $l(w)$ ; the case  $l(w) = 0$  is trivial. Suppose that  $l(w) > 0$  and find a simple root  $\alpha_j$  so that  $w = w's_j$  where  $l(w) = l(w') + 1$  ( $s_j$  is the fundamental reflection associated to  $\alpha_j$ ). Let  $I'$  be the support of  $\gamma - w'\gamma$ ; we know by induction that  $w' \in \mathcal{W}_{I'}$ . Thus,  $w \in \mathcal{W}_{I' \cup \{\alpha_j\}}$ . It suffices to show that  $I' \subset I$  and  $\alpha_j \in I$ . Notice that  $s_j\gamma = \gamma - c_j\alpha_j$  where  $c_j = \langle \gamma, \alpha_j \rangle > 0$  since  $\gamma$  is regular. We have  $\gamma - w\gamma = \gamma - w's_j\gamma = \gamma - w'\gamma + c_j w'\alpha_j$ . Since  $l(w') < l(w's_j)$ , we know that  $w'\alpha_j > 0$ ; it follows that  $I$  contains  $I'$ . If  $I'$  contains  $\alpha_j$ , we are done. Otherwise, the support of  $w'\alpha_j$  contains  $\alpha_j$  since  $w \in \mathcal{W}'_I$ ; it follows that  $\alpha_j \in I$ . This completes the proof.  $\square$

**Lemma 4.5.6.** *Let  $\lambda > \mu$  be dominant weights. Let  $\alpha$  be a positive weight and suppose that  $\lambda - r\alpha \in \mathcal{W}.\mu$  for some  $0 < r < \langle \lambda + \rho, \alpha^\vee \rangle$ . Then  $\alpha$  and  $\lambda - \mu$  have the same support.*

*Proof.* Since  $\mu + \rho$  is dominant,  $\lambda + \rho - r\alpha \leq \mu + \rho \leq \lambda + \rho$ , hence  $r\alpha \geq \lambda - \mu$ . This proves that the support of  $\lambda - \mu$  is contained in the support of  $\alpha$ .

To obtain the other inclusion, let  $w \in \mathcal{W}$  satisfy  $\mu + \rho = w(\lambda + \rho - r\alpha)$ . Let  $I$  be the support of  $\lambda - \mu$  and set  $\mathcal{W}_I = \langle s_j \mid j \in I \rangle$ . Assume for the moment that  $w\alpha > 0$ . We have  $\lambda - \mu = \lambda + \rho - w(\lambda + \rho) + rw\alpha$ . Both  $(\lambda + \rho) - w(\lambda + \rho)$  and  $rw\alpha$  are positive. Since their sum has support in  $I$ , both terms have support in  $I$ .

Since  $\lambda + \rho$  is regular, we may apply Lemma 4.5.5 to deduce that  $w \in \mathcal{W}_I$ . We may now observe that since  $w\alpha$  has support in  $I$  and  $w \in \mathcal{W}_I$ ,  $\alpha$  must have support in  $I$ .

It now only remains to handle the case  $w\alpha < 0$ . In this case, we take  $r' = \langle \lambda + \rho, \alpha^\vee \rangle - r$ . We have  $\lambda + \rho - r'\alpha = s_\alpha(\lambda + \rho - r\alpha)$  so that  $\mu + \rho = ws_\alpha(\lambda + \rho - r'\alpha)$ ; the preceding result applied to  $r'$  and  $ws_\alpha$  now yields the claim.  $\square$

The proof of the preceding lemma was sketched to me by Jantzen in a personal communication; I include the argument here for completeness.

**Lemma 4.5.7.** *Let  $\lambda = \varpi_1 + \varpi_2$ . We have the following values for  $\nu^c(T_\lambda)$ :*

- (a)  $\text{div}(3)\chi(\varpi_3) + \text{div}(2\ell + 1)\chi(\varpi_1)$  when  $\Phi = C_\ell$ ,  $\ell \geq 3$ .
- (b)  $\text{div}(3)\chi(\varpi_3) + \text{div}(2\ell - 1)\chi(\varpi_1)$  when  $\Phi = D_\ell$ ,  $\ell \geq 5$ .
- (c)  $\text{div}(3)\chi(\varpi_3 + \varpi_4) + \text{div}(7)\chi(\varpi_1)$  when  $\Phi = D_4$ .
- (d)  $\text{div}(3)\chi(\varpi_3) + \text{div}(2\ell)\chi(\varpi_1) + \text{div}(2)\chi(2\varpi_1) + \text{div}(2)\chi(\varpi_2) - \text{div}(2)\chi(0)$  when  $\Phi = B_\ell$ ,  $\ell \geq 4$ .
- (e)  $\text{div}(3)\chi(2\varpi_3) + \text{div}(6)\chi(\varpi_1) + \text{div}(2)\chi(2\varpi_1) + \text{div}(2)\chi(\varpi_2) - \text{div}(2)\chi(0)$  when  $\Phi = B_3$ .

*Proof.* Let me first describe the calculation of  $\nu^c(T_\lambda)$  for type  $C_\ell$ . The only weights which are subdominant to  $\lambda$  are  $\varpi_1$  and  $\varpi_3$ . Let  $\mu \leq \lambda$  be a dominant weight. To compute the coefficient of  $\chi(\mu)$  in  $\nu^c(T_\lambda)$ , we apply (4.5.b); according to this formula, we must find each  $\alpha > 0$  and  $0 < r < \langle \lambda + \rho, \alpha^\vee \rangle$  for which  $\mu \in \mathcal{W}(\lambda - r\alpha)$ . Write  $\lambda + \rho - r\alpha = \sum_i a_i \varepsilon_i$ ,  $\mu + \rho = \sum_i b_i \varepsilon_i$  with  $a_i, b_i \in \mathbb{N}$ , and let  $A = \{|a_i|\}$ ,  $B = \{|b_i|\}$ . Since  $\mathcal{W}$  acts by permutations and sign changes of the  $\varepsilon_i$ , we have  $\mu + \rho \in \mathcal{V}(\lambda + \rho - r\alpha)$  if and only if  $A = B$ . Suppose first that  $\mu = \varpi_3$ . Then  $\lambda - \mu = \alpha_1 + \alpha_2$ . Lemma 4.5.6 shows that there are only 3 possibilities for  $\alpha$ ; it is straightforward to argue that  $A \neq B$  unless  $\alpha = \alpha_1 + \alpha_2$  and  $r = 1, 3$  (and that conjugacy holds in these cases). When  $\mu = \varpi_1$ , we have  $\lambda - \mu = \varpi_2 = \varepsilon_1 + \varepsilon_2$ ; the support of this root is  $\Delta$ . According to Lemma 4.5.6, the possibilities for  $\alpha$  are  $\varepsilon_1 + \varepsilon_j$  for  $1 \leq j \leq \ell$ . Note that in this case  $B = \{\ell + 1, \ell, \ell - 2, \ell - 3, \dots, 2, 1\}$ . When  $j > 2$ , we must have  $r = 1$  in order that  $(\ell + 1) \in A$ , but then  $(\ell - j + 1) \notin A$ . When  $j = 1$ , note that  $a_0 = \ell + 2 - 2r$ ; in particular, we never get  $\ell + 1 \in A$  for any  $r$ . When  $j = 2$ , it is easy to see that  $\lambda - r\alpha \in \mathcal{W} \cdot \mu$  if and only if  $r = 1, 2\ell + 1$ . The formula given above is now verified.

The calculation for type  $D_\ell$  is very similar to that for  $C_\ell$ ; we omit the details here. The calculation for  $B_\ell$  is somewhat more involved. We content ourselves here with listing those  $\alpha$  which make a non-zero contribution to  $\nu^c(T_\lambda)$ ; they are as follows:

Table 4.5.1. Sum Formula Contributions.

$\alpha$	$n_\alpha = \langle \lambda + \rho, \alpha^\vee \rangle$	$\sum_{j=1}^{n_\alpha-1} \text{div}(j)\chi(\lambda - j\alpha)$
$\alpha_1 + \alpha_2$	4	$\text{div}(3)\chi(\varpi_3)$
$\varepsilon_1$	$2\ell + 3$	$\text{div}(2\ell + 2)\chi(\varpi_2) + \text{div}\left(\frac{3}{2\ell}\right)\chi(0)$
$\varepsilon_1 + \varepsilon_2$	$2\ell + 1$	$\text{div}(2\ell)\chi(\varpi_1)$
$\varepsilon_1 + \varepsilon_{\ell-1}$	$\ell + 3$	$\text{div}\left(\frac{\ell}{3}\right)\chi(0)$
$\varepsilon_1 + \varepsilon_\ell$	$\ell + 2$	$\text{div}\left(\frac{1}{\ell + 1}\right)\chi(\varpi_2)$
$\varepsilon_2$	$2\ell - 1$	$\text{div}(2\ell - 2)\chi(2\varpi_1)$
$\varepsilon_2 + \varepsilon_\ell$	$\ell$	$\text{div}\left(\frac{1}{\ell - 1}\right)\chi(2\varpi_1)$

One may easily reconstruct the formula for  $\nu^c(T_\lambda)$  from this list. □

*Remark 4.5.8.* J. Jantzen has written a computer implementation of an algorithm which evaluates the sum formula (for fixed and reasonably small  $\ell$ ). This algorithm computes  $\nu^c(T_\lambda)$  as a  $D$ -linear combination of the characters  $\chi(\mu)$  for  $\mu \in X_+$ . Let me briefly describe this algorithm. One wishes to use equation (4.5.b); thus, one is required to express the various summands  $\chi(\lambda - j\alpha)$  in terms of characters corresponding to dominant weights. For this, one uses Lemma 4.3.1. To use the conditions in this lemma, one only needs to compute the conjugate  $\mu \in \mathcal{W}(\lambda - j\alpha) \cap \mathcal{D}$ . To find  $\mu$ , one must find the *dominant* conjugate of  $\lambda - j\alpha + \rho$  under the ordinary action of  $\mathcal{W}$ ; this dominant conjugate is then  $\mu + \rho$ . Using the construction of the root systems described in [3] in terms of the Euclidean basis  $\varepsilon_i$ , and the action of the Weyl group on the  $\varepsilon_i$ , however, this computation is straightforward to implement. For example, when  $\Phi = A_\ell$ , any weight  $x$  has the form  $\sum_{i=1}^{\ell+1} x_i \varepsilon_i$  ( $x_i \in \mathbb{Z}$ ). The condition for  $x$  to be dominant is that  $x_1 \geq x_2 \geq \cdots \geq x_\ell \geq x_{\ell+1}$ . Since the Weyl group is isomorphic to  $\text{Sym}_{\ell+1}$  and acts by permuting the indices of the  $\varepsilon_i$ , it is clear that a sorting algorithm can be used to compute the dominant conjugate of  $x$ . Similar techniques work for each (classical) type of root system.

We apply this algorithm to obtain the computations of  $\nu = \nu^c(T_\lambda)$  given below in table 4.5.2. In reading table 4.5.2, one must interpret  $\omega_0$  and  $\omega_{\ell+1}$  as 0. To obtain the dimension assertions in table 4.5.2, one applies the Weyl degree formula.

*Remark 4.5.9.* In general, the calculations in table 4.5.2 should only be regarded as valid for the ranks  $\ell$  specified. Let me remark that for  $\lambda$  either  $2\omega_2$  or  $2\omega_1 + \omega_2$  when  $\Phi = A_\ell$ , the above computations for  $\nu^c(T_\lambda)$  are valid for all  $\ell \geq 3$ . Jantzen has pointed out to me the following argument. First, the more general computations follow from those given provided that each weight  $\mu$  subdominant to  $\lambda$  has the property that  $\lambda - \mu$  lies in the root subsystem generated by  $\alpha_1, \alpha_2, \alpha_3$ . Second, one can check this latter fact first for the minimal weight  $\omega_4 \leq \lambda$ ; the more general statement follows from this at once.

Table 4.5.2. Bounded rank sum formula calculations.

$\Phi$	$\lambda$ $\nu^c(T_\lambda)$	$\dim_k V(\lambda)$
$A_\ell$ $3 \leq \ell \leq 5$	$2\mathfrak{w}_2$ $\text{div}(2)\chi(\mathfrak{w}_1 + \mathfrak{w}_3) - \text{div}(2)\chi(\mathfrak{w}_4) + \text{div}(3)\chi(\mathfrak{w}_4)$	$(1/3) \binom{\ell+2}{2} \binom{\ell+1}{2}$
$A_\ell$ $\ell = 3, 4, 5$	$2\mathfrak{w}_1 + \mathfrak{w}_2$ $\text{div}(2)(\chi(2\mathfrak{w}_2) + 2\chi(\mathfrak{w}_1 + \mathfrak{w}_3) - \chi(\mathfrak{w}_4))$	45, 105, 210
$A_2$	$3\mathfrak{w}_1 + \mathfrak{w}_2$ $\text{div}(3)\chi(\mathfrak{w}_1 + 2\mathfrak{w}_2) + \text{div}(5)\chi(2\mathfrak{w}_1)$	24
$A_4$	$\mathfrak{w}_2 + \mathfrak{w}_3$ $\text{div}(2)\chi(0) + \text{div}(3)\chi(\mathfrak{w}_1 + \mathfrak{w}_4)$	75
$B_\ell$ $\ell = 3, 4, 5, 6$	$\mathfrak{w}_1 + \mathfrak{w}_\ell$ $\text{div}(2\ell + 1)\chi(\mathfrak{w}_\ell)$	48, 128, 320, 760
$B_\ell$ $\ell = 3, 4$	$2\mathfrak{w}_\ell$ $\text{div}(2) \left( \sum_{j=0}^{\ell} \chi(\mathfrak{w}_j) \right)$	
$B_\ell$ $\ell = 3, 4$	$\mathfrak{w}_2 + \mathfrak{w}_\ell$ $\text{div}(\ell)\chi(\mathfrak{w}_\ell) + \text{div}(2\ell - 1)\chi(\mathfrak{w}_1 + \mathfrak{w}_\ell)$	112, 432
$C_\ell$ $\ell = 2, 3$	$2\mathfrak{w}_2$ $\text{div}(2)\chi(2\mathfrak{w}_1) + \text{div}(2)\chi(\mathfrak{w}_2) + \text{div}(5/2)\chi(0)$ ( $\ell = 2$ ) $\text{div}(2)(\chi(\mathfrak{w}_1 + \mathfrak{w}_3) + \chi(2\mathfrak{w}_1) + 2\chi(\mathfrak{w}_2)) + \text{div}(7/2)\chi(0)$ ( $\ell = 3$ )	14, 90
$C_3$	$\mathfrak{w}_1 + \mathfrak{w}_3$ $\text{div}(2)(\chi(2\mathfrak{w}_1) + \chi(\mathfrak{w}_2) + \chi(0)) + \text{div}(3)(\chi(\mathfrak{w}_2) - \chi(0))$	70
$D_5$	$2\mathfrak{w}_4$ $\text{div}(2)(\chi(\mathfrak{w}_3) + \chi(\mathfrak{w}_1))$	126
$D_5$	$\mathfrak{w}_4 + \mathfrak{w}_5$ $\text{div}(2)\chi(\mathfrak{w}_2) + \text{div}(2)\chi(0)$	210
$E_7$	$2\mathfrak{w}_7$ $\text{div}(2)\chi(\mathfrak{w}_6) + \text{div}(4)\chi(0) + \text{div}(3)\chi(\mathfrak{w}_1)$	1463

**4.6. Modules with “Good Filtrations”.** Let  $M$  be a rational  $G$  module. For simplicity, we shall always assume  $M$  to be finite dimensional even though this restriction is not always needed. A *filtration* of  $M$  is just an ascending chain of submodules  $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r = M$  for some  $r \in \mathbb{N}$ . We say that  $M$  has a  $V$ -filtration provided  $M$  has a filtration as above so that for each  $i = 1, 2, \dots, r$  there is a weight  $\lambda_i \in X_+$  with  $M_i/M_{i-1} \simeq V(\lambda_i)$ ;  $M$  has an  $H$ -filtration provided instead that  $M_i/M_{i-1} \simeq H^0(\lambda_i)$ . In the literature, an  $H$ -filtration is sometimes referred to as a “good filtration.”

Recall that the characters  $\chi(\lambda)$  for  $\lambda \in X_+$  constitute a  $\mathbb{Z}$  basis for the  $\mathcal{W}$  invariants of  $\mathbb{Z}[X]$ ; see [10] Lemma II.5.8 (and the remark following that lemma). Thus, one can uniquely represent  $\text{ch } M = \sum_{\lambda} m_{\lambda} \chi(\lambda)$  where  $M$  is a rational  $G$  module and  $m_{\lambda}$  are integers. It follows that one can read off the factors occurring in a module  $M$  with either a  $V$ - or an  $H$ -filtration from knowledge of  $\text{ch } M$ .

We shall be interested in the situation where  $M$  satisfies the following special hypothesis with respect to a dominant weight  $\tau$ :

- †
- (1)  $M$  has a  $V$ -filtration.
  - (2)  $M$  has an  $H$ -filtration.
  - (3)  $\chi(\tau)$  occurs with multiplicity 1 in  $\text{ch } M$ .
  - (4)  $\tau$  is the *highest weight* of  $M$ ; i.e. any dominant weight  $\sigma$  with  $M_\sigma \neq 0$  satisfies  $\sigma \leq \tau$ .

*Remark 4.6.1.* Applying the result contained in the last paragraph of [10] II.2.14, one obtains immediately the following:

$$\text{Ext}_G^1(V(\tau), V(\mu)) = 0 \text{ whenever } \mu \not\geq \tau.$$

Using this result, it is easy to see that if the  $G$ -module  $M$  satisfies (1) and (4) of the hypothesis †, then there is an injection  $V(\tau) \hookrightarrow M$ .

Dualizing, we have immediately the corresponding result for the induced modules:

$$\text{Ext}_G^1(H^0(\mu), H^0(\tau)) = 0 \text{ whenever } \mu \not\geq \tau.$$

This result implies that if the  $G$ -module  $M$  satisfies (2) and (4) of †, then there is a surjection  $M \rightarrow H^0(\tau)$ .

**Proposition 4.6.2.** (*Tensor Products of Weyl and induced modules*) Fix weights  $\lambda, \mu \in X_+$  and let  $\tau = \lambda + \mu$ . Put  $V_{\lambda, \mu} = V(\lambda) \otimes V(\mu)$  and  $H_{\lambda, \mu} = H^0(\lambda) \otimes H^0(\mu)$ .

- (a) Let  $m_\lambda(\sigma) = \dim_k V(\mu)_\sigma$  for  $\sigma \in X$ . Then  $\text{ch } V_{\lambda, \mu} = \sum_{\sigma \in \Pi(\lambda)} m_\lambda(\sigma) \chi(\mu + \sigma)$ . (Refer to section 2.1 for the definition of  $\chi(?)$ .)
- (b)  $V_{\lambda, \mu}$  satisfies (1), (3), and (4) of hypothesis † for the weight  $\tau$ .
- (c)  $H_{\lambda, \mu}$  satisfies (2), (3), and (4) of hypothesis † for the weight  $\tau$ .
- (d) Suppose that  $V(\lambda)$  and  $V(\mu)$  are simple. Then  $V_{\lambda, \mu} = H_{\lambda, \mu}$  satisfies † for  $\tau$ .

*Proof.* For part (a), the equality  $\text{ch } V_{\lambda, \mu} = \text{ch } H_{\lambda, \mu}$  is clear because the computation can be done in characteristic 0 where  $V(?)$  and  $H^0(?)$  coincide. It is also evident that these characters are given by  $\chi(\lambda) \cdot \chi(\mu)$  where the multiplication occurs in the ring  $\mathbb{Z}[X]$ . The formula for multiplying these characters is given in [8], exercise 9 of §24.4 (Lemma 4.3.1 is useful in translating from Humphreys' notation).

Part (a) shows that (3) and (4) of hypothesis † hold for the modules  $V_{\lambda, \mu}$  and  $H_{\lambda, \mu}$ . Furthermore, the validity of (1) of hypothesis † for every  $H_{\lambda, \mu}$  implies the validity of (2) of † for every  $V_{\lambda, \mu}$  by duality. The validity of part (1) of hypothesis † for  $H_{\lambda, \mu}$  is a result due to S. Donkin, O. Mathieu, and J.-P. Wang; we cite [5] and [16] for a complete proof.

Part (d) follows from (b) and (c) together with the observations that, in this case,  $V(\lambda) = H^0(\lambda)$  and  $V(\mu) = H^0(\mu)$ .  $\square$

**Lemma 4.6.3.** Suppose that  $M$  satisfies either (1) or (2) of hypothesis † for the weight  $\tau$ , assume that  $M^* \simeq M$  as  $G$ -modules, and suppose that  $\tau = \tau^*$ . Then  $M$  satisfies both (1) and (2) of hypothesis †.

*Proof.* The condition  $\tau = \tau^*$  shows that  $H^0(\tau)^* = V(\tau)$  (and vice versa). The result follows by dualizing.  $\square$

**Lemma 4.6.4.** (a) *Let  $M$  and  $N$  be finite dimensional rational  $G$ -modules. Suppose that  $N$  has a filtration with factor modules  $P_i$  ( $i = 1, 2, \dots, r$ ), and assume that  $\text{Ext}_G^1(M, P_i) = \text{Ext}_G^1(P_i, M) = 0$ . Then  $\text{Ext}_G^1(M, N) = \text{Ext}_G^1(N, M) = 0$ .*  
 (b) *Let  $V$  have a  $V$ -filtration and let  $H$  have an  $H$ -filtration. Then  $\text{Ext}_G^1(V, H) = 0$ . In particular, if  $M$  and  $N$  each have a  $V$ -filtration and an  $H$ -filtration, then  $\text{Ext}_G^1(M, N) = \text{Ext}_G^1(N, M) = 0$ .*

*Proof.* Part (a) is straightforward. For (b), first observe that, according to Proposition II.4.12 of [10], we have

$$\text{Ext}_G^1(V(\lambda), H^0(\mu)) = 0$$

for every  $\lambda, \mu \in X_+$ . Part (b) now follows from two applications of (a).  $\square$

*Definition 4.6.5.* Let  $N$  be a rational  $G$ -module. The notation  $N = L(\mu_1) \mid L(\mu_2) \mid \cdots \mid L(\mu_r)$  indicates that  $N$  has a composition series  $0 = N_0 \subset N_1 \subset \cdots \subset N_r = N$  so that  $N_i/N_{i-1} \simeq L(\mu_i)$  for  $i = 1, 2, \dots, r$ . Note that, in general,  $N$  is not uniquely determined by the sequence  $\{\mu_i\}$ . We write  $N = I(\mu_1, \mu_2, \dots, \mu_r)$  to indicate that  $N$  is *indecomposable* and has a composition series as above.

**Proposition 4.6.6.** *Let  $M$  be a rational  $G$  module satisfying hypothesis  $\dagger$  for the weight  $\tau$ .*

- (a) *If  $V(\tau)$  is simple, then  $V(\tau)$  is a split summand of  $M$ .*
- (b) *If  $\text{rad } V(\tau) = V(\sigma)$  where  $V(\sigma)$  is simple, then  $M$  has an indecomposable summand  $N = I(\sigma, \tau, \sigma)$ . Furthermore,  $\text{rad } N = V(\tau)$ .*

*Proof.* By hypothesis  $\dagger$  and remark 4.6.1, we have an injection  $\phi : V(\tau) \rightarrow M$  and a surjection  $\psi : M \rightarrow H^0(\tau)$ . By (2.1.b), one knows that  $\text{Hom}_G(V(\tau), H^0(\tau)) \simeq k$ . Property (4) of  $\dagger$  guarantees that the multiplicity of  $\chi(\tau)$  in  $\text{ch } M$  is the same as the multiplicity with which  $L(\tau)$  occurs as a composition factor in  $M$ . By (3), the coefficient of  $\chi(\tau)$  in  $M$  is 1; there is thus only one such composition factor. One now deduces that  $\psi \circ \phi \neq 0$ , so  $\psi \circ \phi$  is determined up to (non-0)scalar.

For (a), one has  $L(\tau) = V(\tau) = H^0(\tau)$ , and one observes that  $\psi$  provides a splitting for the sequence  $0 \rightarrow V(\lambda) \xrightarrow{\phi} M$ , so that  $M \simeq V(\lambda) \oplus N$ .

For (b), one notes that since  $H^0(\tau)$  is a top  $H$ -filtration factor of  $M$ ,  $\ker \psi$  has an  $H$ -filtration. Similarly,  $\text{coker } \phi$  has a  $V$ -filtration.

Let

$$Q = \ker \psi / \text{rad } V(\tau) = \ker \psi / V(\sigma)$$

Since  $V(\sigma) = H^0(\sigma)$ , it follows from [10] Corollary II.4.7 that  $Q$  has an  $H$ -filtration.

The map  $M \xrightarrow{\psi} H^0(\tau) \rightarrow H^0(\tau) / \text{soc } H^0(\tau) = V(\sigma)$  factors through the image of  $\phi$  giving a map  $\text{coker } \phi \rightarrow V(\sigma)$ . One then checks that

$$V(\sigma) \rightarrow \ker \psi \rightarrow \text{coker } \phi \rightarrow V(\sigma)$$

is exact. It follows that  $Q \simeq \ker(\text{coker } \phi \rightarrow V(\sigma))$ . The dual to the corollary just invoked therefore shows that  $Q$  has a  $V$ -filtration as well.

It is clear that  $M$  has a filtration with the following factors:

$$V(\sigma) \mid Q \mid L(\tau) \mid V(\sigma)$$

Lemma 4.6.4 shows that  $\text{Ext}_G^1(Q, V(\sigma)) = \text{Ext}_G^1(V(\sigma), Q) = 0$ . Let us denote by  $S$  the section  $Q \mid L(\tau)$ ;  $Q$  will be a summand of  $M$  if we can show that  $S \simeq Q \oplus L(\tau)$ .

Let  $\hat{\psi} : M \rightarrow H^0(\tau)/\text{soc } H^0(\tau) = V(\sigma)$  be the map obtained from  $\psi$ . Then  $S$  is determined as  $\ker \hat{\psi}/\text{rad } V(\tau) = \ker \hat{\psi}/V(\sigma)$ . In particular, the map  $\phi : V(\tau) \rightarrow M$  induces a map  $\hat{\phi} : L(\tau) \rightarrow S$ . The map  $\hat{\phi}$  then provides a section to the sequence

$$0 \longrightarrow Q \longrightarrow S \longrightarrow L(\tau) \longrightarrow 0$$

so that the claim on  $S$  follows.

We now have the requisite summand  $N = L(\sigma) \mid L(\tau) \mid L(\sigma)$  of  $M$ . To see that  $N$  is indecomposable, first note that it is enough to show that  $\text{soc } N$  is simple. Observe that we have a surjection  $N \rightarrow H^0(\tau)$ . Given any module epimorphism  $A \xrightarrow{f} B$ , one knows that  $\text{soc } A = \text{soc}(f^{-1}(\text{soc } B))$ ; in our situation, this shows that  $\text{soc } N = \text{soc } V(\tau) = L(\sigma)$  which is simple as desired.  $\square$

*Remark 4.6.7.* In the literature, modules with an  $H$ -filtration are sometimes said to have a *good* filtration. A module  $M$  satisfying (1) and (2) of  $\dagger$  is sometimes called a (partial) tilting module.

S. Donkin [6] and others have studied tilting modules. For example, in [6] Theorem 1.1, Donkin shows that results of Ringel [19] yield the following:

For any  $\lambda \in X_+$ , there is an essentially unique indecomposable tilting module  $T(\lambda)$  satisfying  $\dagger$ . Any module  $M$  satisfying (1) and (2) of  $\dagger$  is a direct sum of modules  $T(\mu)$  for various dominant weights  $\mu$ .

Thus, Proposition 4.6.6 describes the modules  $T(\tau)$  for those dominant weights  $\tau$  with the property that  $\text{rad } V(\tau)$  is simple.

We present various tools for understanding the structure of the Weyl module, and then we apply these tools together with the above filtration theory to several situations of interest.

**Proposition 4.6.8.** *Assume that  $G$  has a root system of type  $A_\ell$ , let  $1 \leq i < j \leq \ell$ , and let  $a, b \in \mathbb{N}$ . Put  $\lambda = a\varpi_i + b\varpi_j$  and  $\mu = \varpi_{i-1} + (a-1)\varpi_i + (b-1)\varpi_j + \varpi_{j+1}$  where  $0 = \varpi_0 = \varpi_{\ell+1}$ . One has*

$$(4.6.a) \quad \begin{aligned} \dim_k V(\lambda)_\mu &= j - i + 1, & \text{and} \\ \dim_k L(\lambda)_\mu &= \begin{cases} j - i + 1 & \text{if } a + b + j - i \not\equiv 0 \pmod{p} \\ j - i & \text{if } a + b + j - i \equiv 0 \pmod{p} \end{cases} \end{aligned}$$

*Proof.* This is Lemma 8.6 of [21].  $\square$

Fix  $I \subset \Delta$ , and form the parabolic subgroup  $P = P_I$ . Fix all notation as in 1.4. We are interested in restricting a simple module  $L(\lambda)$  for  $G$  to  $L'$ .

**Lemma 4.6.9.** *(The parabolic argument) Let  $\mu, \lambda \in X_+$  satisfy  $\mu \leq \lambda$ . Suppose that  $\lambda - \mu$  is contained in  $\mathbb{Z}_{\geq 0}\Phi_I$ . Let  $V$  denote the Weyl module of high weight  $\lambda \mid_{T'}$  for  $L'$ , and let  $W$  denote the Weyl module  $V(\lambda)$ . Write  $\mu'$  for  $\mu \mid_{T'}$ . Then  $\dim_k W_\mu = \dim_k V_{\mu'}$  and  $\dim_k(\text{rad } W)_\mu = \dim_k(\text{rad } V)_{\mu'}$ .*

*Proof.* This follows from [21] (2.1) and (2.3).  $\square$

**Proposition 4.6.10.** *Let  $\Phi = A_\ell$ . For dominant weights  $\lambda$  and  $\mu$ , let  $M = V_{\lambda, \mu}$ . Write  $\varpi_0 = 0$ .*

- (a) *Fix  $1 \leq i \leq \ell$ . Put  $\lambda = \varpi_\ell$  and  $\mu = \varpi_i$ .  
If  $\ell - i + 2 \equiv 0 \pmod{p}$ ,  $M = I(\varpi_{i-1}, \varpi_i + \varpi_\ell, \varpi_{i-1})$  and  $\text{rad } M = V(\varpi_i + \varpi_\ell)$ .  
Otherwise,  $M \simeq V(\varpi_i + \varpi_\ell) \oplus V(\varpi_{i-1})$  and  $V(\varpi_i + \varpi_\ell)$  is simple.*
- (b) *Let  $\lambda = 2\varpi_1$  and  $\mu = \varpi_\ell$ , and suppose that  $p > 2$ .  
If  $\ell + 2 \equiv 0 \pmod{p}$ ,  $M = I(\varpi_1, 2\varpi_1 + \varpi_\ell, \varpi_1)$  and  $\text{rad } M = V(2\varpi_1 + \varpi_\ell)$ .  
If  $\ell + 2 \not\equiv 0 \pmod{p}$ ,  $M \simeq V(2\varpi_1 + \varpi_\ell) \oplus V(\varpi_1)$  and  $V(2\varpi_1 + \varpi_\ell)$  is simple.*

*Proof.* Note in both situations Proposition 4.2.2 together with part (d) of Proposition 4.6.2 show that hypothesis  $\dagger$  is valid for  $M$  (for the weight  $\tau = \lambda + \mu$ ). For (a), we first apply Proposition 4.6.2(a) to the weights  $\lambda$  and  $\mu$ . Observe that  $\Pi(\lambda) = \{-\varepsilon_s \mid 1 \leq s \leq \ell + 1\}$  and that  $m_\lambda(-\varepsilon_s) = 1$  for all  $s$ . It is clear that  $-\varepsilon_s + \varpi_i = \varpi_i + \varpi_s - \varpi_{s+1} \in \mathcal{D}$  for  $1 \leq i \leq \ell + 1$  (where  $\varpi_{\ell+1} = 0$ ); it is also clear the  $-\varepsilon_s + \varpi_i$  is not dominant unless  $i = s + 1$  or  $s = \ell$ . Proposition 4.6.2, (a) together with Lemma 4.3.1 then gives  $\text{ch } M = \chi(\lambda + \mu) + \chi(\varpi_{i-1})$ . Observe that the only weight subdominant to  $\varpi_i + \varpi_\ell$  is  $\varpi_{i-1}$ . Proposition 4.6.8 shows that  $\text{rad } V(\varpi_i + \varpi_\ell) = V(\varpi_i)$  if  $\ell - i + 2 \equiv 0 \pmod{p}$ , and  $V(\varpi_i + \varpi_\ell)$  is simple otherwise. (a) now follows immediately from Proposition 4.6.6.

For (b), we again apply Proposition 4.6.2(a); essentially the same argument as above yields  $\text{ch } M = \chi(\lambda + \mu) + \chi(\varpi_1)$  in this case. Observe that the weights subdominant to  $2\varpi_1 + \varpi_\ell$  are  $\varpi_1$  and  $\varpi_2 + \varpi_\ell$ . Since  $p \neq 2$ , we have

$$\dim_k V(2\varpi_1 + \varpi_\ell)_{\varpi_2 + \varpi_\ell} = \dim_k L(2\varpi_1 + \varpi_\ell)_{\varpi_2 + \varpi_\ell} = 1.$$

In particular,  $\text{rad } V(2\varpi_1 + \varpi_\ell)$  must be  $L(\varpi_1)$  isotypic. Proposition 4.6.8 now shows that

$$\text{rad } V(2\varpi_1 + \varpi_\ell) = L(\varpi_1)$$

when  $\ell + 2 \equiv 0 \pmod{p}$ , and that  $V(2\varpi_1 + \varpi_\ell)$  is simple otherwise. (b) now follows from Proposition 4.6.6.  $\square$

*Remark 4.6.11.* Let  $G$  be as in the preceding proposition. Let  $d_\zeta = \dim_k V(\zeta)$ . In order to compute the dimensions of  $L(\lambda + \mu)$  for  $\lambda, \mu$  as in the previous lemma, one only needs to know the following dimensions:  $d_{\varpi_\ell} = \ell + 1$ ,  $d_{\varpi_j} = \binom{\ell + 1}{j}$ ,  $d_{2\varpi_1} = \binom{\ell + 2}{2}$ .

**Proposition 4.6.12.** *Assume that  $\Phi = D_\ell$ . Let  $M = V_{\varpi_1, \varpi_\ell}$ . Let  $\tau = \varpi_1 + \varpi_\ell$  and  $\sigma = \varpi_{\ell-1}$ . If  $\ell \equiv 0 \pmod{p}$ ,  $M = I(\sigma, \tau, \sigma)$  and  $\text{rad } M = V(\tau)$ . Otherwise,  $M \simeq V(\sigma) \oplus V(\tau)$ , and  $V(\tau)$  is simple.*

*Proof.* Note that  $\sigma$  is the only weight subdominant to  $\tau$ . We apply Proposition 4.6.9 with  $I = \Delta \setminus \{\alpha_{\ell-1}\}$ ; the resulting parabolic subgroup has type  $A_{\ell-1}$ . As in that proposition, let  $V$  be the Weyl module for  $L'$  with high weight  $\tau|_{T'}$ . According to (a) of Proposition 4.6.10,  $(\text{rad } V)_0$  has dimension 1 when  $\ell \equiv 0 \pmod{p}$ , and has dimension 0 otherwise. It follows from Proposition 4.6.9 that  $\text{rad } V(\tau) \simeq V(\varpi_{\ell-1})$  when  $\ell \equiv 0 \pmod{p}$ , and  $\text{rad } V(\tau) = 0$  otherwise.

According to 4.2.2, the modules  $V(\varpi_1)$  and  $V(\varpi_\ell)$  are simple. According to (d) of Proposition 4.6.2, hypothesis  $\dagger$  is valid. One calculates using (a) of Proposition 4.6.2 the character of  $M$ ; one obtains  $\text{ch } M = \chi(\sigma) + \chi(\tau)$ . The proposition now follows from Proposition 4.6.6.  $\square$

*Remark 4.6.13.* Let  $G$  be as in the preceding result. In order to compute the dimensions of  $L(\varpi_1 + \varpi_\ell)$ , one only needs to know the following dimensions:  $d_{\varpi_\ell} = d_{\varpi_{\ell-1}} = 2^{\ell-1}$ ,  $d_{\varpi_1} = 2\ell$ .



**4.7. Orthogonal Symmetric Powers.** The symmetric powers of the natural module for the orthogonal groups are closely related to a certain Weyl module. Let  $\Omega = \Omega(V)$  be a group of type  $B_\ell$  or  $D_\ell$  as in section 1.5; in particular, we assume that  $p \neq 2$ .

The invariant form on  $V$  gives an isomorphism  $V \simeq V^*$  as  $\Omega$  modules. *A priori*, the  $\Omega$  invariant quadratic form  $\phi$  on  $V$  is an element of  $S^2(V^*)$ . However, we may and shall choose to identify  $S^2(V^*)$  with  $S^2V$ ; this being done, we let  $Q \in S^2V$  denote the vector identified with  $\phi$ . Section 1.5 describes a basis of  $V$ ; in the odd dimensional case, we normalize the element  $u$  of this basis so that  $(u, u) = 1$ , i.e.  $\phi(u) = \frac{1}{2}$ . Expressed in this basis, we have  $Q = \sum_{i=1}^{\ell} e_i \cdot e_{-i}$  when  $V$  is even dimensional, and  $Q = \sum_{i=1}^{\ell} e_i \cdot e_{-i} + \frac{1}{2}u^2$  when  $V$  is odd dimensional.

We shall exploit a certain invariant bilinear form on the modules  $S^rV$ ; this form is described by the following lemma.

**Lemma 4.7.1.** *Fix  $r \in \mathbb{N}$  with  $p > r$ .*

(a) *There is an  $\mathrm{SL}(V)$ -linear (and hence  $\Omega$ -linear) splitting  $s$  of the exact sequence*

$$(4.7.a) \quad \begin{array}{ccccc} V^{\otimes r} & \longrightarrow & S^rV & \longrightarrow & 0 \\ & & \swarrow^s & & \end{array}$$

*given by  $s(v_1v_2 \cdots v_r) = \frac{1}{r!} \sum_{\tau \in \mathrm{Sym}_r} v_{\tau(1)} \otimes v_{\tau(2)} \otimes \cdots \otimes v_{\tau(r)}$  for  $v_1, v_2, \dots, v_r \in V$ .*

(b) *There is an isomorphism  $S^rV \simeq (S^rV)^*$  as  $\Omega$  modules.*

(c) *The restriction of the product form on  $V^{\otimes r}$  gives a nondegenerate  $\Omega$  invariant form  $\kappa$  on  $S^rV$ .*

*Proof.* For (a), [4] Proposition (12.3) shows that the rule above determines a well-defined splitting  $s$ . One must also verify that  $s$  commutes with the action of  $\mathrm{SL}(V)$ .

As to (b), let us recall the  $\mathrm{SL}(V)$  invariant perfect pairing  $V \times V^* \rightarrow k$ . We have then the product pairing  $V^{\otimes r} \times (V^*)^{\otimes r} \rightarrow k$  which is again perfect and  $\mathrm{SL}(V)$  invariant. By (a), we can consider  $S^rV$  and  $S^r(V^*)$  as subspaces of  $V^{\otimes r}$  and  $(V^*)^{\otimes r}$ . We can therefore restrict the product pairing to form a pairing  $S^rV \times S^r(V^*) \rightarrow k$ ; one checks that this form is non-zero. Proposition 4.2.2 shows that  $S^rV$  and  $S^rV^*$  are simple  $\mathrm{SL}(V)$  modules. It follows that the restriction of the product pairing to  $S^rV \times S^r(V^*)$  is perfect; i.e.  $(S^rV)^* \simeq S^r(V^*)$  as  $\mathrm{SL}(V)$  modules. Since the original form  $\beta$  on  $V$  gives an  $\Omega$  isomorphism  $V \xrightarrow{f_\beta} V^*$ , and hence an  $\Omega$  isomorphism  $S^rV \xrightarrow{S^r(f_\beta)} S^r(V^*)$ , (b) now follows.

We obtain (c) by observing that the product form is obtained as the composition

$$V^{\otimes r} \times V^{\otimes r} \xrightarrow[\simeq]{1 \times (f_\beta)^{\otimes r}} V^{\otimes r} \times (V^*)^{\otimes r} \rightarrow k.$$

The proof of (b) shows that this pairing is  $\Omega$  invariant and non-degenerate. □

The relationship between  $V(r\mathfrak{w}_1)$  and  $S^rV$  is determined by the following:

**Proposition 4.7.2.** *Fix  $r \in \mathbb{N}_{\geq 2}$ .*

(a) *There is an exact sequence*

$$0 \longrightarrow S^{r-2}V \xrightarrow{\gamma_Q} S^rV \longrightarrow H^0(r\mathfrak{w}_1) \longrightarrow 0$$

where  $\gamma_Q$  is multiplication by  $Q$ .

(b) *The module  $S^rV$  has an  $H$ -filtration with  $H^0(r\mathfrak{w}_1)$  occurring as the top factor.*

(c) *Assume that  $r < p$ . Then  $S^rV$  has a  $W$ -filtration with  $V(r\mathfrak{w}_1)$  occurring as the bottom factor.*

*Proof.* Both assertions of (a) are proved in [10], II.2.18; (b) follows immediately. Since  $r < p$ , Lemma 4.7.2 shows that  $S^rV$  is self dual. (c) now follows from Lemma 4.6.3.  $\square$

**Proposition 4.7.3.** *Let  $p > 2$ . If  $\dim_k V = 2\ell$ , let  $c = \ell$ ; if  $\dim_k V = 2\ell + 1$ , let  $c = 2\ell + 1$ . If  $c \not\equiv 0 \pmod{p}$ , then  $S^2V \simeq V(2\mathfrak{w}_1) \oplus V(0)$  and  $V(2\mathfrak{w}_1)$  is simple. If  $c \equiv 0 \pmod{p}$ , then  $S^2V = I(0, 2\mathfrak{w}_1, 0)$  and  $\text{rad } S^2V = V(2\mathfrak{w}_1)$ .*

*Proof.* According to (a) of Proposition 4.7.2,  $\text{ch } S^2V = \chi(2\mathfrak{w}_1) + \chi(0)$ ; parts (b) and (c) guarantee that hypothesis  $\dagger$  holds. In order to apply Proposition 4.6.6, we need to understand  $\text{rad } V(2\mathfrak{w}_1)$ . However, note that in this case,  $\text{rad } V(2\mathfrak{w}_1)$  is in the kernel of the map  $S^2V \rightarrow H^0(2\mathfrak{w}_1)$ ; since this kernel has character  $\chi(0)$ , it is clear that  $\text{rad } V(2\mathfrak{w}_1)$  is either 0 or the trivial module  $L(0)$ . We may thus apply 4.6.6; this result guarantees that  $S^2V$  has the form  $V(2\mathfrak{w}_1) \oplus V(0)$  when  $V(2\mathfrak{w}_1)$  is simple, and that otherwise  $\text{rad } V(2\mathfrak{w}_1) = V(0)$  and  $S^2V$  has a composition series with factors  $V(0) \mid L(2\mathfrak{w}_1) \mid V(0)$ .

Since  $p > 2$ , Lemma 4.7.1 shows that there is a non-degenerate bilinear form  $\kappa$  on  $S^2V$ . Since the inclusion  $V(2\mathfrak{w}_1) \subset S^2V$  is dual to the map  $S^2V \xrightarrow{\psi} H^0(2\mathfrak{w}_1)$ , the image of  $V(2\mathfrak{w}_1)$  is realized as  $(\ker \psi)^\perp$ ; according to 4.7.2,  $\ker \psi = kQ$ .

Assume that  $Q \notin Q^\perp$ . Then  $\psi(V(2\mathfrak{w}_1)) = H^0(2\mathfrak{w}_1)$  so that  $V(2\mathfrak{w}_1)$  is simple. otherwise,  $Q \in Q^\perp = V(2\mathfrak{w}_1)$  is an invariant vector. Evidently it suffices to show that  $Q \in Q^\perp$  gives rise to the congruence  $c \equiv 0 \pmod{p}$ . This congruence will follow provided that we show  $\kappa(Q, Q) = t \cdot c$  where  $c$  is as above and  $t \in k^*$ .

We now observe the following: When  $V$  is even dimensional,  $\kappa(Q, Q) = \frac{\ell}{2} \cdot 1_k$  (since  $\kappa(e_i \cdot e_{-i}, e_i \cdot e_{-i}) = \frac{1}{2}$ ). When  $V$  is odd dimensional,  $\kappa(Q, Q) = \frac{\ell}{2} + \frac{1}{4}\kappa(u, u) = \frac{\ell}{2} + \frac{1}{4} = \frac{2\ell + 1}{4}$ . The result follows.  $\square$

**Proposition 4.7.4.** *Assume that  $p > 3$ . If  $\dim_k V = 2\ell$ , let  $c = \ell + 1$ ; if  $\dim_k V = 2\ell + 1$ , let  $c = 2\ell + 3$ . If  $c \not\equiv 0 \pmod{p}$ , then  $S^3V \simeq V(3\mathfrak{w}_1) \oplus V(\mathfrak{w}_1)$ . If  $c \equiv 0 \pmod{p}$ , then  $S^3V = I(\mathfrak{w}_1, 3\mathfrak{w}_1, \mathfrak{w}_1)$  and  $\text{rad } S^3V = V(3\mathfrak{w}_1)$ .*

*Proof.* The proof of this proposition proceeds precisely as for Proposition 4.7.3. In this case, we have  $\text{ch } S^3V = \chi(3\mathfrak{w}_1) + \chi(\mathfrak{w}_1)$ . Again,  $\text{rad } V(2\mathfrak{w}_1)$  is contained in the kernel of the map  $S^3V \rightarrow H^0(2\mathfrak{w}_1)$ ; a character argument shows that  $\text{rad } V(3\mathfrak{w}_1)$  may be only 0 or  $V(\mathfrak{w}_1)$ . Proposition 4.6.6 thus guarantees that  $S^3V$  has the claimed structure; it only remains to verify the congruence assertions of the lemma.

Since  $p > 3$ , Lemma 4.7.1 yields a non-degenerate bilinear form  $\kappa$  on  $S^3V$ . We deduce that  $V(3\mathfrak{w}_1) = (\ker \psi)^\perp$ ; in this case, we have  $\ker \psi = Q \cdot V$ .

Since  $p \neq 2$ ,  $Q \cdot V$  is simple, so that  $Q \cdot V \cap (Q \cdot V)^\perp$  is 0 or  $Q \cdot V$ . If  $Q \cdot V \not\subset (Q \cdot V)^\perp$  then  $\psi(V(3\omega_1)) = H^0(3\omega_1)$  so that  $V(3\omega_1)$  is simple. Otherwise,  $Q \cdot V \subset (Q \cdot V)^\perp = V(3\omega_1)$  so that  $Q \cdot V$  is the radical. It suffices to show that  $Q \cdot V \subset (Q \cdot V)^\perp$  gives rise to the congruence  $c \equiv 0 \pmod{p}$ .

Observe that a basis of weight vectors for  $Q \cdot V$  consists in  $Q \cdot e_i$  for  $\pm i = 0, 1, \dots, \ell$  with  $i = 0$  dropped in the even dimensional case. Due to invariance of the form  $\kappa$ ,  $Q \cdot e_i$  is automatically orthogonal to  $Q \cdot e_j$  unless  $i + j = 0$ . When  $V$  is even dimensional,

$$\kappa(Q \cdot e_i, Q \cdot e_{-i}) = \frac{\ell+1}{6} \cdot 1_k \quad (i = 1, 2, \dots, \ell).$$

When  $V$  is odd dimensional,

$$\kappa(e_i \cdot Q, e_{-i} \cdot Q) = \left( \frac{\ell+1}{6} + \frac{1}{12} \kappa(e_i \cdot u, e_{-i} \cdot u) \right) 1_k = \left( \frac{2\ell+3}{12} \right) 1_k. \quad (i = 0, 1, 2, \dots, \ell).$$

In each case, it is easy to see that  $\kappa$  restricts to the 0 form on  $Q \cdot V$  if and only if  $c \equiv 0 \pmod{p}$ .  $\square$

**4.8. Symplectic Exterior powers.** Let  $\Omega = \Omega(V)$  be a group of type  $C_\ell$  (so  $V$  is a symplectic vector space). The Weyl modules  $V(\omega_i)$  are closely related to the exterior powers of the natural symplectic module  $V$ . We shall exploit the calculations given in Lemma 4.5.3 to decompose the exterior powers  $\bigwedge^2 V$  and  $\bigwedge^3 V$ .

To simplify some formulas, we write  $\omega_0$  for the weight 0. We need the following general facts:

**Lemma 4.8.1.** *Let  $\Phi = C_\ell$ , and fix  $1 \leq s \leq \ell$ .*

- (a)  $\text{ch } \bigwedge^s V = \sum_{j=0}^i \chi(\omega_{2j})$  if  $s = 2i$ .
- (b)  $\text{ch } \bigwedge^s V = \sum_{j=0}^i \chi(\omega_{2j+1})$  if  $s = 2i + 1$ .
- (c) *The module  $\bigwedge^s V$  satisfies hypothesis  $\dagger$  for the weight  $\omega_s$ .*

*Proof.* (a) and (b) are given in [26], Chap. 4, exercise 24. Since  $\bigwedge^s V$  is self-dual, part (c) follows from [15] Theorem 1.1.  $\square$

**Lemma 4.8.2.** *Let  $\Phi = C_\ell$ , and let  $i = 2, 3$ . If  $\ell + 2 - i \equiv 0 \pmod{p}$ , then  $\bigwedge^i V = I(\omega_{i-2}, \omega_i, \omega_{i-2})$  and  $\text{rad } \bigwedge^i V = V(\omega_i)$ . Otherwise,  $\bigwedge^i V \simeq V(\omega_i) \oplus L(\omega_{i-2})$  and  $V(\omega_i)$  is simple.*

*Proof.* Part (c) of Lemma 4.8.1 guarantees that  $M = \bigwedge^i V$  satisfies hypothesis  $\dagger$  with respect to the weight  $\omega_i$ . Observe that  $\text{rad } V(\omega_i)$  is contained in the kernel of the map  $\bigwedge^i V \rightarrow H^0(\omega_i)$ ; thus, we learn that  $\text{rad } V(\omega_2)$  can be either 0 or  $L(0)$  and  $\text{rad } V(\omega_3)$  can be either 0 or  $L(\omega_1)$ . We may therefore apply Proposition 4.6.6. Proposition 4.5.3 shows that when  $\ell + 2 - i \not\equiv 0 \pmod{p}$ ,  $V(\omega_i)$  is simple. When  $\ell + 2 - i \equiv 0 \pmod{p}$ ,  $\text{rad } V(\omega_i)$  is  $L(\omega_{i-2})$  isotypic. Since  $\text{ch } \bigwedge^i V = \chi(\omega_i) + \chi(\omega_{i-2})$  by (a) and (b) of Lemma 4.8.1, Proposition 4.6.6 now yields the result.  $\square$

*Remark 4.8.3.* One can obtain 4.8.2 also from the results in [18]; there, the dimension of  $L(\omega_i)$  is computed for  $1 \leq i \leq \ell$ .

*Remark 4.8.4.* Continue to assume  $\Phi = C_\ell$ . Let us recall the map  $T_{\mathfrak{w}_i} : W_{\mathbb{Z}}(\mathfrak{w}_i) \rightarrow H_{\mathbb{Z}}^0(\mathfrak{w}_i)$ . The preceding lemma implies that  $\text{coker } T_{\mathfrak{w}_2} \otimes \mathbb{F}_p$  has dimension 1 for every  $p \mid \ell$ ; we may conclude  $\text{coker}(T_{\mathfrak{w}_2}) \simeq \mathbb{Z}/\ell\mathbb{Z}$  as  $\mathbb{Z}$  modules. Similarly, we deduce  $(\text{coker}(T_{\mathfrak{w}_3}))_{\mathfrak{w}_3} \simeq \mathbb{Z}/\ell\mathbb{Z}$  as  $\mathbb{Z}$  modules.

**4.9. The weight  $\mathfrak{w}_1 + \mathfrak{w}_2$ .** Let  $\Omega = \Omega(V)$  be a group of type  $B_\ell$ ,  $C_\ell$  or  $D_\ell$  realized as a group of isometries as in section 1.5.

**Lemma 4.9.1.** *Let  $\Phi = B_\ell$ ,  $C_\ell$  or  $D_\ell$ , and suppose that  $p \neq 2$ .*

- (a) *If  $\Phi = B_\ell$  or  $D_\ell$  then  $M = V_{\mathfrak{w}_1, \mathfrak{w}_2}$  satisfies hypothesis  $\dagger$  for the weight  $\mathfrak{w}_1 + \mathfrak{w}_2$ . For  $\Phi = B_{\geq 4}$  and  $\Phi = D_{\geq 5}$ ,  $\text{ch } M = \chi(\mathfrak{w}_1 + \mathfrak{w}_2) + \chi(\mathfrak{w}_3) + \chi(\mathfrak{w}_1)$ . For  $\Phi = B_3$ ,  $\text{ch } M = \chi(\mathfrak{w}_1 + \mathfrak{w}_2) + \chi(2\mathfrak{w}_3) + \chi(\mathfrak{w}_1)$ . For  $\Phi = B_4$ ,  $\text{ch } M = \chi(\mathfrak{w}_1 + \mathfrak{w}_2) + \chi(\mathfrak{w}_3 + \mathfrak{w}_4) + \chi(\mathfrak{w}_1)$ .*
- (b) *If  $\Phi = C_\ell$  then  $M = V(\mathfrak{w}_1) \otimes \bigwedge^2 V$  satisfies hypothesis  $\dagger$  for the weight  $\mathfrak{w}_1 + \mathfrak{w}_2$ . Furthermore,  $\text{ch } M = \chi(\mathfrak{w}_1 + \mathfrak{w}_2) + \chi(\mathfrak{w}_3) + 2\chi(\mathfrak{w}_1)$ .*

*Proof.* For (a), one applies Propositions 4.2.2 and 4.2.2 to deduce that  $V(\mathfrak{w}_1)$  and  $V(\mathfrak{w}_2)$  are simple (since  $p \neq 2$ ). Hypothesis  $\dagger$  then holds for  $M$  (as above) with respect to the weight  $\mathfrak{w}_1 + \mathfrak{w}_2$  by an application of Proposition 4.6.2. To obtain the statement about the character of  $M$ , we apply the formula in part (a) of Proposition 4.6.2. We have  $\Pi(\lambda) = \{\pm \varepsilon_i \mid i = 0, 1, \dots, \ell\}$  where  $i = 0$  is omitted for type  $D_\ell$ . One observes that  $\varepsilon_i + \mathfrak{w}_2 = -\mathfrak{w}_{i-1} + \mathfrak{w}_i + \mathfrak{w}_2 \in \mathcal{D}$  and  $-\varepsilon_i + \mathfrak{w}_2 = \mathfrak{w}_{i-1} = \mathfrak{w}_i \in \mathcal{D}$  for  $i = 0, 1, 2, \dots, \ell$ , and that these weights are only dominant for  $i = 3$  in the first case and  $i = 2$  in the second. The character formula above now follows from an application of Proposition 4.6.2 (a) and Lemma 4.3.1.

In case  $\Phi = C_\ell$ , (c) of Lemma 4.8.1 shows that  $\bigwedge^r V$  satisfies hypothesis  $\dagger$  for  $r \geq 1$ . Since  $V = V(\mathfrak{w}_1) = H^0(\mathfrak{w}_1)$  also satisfies  $\dagger$ , part (d) of Proposition 4.6.2 shows that  $V \otimes \bigwedge^2 V$  satisfies  $\dagger$  for  $\mathfrak{w}_1 + \mathfrak{w}_2$ .

Part (a) of Lemma 4.8.1 yields  $\text{ch } \bigwedge^2 V = \chi(\mathfrak{w}_2) + \chi(0)$ . To obtain the character formula, one notes that  $\text{ch } M = \text{ch } V_{\mathfrak{w}_1, \mathfrak{w}_2} + \chi(\mathfrak{w}_1)$  by the construction of  $M$ . Arguing as in (a), one obtains  $\text{ch } V_{\mathfrak{w}_1, \mathfrak{w}_2} = \chi(\mathfrak{w}_1 + \mathfrak{w}_2) + \chi(\mathfrak{w}_3) + \chi(\mathfrak{w}_1)$ ; the claim now follows.  $\square$

**Lemma 4.9.2.** *Let  $\lambda = \mathfrak{w}_1 + \mathfrak{w}_2$ , and let  $M$  be as in the preceding lemma. Assume that  $p > 3$ . Let  $c = 2\ell + 1, 2\ell - 1, \ell$  if  $\Phi = C_\ell, D_\ell, B_\ell$ .*

- (a) *Let  $\Phi = B_\ell$  or  $\Phi = D_\ell$ . If  $c \equiv 0 \pmod{p}$ , then  $M \simeq V(\mu) \oplus I(\mathfrak{w}_1, \mathfrak{w}_1 + \mathfrak{w}_2, \mathfrak{w}_1)$  and  $\text{rad } V(\lambda) = L(\mathfrak{w}_1)$ . If  $c \not\equiv 0 \pmod{p}$ , then  $M \simeq V(\mu) \oplus V(\lambda)$  and  $V(\lambda)$  is simple. Here,  $\mu = \mathfrak{w}_3$  if  $\Phi = D_\ell$ , or  $\Phi = B_\ell$  and  $\ell \geq 5$ . If  $\Phi = B_4$ ,  $\mu = \mathfrak{w}_3 + \mathfrak{w}_4$ . If  $\Phi = B_3$ ,  $\mu = 2\mathfrak{w}_3$ .*
- (b) *If  $\Phi = C_\ell$  and  $c \equiv 0 \pmod{p}$ , then  $M \simeq N \oplus I(\mathfrak{w}_1, \mathfrak{w}_1 + \mathfrak{w}_2, \mathfrak{w}_1)$  and  $\text{rad } V(\lambda) = L(\mathfrak{w}_1)$ . If  $c \not\equiv 0 \pmod{p}$ , then  $M \simeq N \oplus V(\lambda)$  and  $V(\lambda)$  is simple. In each case  $\text{ch } N = \chi(\mathfrak{w}_3) + \chi(\mathfrak{w}_1)$ .*

*Proof.* We evaluated the general sum formula for  $\lambda$  in Lemma 4.5.7; for  $p > 3$  this lemma shows that  $\text{rad } V(\lambda)$  is  $L(\mathfrak{w}_1)$  isotypic, and that  $\text{rad } V(\lambda)$  is 0 if  $c \not\equiv 0 \pmod{p}$ .

By Lemma 4.9.1, we know that hypothesis  $\dagger$  holds for the module  $M$ . Observe that  $\text{rad } V(\lambda)$  is contained in the kernel of the map  $M \rightarrow H^0(\lambda)$ . For types  $B_\ell$  and  $D_\ell$ , we now see that  $\text{rad } V(\lambda)$  can be only 0 or  $L(\mathfrak{w}_1)$ ; Proposition 4.6.6 now gives the result for these types.

For type  $C_\ell$ , assume that  $c \equiv 0 \pmod{p}$ . A character argument as above shows only that  $\text{rad } V(\lambda)$  can have character 0,  $\chi(\mathfrak{w}_1)$  or  $2\chi(\mathfrak{w}_1)$ . Note that  $\ell \not\equiv 0 \pmod{p}$ ; hence

$\bigwedge^2 V \simeq V(\varpi_2) \oplus L(0)$  and  $V(\varpi_2)$  is simple by Lemma 4.8.2. It follows that

$$M \simeq V(\varpi_1) \oplus (V(\varpi_2) \otimes V(\varpi_1));$$

one now sees that  $\text{chrad} V(\lambda)$  can only be 0 or  $\chi(\varpi_1)$ . An application of Proposition 4.6.6 gives the result now for  $C_\ell$ .  $\square$

*Remark 4.9.3.* Let  $\lambda = \varpi_1 + \varpi_2$  and let  $c$  be as in the preceding lemma. We have the following dimension formulas.

(a) If  $\Phi = B_\ell$ ,  $\dim_k V(\lambda) = 2^4 \binom{\ell + \frac{3}{2}}{3}$ . If  $c \equiv 0 \pmod{p}$ , we have  $\dim_k L(\lambda) = \dim_k V(\lambda) - 2\ell - 1$ .

(b) If  $\Phi = C_\ell$  or  $D_\ell$ ,  $\dim_k V(\lambda) = 2^4 \binom{\ell + 1}{3}$ . If  $c \equiv 0 \pmod{p}$ , we have  $\dim_k L(\lambda) = \dim_k V(\lambda) - 2\ell$ .

Indeed, the Weyl module dimension formulas follow from the character calculations for  $M$  and the structural results for  $M$  given in Lemma 4.9.2; the statements on the dimensions of the irreducibles are immediate.

**4.10. Summarizing the Weyl module results.** When the following weights  $\lambda$  are restricted, the results in the previous sections yield in particular the indicated results in table 4.10.1 on the structure of  $V(\lambda)$  and the dimension of  $L(\lambda)$ .

Table 4.10.1. Non-simple Weyl modules.

$\Phi$	$\lambda$	Condition on $p$	$\text{rad } V(\lambda)$	$\dim_k V(\lambda)$
$A_\ell$	$\varpi_1 + \varpi_\ell$	$\ell + 1 \equiv 0 \pmod{p}$	$L(0)$	$\ell(\ell + 2)$
$A_3$	$\varpi_1 + \varpi_2, \varpi_2 + \varpi_3$	$p = 3$	$L(\varpi_3), L(\varpi_1)$	20
$B_\ell$	$\varpi_1$	$p = 2$	$L(0)$	$2\ell$
$B_\ell$	$2\varpi_1$	$2\ell + 1 \equiv 0 \pmod{p}$	$L(0)$	$\binom{2\ell+2}{2} - 1$
$B_\ell$	$3\varpi_1$	$2\ell + 3 \equiv 0 \pmod{p}$	$L(\varpi_1)$	$\binom{2\ell+3}{3} - 2\ell - 1$
$B_3$	$\varpi_1 + \varpi_3$	$p = 7$	$L(\varpi_3)$	40
$C_\ell$	$\varpi_2$	$\ell \equiv 0 \pmod{p}$	$L(0)$	$\binom{2\ell}{2} - 1$
$D_\ell$	$2\varpi_1$	$\ell \equiv 0 \pmod{p}$	$L(0)$	$\binom{2\ell}{2} - 1$
$E_6$	$\varpi_2$	$p = 3$	$L(0)$	78
$F_4$	$\varpi_4$	$p = 3$	$L(0)$	26
$G_2$	$\varpi_1$	$p = 3$	$L(\varpi_1)$	14
$G_2$	$\varpi_2$	$p = 2$	$L(0)$	7

Indeed, when  $\Phi = B_\ell$  and  $p = 2$ , it is well known that the Weyl module  $V(\varpi_1)$  is the dual of the “natural” orthogonal module for  $\Omega(V, \phi)$  (the stability group of the non-singular quadratic form  $\phi$ ); the radical of  $V(\varpi_1)$  is therefore  $L(0)$  (and the dimension assertion is obvious). The remaining assertions about  $\text{rad } V(\lambda)$  and  $\dim_k L(\lambda)$  are verified in the following results (for the classical types): 4.6.10, 4.7.3, 4.7.4, 4.8.2. For the exceptional types, the assertions are verified in the tables at the end of [7]. When  $\lambda$  is restricted, one should observe in all cases, that these are the only primes for which  $V(\lambda)$  reduces. While not every pair  $(\lambda, p)$  listed in table 4.10.1 gives rise to a simple module with dimension  $\leq \mathcal{C}p$ , one may observe that for each  $\lambda$  appearing, one can find  $\ell$  and  $p$  so that  $\dim_k L(\lambda) \leq \mathcal{C}p$ .

**Lemma 4.10.1.** *Assume that  $p$  is not special, and let  $\ell \geq 2$ . Fix  $\lambda \in X_1$ , and assume that  $\dim_k L(\lambda) \leq \mathcal{C}p$ . Then  $V(\lambda) = L(\lambda) = H^0(\lambda)$  unless  $\lambda$  is in table 4.10.1 above.*

*Proof.* By Lemma 4.1.1, we know that any  $\lambda \in X_1$  with  $\dim_k L(\lambda) \leq \mathcal{C}p$  is either in the interior of the first alcove or in  $\mathcal{I}$ . In the first case,  $V(\lambda)$  is simple; hence we may suppose that  $\lambda \in \mathcal{I}$ .

When  $p = 2$ , we note the following further restrictions on  $\lambda$ . When  $\Phi = D_\ell$  and  $p = 2$ , one can use the information in Lemma 5.4.4 together with the assumption that  $\lambda$  is restricted to deduce that, up to diagram automorphism,  $\lambda$  is one of:  $\varpi_1$ ,  $\varpi_2$ , or  $\varpi_\ell$  with the latter possibility occurring only when  $\ell \leq 7$ . When  $\Phi = B_\ell$  or  $C_\ell$ , recall that  $p = 2$  is a special prime; it is thus excluded from consideration in this result.

Notice that if  $V(\lambda) = L(\lambda)$ , then automatically  $H^0(\lambda) = L(\lambda)$ . Fix a weight  $\lambda$  not listed in table 4.10.1 (and satisfying the hypothesis of the lemma); we shall verify that  $L(\lambda) = V(\lambda)$ . Observe that by preceding remarks, we can suppose that  $\lambda$  does not appear (for any  $\ell$  or  $p$ ) in table 4.10.1. One ideally wishes to know the precise dimension of  $L(\lambda)$  and the primes for which  $V(\lambda)$  fails to be simple; one then hopes to show that the dimension of  $L(\lambda)$  exceeds the bound  $\mathcal{C}p$  for the decomposing primes  $p$ .

When the group is of exceptional type ( $\Phi = E_\ell, F_4$ , or  $G_2$ ), the tables in [7] contain the needed information with the single omission of the weight  $2\varpi_7$  for type  $E_7$ . In table 4.5.2, we have evaluated the sum formula for this weight. According to this calculation,  $V(2\varpi_7)$  is simple of dimension 1463 when  $p > 3$ . When  $p = 3$ , the radical of  $V(2\varpi_7)$  is  $V(\varpi_1)$  which is simple of dimension 133 so that  $\dim_k L(2\varpi_7)$  is 1330 in this case.

For the classical types, we compile information on Weyl module decompositions in the following tables: 4.10.2, 4.10.3, 4.10.5, and 4.10.4. The entries in these tables correspond to the weights  $\lambda$  under consideration. For each such  $\lambda$ , we list or characterize the non-special primes  $\mathfrak{P}$  for which  $\lambda$  is restricted and for which  $V(\lambda)$  may fail to be simple. We then list some data concerning the dimension of  $L(\lambda)$  for the the primes  $p \in \mathfrak{P}$ .

We claim that these tables settle the lemma; indeed, using the data either in the tables below or in [7], one obtains a lower bound for  $p$  from the inequality  $\dim_k L(\lambda) \leq \mathcal{C}p$ ; given this resulting lower bound, the tables show in turn that  $V(\lambda)$  is simple. As an example, take  $\Phi = E_6$  and the weight  $\lambda = \varpi_1 + \varpi_5$ . For this weight we have  $\dim_k L(\lambda) \geq 26244$ . This easily leads to  $p > 13$  so that  $V(\lambda)$  is simple. It is a straightforward task (which is left to the reader) to check that this procedure verifies the lemma in all cases.

As to the verification of the data in the tables, the reducibility information may be found in one of the results indicated below; for the dimension information, we either apply one of the results below or we utilize Premet’s Theorem 2.2.3 to obtain a lower bound for  $\dim_k L(\lambda)$ .

The data in the tables may be found in the following results: 4.2.2, 4.5.2, 4.6.10, 4.9.2, 4.9.3, 4.8.2.  $\square$

 Table 4.10.2. Type  $A_\ell$ .

$\lambda$	$\mathfrak{P}$	$\dim_k L(\lambda), \quad (p \in \mathfrak{P})$
$\mathfrak{w}_i \ (1 \leq i \leq 5)$	$\emptyset$	
$r\mathfrak{w}_1 \ (1 \leq r \leq 4)$	$\emptyset$	
$\mathfrak{w}_1 + \mathfrak{w}_2$	$\{3\}$	$\mathfrak{C} \cdot \frac{1}{3} \cdot (\ell + 5)$
$2\mathfrak{w}_1 + \mathfrak{w}_\ell$	$\{p \mid \ell + 2 \equiv 0 \pmod{p}\}$	$(\ell + 1) \left( \binom{\ell+2}{2} - 2 \right)$
$\mathfrak{w}_2 + \mathfrak{w}_\ell$	$\{p \mid \ell \equiv 0 \pmod{p}\}$	$\mathfrak{C}(\ell + 1 - (4/\ell))$
$\mathfrak{w}_3 + \mathfrak{w}_\ell$	$\{p \mid \ell - 1 \equiv 0 \pmod{p}\}$	$(\ell + 1) \binom{\ell+1}{3} - 2 \binom{\ell+1}{2}$
$\mathfrak{w}_4 + \mathfrak{w}_\ell$	$\{p \mid \ell - 2 \equiv 0 \pmod{p}\}$	$(\ell + 1) \binom{\ell+1}{4} - 2 \binom{\ell+1}{3}$
$2\mathfrak{w}_2$	$\{3\}$	$(1/3) \binom{\ell+2}{2} \binom{\ell+1}{2} - \binom{\ell+1}{4}$
$2\mathfrak{w}_1 + \mathfrak{w}_2 \ (\ell = 3, 4, 5)$	$\emptyset$	
$3\mathfrak{w}_1 + \mathfrak{w}_2 \ (\ell = 2)$	$\{5\}$	$18 > 5\mathfrak{C} = 15$
$\mathfrak{w}_2 + \mathfrak{w}_3 \ (\ell = 4)$	$\{2, 3\}$	$74 \quad (p = 2)$ $60 \quad (p = 3)$

 Table 4.10.3. Type  $D_\ell$ .

$\lambda$	$\mathfrak{P}$	$\dim_k L(\lambda), \quad (p \in \mathfrak{P})$
$\mathfrak{w}_i \ (i = 1, \ell)$	$\emptyset$	
$\mathfrak{w}_2$	$\{2\}$	$\geq 1 +  \mathcal{W}\mathfrak{w}_2  > 2\mathfrak{C}$
$\mathfrak{w}_i \ (3 \leq i \leq \ell - 1)$	$\{2\}$	
$\mathfrak{w}_1 + \mathfrak{w}_2$	$\{3, p \mid 2\ell - 1 \equiv 0 \pmod{p}\}$	$\geq 2^4 \binom{\ell+1}{3} - 2\ell, \quad (p \neq 3, p \in \mathfrak{P})$ $\geq  \mathcal{W}(\lambda)  +  \mathcal{W}\mathfrak{w}_3  = \frac{4}{3}\mathfrak{C}(\ell + 1)$ $(p = 3)$
$2\mathfrak{w}_4 \ (\ell = 5)$	$\emptyset$	
$\mathfrak{w}_1 + \mathfrak{w}_\ell$	$\{p \mid \ell \equiv 0 \pmod{p}\}$	$2^\ell(\ell - 1)$
$\mathfrak{w}_4 + \mathfrak{w}_5 \ (\ell = 5)$	$\{2\}$	164
$2\mathfrak{w}_4$	$\emptyset$	

Table 4.10.4. Type  $C_\ell$ .

$\lambda$	$\mathfrak{P}$	$\dim_k L(\lambda), \quad (p \in \mathfrak{P})$
$r\omega_1 \quad (r = 1, 2, 3)$	$\emptyset$	
$\omega_3$	$\{p \mid \ell - 1 \equiv 0 \pmod{p}\}$	$\binom{2\ell}{3} - 4\ell$
$\omega_4$	$\{p \mid \ell - i \equiv 0 \pmod{p}, \quad i = 1, 2\}$	$\geq  \mathcal{W}\lambda  +  \mathcal{W}\omega_2 $ $= \mathcal{C} \frac{2}{3}(\ell^2 - 5\ell + 9) > \mathcal{C}(\ell - 1)$ $\geq 2^5 + 2^3 \binom{5}{3} = 112$
$\omega_5 \quad (\ell = 5)$	$\{3\}$	$\geq 3 \cdot 2^6$
$\omega_5 \quad (\ell = 6)$	$\{3\}$	$\geq 2^6 + 2^4 \binom{6}{4} = 2^4 \cdot 19$
$\omega_6 \quad (\ell = 6)$	$\{3\}$	$\geq 2^4 \binom{\ell+1}{3} - 2\ell$ $(p \neq 3, p \in \mathfrak{P})$ $\geq  \mathcal{W}(\omega_1 + \omega_2)  +  \mathcal{W}\omega_3 $ $= \frac{4}{3}\mathcal{C}(\ell + 1) \quad (p = 3, \ell > 2)$ $\geq 8 > 3\mathcal{C} \quad (p = 3, \ell = 2)$
$\omega_1 + \omega_2$	$\{3, p \mid 2\ell + 1 \equiv 0 \pmod{p}\}$	$13 > 5\mathcal{C}$
$2\omega_2 \quad (\ell = 2)$	$\{5\}$	$89 > 7\mathcal{C}$
$2\omega_2 \quad (\ell = 3)$	$\{7\}$	$\geq  \mathcal{W}\lambda  = 96 > 7\mathcal{C}$
$\omega_2 + \omega_4 \quad (\ell = 4)$	$\{3, 7\}$	57
$\omega_1 + \omega_3 \quad (\ell = 3)$	$\{3\}$	

Table 4.10.5. Type  $B_\ell$ .

$\lambda$	$\mathfrak{P}$	$\dim_k L(\lambda), \quad (p \in \mathfrak{P})$
$\omega_i$	$\emptyset$	
$\omega_1 + \omega_2$	$\{3, p \mid \ell \equiv 0 \pmod{p}\}$	$\geq 2^4 \binom{\ell+\frac{3}{2}}{3} - 2\ell - 1 \quad (p \neq 3, p \in \mathfrak{P})$ $\geq  \mathcal{W}(\lambda)  +  \mathcal{W}\omega_3  = (4/3)\mathcal{C}(\ell + 1)$ $(p = 3)$
$\omega_2 + \omega_3 \quad (\ell = 3)$	$\{3, 5\}$	$77 > 18 = \mathcal{C}p \quad (p = 3)$ $64 > 30 = \mathcal{C}p \quad (p = 5)$
$\omega_2 + \omega_4 \quad (\ell = 4)$	$\{7\}$	$p = 2$ is special $306 > 84 = \mathcal{C}p \quad (p = 7)$
$2\omega_\ell \quad (\ell = 3, 4)$	$\emptyset$	
$\omega_1 + \omega_3 \quad (\ell = 3)$	$\{7\}$	excluded
$\omega_1 + \omega_4 \quad (\ell = 4)$	$\{3\}$	$112 > 36 = 3\mathcal{C}$
$\omega_1 + \omega_5 \quad (\ell = 5)$	$\{11\}$	$288 > 220 = 11\mathcal{C}$
$\omega_1 + \omega_6 \quad (\ell = 6)$	$\{13\}$	$696 > 390 = 13\mathcal{C}$



## 5. PROOF OF THEOREM 1.

Unless otherwise indicated, assume throughout this section that  $G$  is an almost simple algebraic  $k$  group; equivalently, assume that  $G$  has an irreducible root system.

**5.1. Exceptions.** The exceptional subquotients referred to in Theorem 1 are described by the following:

**Proposition 5.1.1.** (*Exceptional Modules.*) *Let  $\sigma$  be a (possibly trivial) diagram automorphism of  $\Phi$ , and suppose that  $\{\xi^\sigma, \zeta^\sigma\}$  appears on the following list. Then there is an indecomposable module  $E = E(\xi, \zeta)$  of length two with composition factors  $L(\xi)$  and  $L(\zeta)$ . The dimension of  $E$  (which is the same as the dimension of  $E(\xi^\sigma, \zeta^\sigma)$ ) is as specified. In each case,  $\text{Ext}_G^1(L(\xi), L(\zeta))$  is 1 dimensional. Furthermore, there exist  $\ell$  and  $p$  so that  $\dim_k E \leq \mathcal{C}p$ .*

Table 5.1.1. Small Indecomposable Modules

$\Phi$	$\{\xi, \zeta\}$ $\dim_k E$	Condition	$\Phi$	$\{\xi, \zeta\}$ $\dim_k E$	Condition
$A_\ell$	$\{\omega_1 + \omega_\ell, 0\}$ $\ell(\ell + 2)$	$\ell + 1 \equiv 0 \pmod{p}$	$A_\ell$	$\{2\omega_1, \omega_2\}$ $\binom{\ell+2}{2}$	$p = 2$
$B_\ell$	$\{2\omega_1, 0\}$ $\binom{2\ell+2}{2} - 1$	$2\ell + 1 \equiv 0 \pmod{p}$	$B_\ell$	$\{3\omega_1, \omega_1\}$ $\binom{2\ell+3}{3} - 2\ell - 1$	$2\ell + 3 \equiv 0 \pmod{p}$ and $p \neq 3$
	$\{\omega_1, 0\}$ $2\ell + 1$	$p = 2$			
$C_\ell$	$\{\omega_2, 0\}$ $\binom{2\ell}{2} - 1$	$\ell \equiv 0 \pmod{p}$	$C_\ell$	$\{2\omega_1, 0\}$ $2\ell + 1$	$p = 2$
$D_\ell$	$\{2\omega_1, 0\}$ $\binom{2\ell+1}{2} - 1$	$\ell \equiv 0 \pmod{p}$ and $p \neq 2$			
$E_6$	$\{\omega_2, 0\}$ 78	$p = 3$	$F_4$	$\{\omega_4, 0\}$ 26	$p = 3$

*Proof.* All relevant information is invariant under the diagram automorphism  $\sigma$ ; hence it is enough to consider the case where  $\{\xi, \zeta\}$  is on the above list. Given the asserted dimension formulas for  $E$  given above, it is not difficult to establish the existence of  $\ell$  and  $p$  for which  $\dim_k E \leq \mathcal{C}p$ . The most interesting case is  $\Phi = B_\ell$  and the pair  $\{3\omega_1, 0\}$ . If  $2\ell + 3 = p$ , then  $\mathcal{C}p = \ell(\ell - 1)(2\ell + 3)$ ; on the other hand  $\dim_k E = \binom{2\ell + 3}{3} - 2\ell - 1$ . A computation shows that for every  $\ell \geq 6$ ,  $\dim_k E \leq \mathcal{C}(2\ell + 3)$ . If  $\ell \geq 6$  and  $2\ell + 3 = p$  is prime, then  $E$  satisfies  $\dim_k E \leq \mathcal{C}p$ . The first such  $\ell$  is 7; we have in this case  $p = 17$  and  $\dim_k E = 665 < \mathcal{C}p = 714$ .

For each pair of weights above, we may assume  $\xi \geq \zeta$  (re-numbering the weights if necessary). With the exception of  $\{2\omega_1, 0\}$  when  $\Phi = C_\ell$  and  $p = 2$ , the information in table 4.10.1 shows that  $\text{rad } V(\zeta) \simeq L(\xi)$ . Part (c) of Lemma 2.3.1 then yields  $\text{Ext}_G^1(L(\zeta), L(\xi)) \simeq k$ ; the

dimensional assertions may be obtained from table 4.10.1. The result now follows for these  $\{\zeta, \xi\}$ .

Now suppose that  $p = 2$  and  $\Phi = C_\ell$ ; we consider the pair  $\{2\varpi_1, 0\}$  when  $p = 2$ . According to [1], 6.19 one has  $H^1(G_1, L(0))^{[-1]} = H^0(\varpi_1)$ . It follows from (a) of Lemma 2.3.3 that

$$\text{Ext}_G^1(L(2\varpi_1), L(0)) \simeq \text{Hom}_G(L(\varpi_1), H^1(G_1, L(0))^{[-1]}) \simeq \text{Hom}_G(L(\varpi_1), H^0(\varpi_1)) \simeq k;$$

this verifies the existence of the module  $E$ . The dimension formula for  $E$  is clear.  $\square$

*Remark 5.1.2.* Since one has always  $\text{Ext}_G^1(L(\xi), L(\zeta)) \simeq \text{Ext}_G^1(L(\zeta), L(\xi))$ , it follows that for each of the pairs of weights above there are indecomposable modules  $E$  and  $E'$  having  $\text{soc } E = L(\xi)$  and  $\text{soc } E' = L(\zeta)$ . Furthermore, since each Ext group is one dimensional, the indecomposable modules  $E$  and  $E'$  are determined up to isomorphism by their socle.

**5.2. Reformulation and an initial case.** We begin with the following reformulation of Theorem 1.

**Theorem 2.** *Let  $L, L'$  be simple  $G$  modules with  $\dim_k L + \dim_k L' \leq \mathcal{C}p$ . Let  $\lambda, \lambda'$  be the respective highest weights. If  $\text{Ext}_G^1(L, L') \neq 0$ , then  $\lambda = p^r \xi$  and  $\lambda' = p^r \zeta$  for some  $r \in \mathbb{N}$  and some  $\{\xi, \zeta\}$  on the list of Proposition 5.1.1.*

We first point out that this is equivalent to the main theorem. Indeed, let  $V$  be a  $G$  module satisfying  $\dim_k V \leq \mathcal{C}p$ . To verify the main theorem, we suppose that  $V$  is not semisimple. We can then find consecutive composition factors  $L$  and  $L'$  which determine an indecomposable subquotient  $E$ . As indicated in our earlier discussion of extension,  $E$  determines then a non-0 element of  $\text{Ext}_G^1(L, L')$ . Theorem 2 (together with Steinberg's Theorem 2.2.1) then shows that the subquotient  $E$  is a Frobenius twist of one of modules from Proposition 5.1.1, as desired.

Let us now observe that when  $\ell = 1, \mathcal{C} = 1$ . Theorem 1 (and Theorem 2) therefore follow from Jantzen's result (Theorem II of [12]).

*We assume from now on that the rank of  $G$  is at least 2.*

We are in a position to prove the following special case of the theorem.

**Proposition 5.2.1.** *Assume that  $p$  is not special. Let  $\lambda, \mu \in X_1$  with  $\dim_k L(\lambda) + \dim_k L(\mu) \leq \mathcal{C}p$ . If*

$$\text{Ext}_G^1(L(\lambda), L(\mu)) \neq 0,$$

*then  $\{\lambda, \mu\}$  is one of the sets  $\{\xi, \zeta\}$  from Proposition 5.1.1.*

*Proof.* Applying Proposition 2.3.1 (a), we can assume (exchanging  $\lambda$  and  $\mu$  if necessary) that the Ext group is isomorphic to  $\text{Hom}_G(\text{rad } W(\lambda), L(\mu))$ . Thus,  $V(\lambda)$  is not simple; by Lemma 4.1.1, this forces  $\lambda \in \mathcal{I}$ . An application of Lemma 4.10.1 now shows in fact that  $\lambda$  is listed in table 4.10.1. The description in (4.10.1) of the structure of  $W(\lambda)$  shows that in all cases  $\text{rad } V(\lambda)$  is simple; there is thus only one possibility for  $\mu$ .

One now observes that all possible  $\lambda, \mu$  occur in Proposition 5.1.1. (Note that four weights appear in table 4.10.1 and not in 5.1.1; in these cases, one can check that  $\dim_k W(\lambda) > \mathcal{C}p$ .)  $\square$

**5.3. Cohomology.** In order to get the result for weights not necessarily in  $X_1$  we want to apply Lemma 2.3.3. For this we need information on  $G_1$  cohomology. Ideally, we should wish to know  $H^1(G_1, L)$  for every restricted simple module with  $\dim_k L \leq \mathcal{C}p$ . According to Lemma 4.1.1, we must consider weights in  $\mathcal{C}$  and weights in  $\mathcal{I}$ .

The following lemma addresses weights in  $\mathcal{C}$ .

**Lemma 5.3.1.** *If  $\gamma \in X_1$  and  $\langle \gamma + \rho, \alpha_0^\vee \rangle < p$ , then  $H^1(G_1, V(\gamma)) = 0$ .*

*Proof.* This is observed in the proof of Lemma 1.7 in [12] (see the end of the second paragraph of the proof).  $\square$

Computing  $H^1(G_1, L)$  for simple modules  $L$  is in general a difficult task. However, Jantzen has computed (in [11]), the  $G_1$  cohomology for the induced modules  $H^0(\lambda)$ . We record here those  $\lambda$  in  $\mathcal{I}$  for which this cohomology group is non zero.

**Proposition 5.3.2.** *Let  $G$  be an almost simple group. Then one has  $H^1(G_1, H^0(\lambda)) = 0$  for  $\lambda \in \mathcal{I}$  unless  $\lambda$ , or a diagram automorphism conjugate of  $\lambda$ , is in the following list:*

- $\Phi = A_\ell$ :
  - $\mathfrak{w}_2$  when  $\ell \geq 2$  and  $p = 2$ ;
  - $\mathfrak{w}_{\ell-1} + \mathfrak{w}_\ell$  when  $\ell = 2, 3, 4, 5$  and  $p = 3$ ;
  - $\mathfrak{w}_{\ell-2} + \mathfrak{w}_\ell$  when  $\ell = 3, 4, 5, 6$  and  $p = 2$ ;
  - $3\mathfrak{w}_1 + \mathfrak{w}_2$  when  $\ell = 2$  and  $p = 5$ .
- $\Phi = C_\ell$ .
  - $\mathfrak{w}_2$  when  $\ell \geq 2$  and  $p = 2$ ;
  - $\mathfrak{w}_{\ell-1}$  when  $\ell = 2, 3, 4, 5, 6$  and  $p = 2$ ;
  - $\mathfrak{w}_1 + \mathfrak{w}_2$  when  $\ell \geq 2$  and  $p = 3$ ;
  - $\mathfrak{w}_1 + \mathfrak{w}_3$  when  $\ell = 3$  and  $p = 2$ .
- $\Phi = B_\ell$ .
  - $\mathfrak{w}_2$  when  $\ell \geq 3$  and  $p = 2$ ;
  - $\mathfrak{w}_1 + \mathfrak{w}_2$  when  $\ell \geq 3$  and  $p = 3$ .
- $\Phi = D_\ell$ .
  - $\mathfrak{w}_2$  when  $\ell \geq 4$  and  $p = 2$ ;
  - $\mathfrak{w}_3$  when  $\ell = 5$  and  $p = 2$ ;
  - $\mathfrak{w}_4$  when  $\ell = 6$  and  $p = 2$ .
  - $\mathfrak{w}_1 + \mathfrak{w}_2$  when  $\ell \geq 4$  and  $p = 3$ .
- $\Phi = F_4$ :  $\mathfrak{w}_3$  when  $p = 2$ ;
- No exceptions for types  $E_6, E_7, E_8$ , and  $G_2$ .

*Proof.* Let  $\xi_{p,i} = p\mathfrak{w}_i - \alpha_i$  for  $1 \leq i \leq \ell$ . According to [11], Proposition 4.1,  $H^1(G_1, H^0(\lambda)) = 0$  unless  $\lambda = \xi_{p,i}$  for some  $i$ . One must now check that the weights enumerated above are the only  $\lambda \in \mathcal{I}$  for which there is a prime  $p$  and  $1 \leq i \leq \ell$  so that  $\lambda = \xi_{p,i}$ .

We sketch the verification of this for type  $C_\ell$ ; the remaining cases are similar and are left to the reader. Note that  $\langle \xi_{p,i}, \alpha_i^\vee \rangle = p - 2$ . Inspecting table 3.1.1, one checks that the maximal value of  $\langle \lambda, \alpha_i^\vee \rangle$  for  $\lambda \in \mathcal{I}$  is 3. This shows  $p \leq 5$ . The only  $\lambda$  such that  $\langle \lambda, \alpha_i^\vee \rangle = 3$  is achieved is  $3\mathfrak{w}_1$ . One easily checks that  $3\mathfrak{w}_1 \neq \xi_{5,i}$  for any  $i$ . Thus, we may assume that  $p = 2, 3$ . When  $p = 3$ , one can check that  $\xi_{3,i}$  has non-zero coefficients on at least 3 fundamental dominant weights when  $i \neq 1, \ell$ ; on the other hand, no weight in  $\mathcal{I}$  has this property. One easily checks that  $\xi_{3,1} = \mathfrak{w}_1 + \mathfrak{w}_2 \in \mathcal{I}$ , and  $\xi_{3,\ell} \notin \mathcal{I}$ . A direct computation of each  $\xi_{2,i}$  now yields the remaining  $\lambda$  on the list.  $\square$

We can now determine the  $G_1$  cohomology of the simple modules of interest.

**Lemma 5.3.3.** *Suppose that  $p$  is not special. Let  $\lambda \in X_1$  satisfy  $\dim_k L(\lambda) \leq \mathcal{C}p$ . Then  $H^1(G_1, L(\lambda)) = 0$  unless  $\lambda$  is on the following list:*

Table 5.3.1. Non-vanishing  $G_1$  Cohomology

$\Phi$	$p$	$\lambda$	$H^1(G_1, L(\lambda))^{[-1]}$
$A_\ell$	2	$\varpi_2$ or $\varpi_{\ell-1}$	$L(\varpi_1)$ or $L(\varpi_\ell)$ ( $\ell \geq 4$ )
$A_\ell, \ell > 2$	$p \mid \ell + 1 \equiv 0 \pmod{p}$	$\varpi_1 + \varpi_\ell$	$k$
$A_2$	$p = 3$	$\varpi_1 + \varpi_2$	$L(0) \oplus L(\varpi_1) \oplus L(\varpi_2)$
$A_3$	$p = 3$	$\varpi_1 + \varpi_2$ or $\varpi_2 + \varpi_3$	$L(\varpi_1)$ or $L(\varpi_2)$
$B_\ell$	$p \mid 2\ell + 1 \equiv 0 \pmod{p}$	$2\varpi_1$	$k$
$C_\ell$	$p \mid \ell \equiv 0 \pmod{p}$	$\varpi_2$	$k$
$D_\ell$	$p \mid \ell \equiv 0 \pmod{p}$	$2\varpi_1$	$k$

*Proof.* Since  $\dim_k L \leq \mathcal{C}p$ , 4.1.1 shows that  $\lambda$  is either in the first alcove or  $\lambda \in \mathcal{I}$ . In the first case,  $H^1(G_1, L(\lambda)) = H^1(G_1, V(\lambda)) = 0$  by Lemma 5.3.1; thus, we may assume that  $\lambda \in \mathcal{I}$ .

Suppose now that  $\lambda \in \mathcal{I}$  and that  $V(\lambda) = H^0(\lambda)$  is simple. In this case Lemma 5.3.2 lists all possibilities for  $\lambda \in \mathcal{I}$  for which  $H^1(G_1, H^0(\lambda)) \neq 0$ ; inspecting this list, one observes (using the dimension calculations in section 4) that  $\dim_k H^0(\lambda) > \mathcal{C}p$  with the exception of  $p = 2$ ,  $\Phi = A_\ell$  and  $\lambda = \varpi_2$  or  $\varpi_{\ell-1}$ ; the isomorphism  $H^1(G_1, L(\varpi_2))^{[-1]} \simeq L(\varpi_1)$  is demonstrated in [11], Proposition 4.1.

We may now suppose that  $\lambda \in \mathcal{I}$  and that  $H^0(\lambda)$  is not simple. Thus,  $\lambda$  is in table 4.10.1. For these  $\lambda$ , Lemma 5.3.2 shows that  $H^1(G_1, H^0(\lambda)) = 0$  with the following exceptions:

- $p = 3$ ,  $\Phi = A_2$  and  $\lambda = \varpi_1 + \varpi_2$
- $p = 3$ ,  $\Phi = A_3$  and  $\lambda = \varpi_1 + \varpi_2$  or  $\varpi_2 + \varpi_3$ .

One has in all cases an exact sequence

$$\begin{array}{ccccccc} (H^0(\lambda))^{G_1} & \longrightarrow & (H^0(\lambda)/L(\lambda))^{G_1} & \longrightarrow & H^1(G_1, L(\lambda)) & \longrightarrow & \\ H^1(G_1, H^0(\lambda)) & \longrightarrow & H^1(G_1, H^0(\lambda)/L(\lambda)) & & & & \end{array}$$

For any  $G$  module  $V$ ,  $V^{G_1}$  is a  $G$  submodule of  $V$ . We claim that  $H^0(\lambda)^{G_1} = 0$ . Indeed,  $H^0(\lambda)$  has  $G$ -socle  $L(\lambda)$ . Were  $H^0(\lambda)^{G_1}$  non-zero, we would have  $L(\lambda) = \text{soc}_G H^0(\lambda) \subseteq H^0(\lambda)^{G_1}$ . However, since  $\lambda \in X_1$ ,  $L(\lambda)$  is a simple *non-trivial*  $G_1$  module. This verifies the claim.

When  $\lambda$  is not one of the two exceptions listed above, we have an isomorphism

$$H^1(G_1, L(\lambda)) \simeq (H^0(\lambda)/L(\lambda))^{G_1};$$

it is now easy to verify the above calculations.

We consider now the two exceptional situations mentioned above; thus  $p = 3$ . First suppose  $\Phi = A_3$  and  $\lambda = \varpi_1 + \varpi_2$ . One observes that in this case  $H^0(\lambda)/L(\lambda)$  is a simple non-trivial  $G_1$  module; the exact sequence together with [11] Proposition 4.1 yields in this case

$$H^1(G_1, L(\lambda)) \simeq H^1(G_1, H^0(\lambda)) \simeq L(\varpi_1)^{[1]}.$$

If  $\Phi = A_2$  and  $\lambda = \varpi_1 + \varpi_2$ , we quote [11] 4.10(3) to get the stated cohomology.  $\square$

We are now able to verify another special case of our main theorem.

**Proposition 5.3.4.** *Suppose that  $p$  is not special. Let  $\lambda \in X_1$ ,  $0 \neq \lambda' \in pX_+$  and write  $L = L(\lambda)$  and  $L' = L(\lambda')$ . Assume that  $\dim_k L \leq \mathcal{C}p$ . Then  $\text{Ext}_G^1(L, L') = 0$  unless  $p = 2$ ,  $\Phi = A_\ell$  and  $\{\lambda, \mu\}$  is either  $\{\varpi_2, 2\varpi_1\}$  or  $\{\varpi_{\ell-1}, 2\varpi_\ell\}$ .*

*Proof.* Suppose that  $\lambda \neq 0$ . By (a) of 2.3.3, we can deduce the vanishing of  $\text{Ext}_G^1(L, L') \simeq \text{Ext}_G^1(L', L)$  provided we show that  $\text{Ext}_{G_1}^1(k, L) = H^1(G_1, L) = 0$ .

In Lemma 5.3.3, we determined the weights  $\lambda$  for which  $\dim_k L(\lambda) \leq \mathcal{C}p$  and  $H^1(G_1, L(\lambda)) \neq 0$ . Thus, we obtain the proposition unless  $\lambda$  is on this list; suppose now that this is so.

Again by (a) of 2.3.3, we have

$$(5.3.a) \quad \text{Ext}_G^1(L(\lambda'), L(\lambda)) \simeq \text{Hom}_G(L(p^{-1}\lambda'), H^1(G_1, L(\lambda))^{[-1]}).$$

Suppose that  $\lambda$  is in table 5.3.1 but that  $\lambda$  is not  $\omega_1 + \omega_2$  when  $\Phi = A_2$  and  $p = 3$ . Then  $H^1(G_1, L(\lambda))^{[-1]}$  is a simple  $G$  module. Thus, the Hom space is non-zero only when  $L(p^{-1}\lambda') \simeq H^1(G_1, L)^{[-1]}$ . For all but one of the remaining weights,  $H^1(G_1, L)^{[-1]} \simeq L(0)$ ; this forces  $\lambda' = 0$  contrary to our hypothesis.

We now handle the remaining cases. When  $p = 2$ ,  $\Phi = A_\ell$  and  $\lambda = \omega_2, \omega_{\ell-1}$ , we have  $H^1(G_1, L)^{[-1]} \simeq L(\omega_1)$ , resp.  $L(\omega_\ell)$ . This forces  $\lambda' = 2\omega_1$ , resp.  $2\omega_\ell$ ; we get in this way the exceptional  $\{\lambda, \mu\}$  of the lemma.

When  $p = 3$ ,  $\Phi = A_2$  and  $\lambda = \omega_1 + \omega_2$ , we have  $H^1(G_1, L)^{[-1]} \simeq L(0) \oplus L(\omega_1) \oplus L(\omega_2)$ . Thus  $\lambda'$  can be  $0, 3\omega_1, 3\omega_2$ ; however, we have already discussed the situation where  $\lambda' = 0$ . If  $\lambda' = 3\omega_1$  or  $3\omega_2$ , then  $\dim_k L + \dim_k L' = 10$  whereas  $\mathcal{C}p = 9$ ; this contradicts our dimension assumption.  $\square$

**5.4. Some technical results on tensor decomposable simple modules.** To prove the general theorem, we shall have to handle weights which may fail to be restricted. To do this, we will reduce to the situation where  $p$  is fairly small; in this setting, we can show that there are relatively few weights with more than one term in their  $p$ -adic expansion.

In order to obtain these reductions, we shall use the following lemma to obtain lower bounds for module dimensions.

**Lemma 5.4.1.** *Let  $V$  be a  $G$  module, and suppose that  $G$  acts non-trivially on  $V$ . Then  $\dim_k V \geq M$  where  $M$  is described by:*

- $\Phi = A_\ell$ :  $M = \ell + 1$ .
- $\Phi = B_\ell$ : when  $p = 2$ ,  $M = 2\ell$ ; when  $p > 2$ ,  $M = 2\ell + 1$ .
- $\Phi = C_\ell, D_\ell$ :  $M = 2\ell$ .
- $\Phi = E_6, E_7, E_8$ :  $M = 27, 56, 248$  resp.
- $\Phi = F_4$ : when  $p = 3$ ,  $M = 25$ ; when  $p \neq 3$ ,  $M = 26$ .
- $\Phi = G_2$ : when  $p = 2$ ,  $M = 6$ ; when  $p > 2$ ,  $M = 7$ .

*Proof.* See [13], 5.4.13.  $\square$

We shall use the following lemma to bound the prime  $p$ .

**Lemma 5.4.2.** *Let  $\ell \geq 2$ . Assume that  $p$  is not special, and that  $\mu$  is a restricted weight. Suppose that  $\langle \mu + \rho, \alpha_0 \rangle \geq p$ . If  $\dim_k L(\mu) \leq \mathcal{C} \cdot p$ , then  $p$  satisfies the following condition*

$$(5.4.a) \quad p \leq \begin{cases} \ell + 4 & \text{if } \Phi = A_\ell. \\ 2\ell + 5 & \text{if } \Phi = B_\ell. \\ 2\ell + 2 & \text{if } \Phi = C_\ell. \\ 2\ell - 1 & \text{if } \Phi = D_\ell. \end{cases} \quad ; \quad p \leq \begin{cases} 13 & \text{if } \Phi = E_6. \\ 19 & \text{if } \Phi = E_7. \\ 31 & \text{if } \Phi = E_8. \\ 13 & \text{if } \Phi = F_4. \\ 13 & \text{if } \Phi = G_2. \end{cases}$$

*Proof.* The assumption on  $\mu$  together with Lemma 4.1.1 shows that  $\lambda \in \mathcal{I}$ . The condition (5.4.a) now follows from a determination of the maximal value that  $\langle \lambda + \rho, \alpha_0 \rangle$  attains as  $\lambda$  ranges over  $\mathcal{I}$  (see table 3.1.1). Observe that we have “rounded down” to the nearest prime for the exceptional types.  $\square$

**Lemma 5.4.3.** *Let  $\Phi$  be an irreducible root system of rank  $\ell \geq 2$ , and suppose that  $p \neq 2$ . Let  $\lambda = \lambda_0 + p^s \lambda_1$  where  $\lambda_0 \in X_1$ ,  $\lambda_1 \notin pX$  and  $s > 0$ . Assume that (5.4.a) holds. If  $\dim_k L(\lambda) \leq \mathcal{C}p$ , then  $\Phi$  must be of type  $A_\ell$ ,  $B_\ell$ ,  $C_\ell$ , or  $D_\ell$ . Furthermore,  $\lambda_1$  is restricted, and we have*

$$\dim_k L(\lambda_0) \leq 2\mathcal{C} \quad \text{and} \quad \dim_k L(\lambda_1) \leq 2\mathcal{C}.$$

*Proof.* Since  $\dim_k L(\lambda_0) < \dim_k L(\lambda)$ , the hypothesis on  $\lambda_0$  together with Lemma 5.4.2 show that  $p$  satisfies (5.4.a).

Let us denote by  $N$  the upper bound for  $p$  described by (5.4.a). Steinberg’s tensor product Theorem 2.2.1 shows that  $L(\lambda) \simeq L(\lambda_0) \otimes L(\lambda_1)^{[s]}$ ; in particular we obtain

$$\dim_k L(\lambda_0) \leq \mathcal{C}p / \dim_k L(\lambda_1) \leq \mathcal{C}N/M.$$

Since  $p \neq 2$ , one has  $M = 27, 56, 248, 25, 7$  for  $G$  is of type  $E_6, E_7, E_8, F_4, G_2$  respectively. One checks in each case that  $\frac{\mathcal{C} \cdot N}{M} < M$ ; according to the above inequality this implies that  $L(\lambda_0)$  is trivial so that  $\lambda_0 = 0$ . This contradicts the hypothesis; we deduce that  $\Phi = A_\ell, B_\ell, C_\ell$ , or  $D_\ell$ .

For the remaining root system types, one checks that  $N/M \leq 2$ ; the dimensional assertions now follow. Finally, observe that  $M^2 > 2\mathcal{C}$  for each root system type; this shows that  $\dim_k L(\lambda_1) < M^2$  so that  $\lambda_1 \in X_1$ .  $\square$

For a dominant weight  $\lambda$ , let  $I_\lambda = \{\alpha_i \in \Delta \mid \langle \lambda, \alpha_i \rangle > 0\}$  be the *support* of  $\lambda$ . The length of the Weyl group orbit of  $\lambda$  is completely determined by  $I_\lambda$ ; indeed, the stabilizer in  $\mathcal{W}$  of  $\lambda$  is the Weyl group whose Dynkin diagram is determined by the complement of the support of  $\lambda$ .

Some Weyl group orbit orders are best expressed as certain binomial coefficients. We observe the following inequality:

$$(5.4.b) \quad \binom{n+1}{j} \geq \binom{n+1}{i} \quad \text{provided } i \leq j \leq n-i.$$

To verify this, it suffices to consider only  $j = i+1$  (and  $i < \frac{n}{2}$ ). One then observe that

$$\binom{n+1}{i+1} = \binom{n+1}{i} \cdot \frac{n+1-i}{i+1} \quad \text{and the inequality follows.}$$

**Lemma 5.4.4.** *Let  $\Phi = A_\ell, B_\ell, C_\ell$  or  $D_\ell$ . Fix  $\lambda \in X_+$ ,  $\lambda \neq 0$ , and assume that  $|\mathcal{W}\lambda| \leq 2\mathcal{C}$ . Then  $I = I_\lambda$  must be one of the following:*

- $\Phi = A_\ell$  and  $\ell \geq 2$ ,  $I = \{\alpha_1\}, \{\alpha_\ell\}, \{\alpha_2\}, \{\alpha_{\ell-1}\}$ , or  $\{\alpha_1, \alpha_\ell\}$ .
- $\Phi = A_5, A_6, A_7$ ,  $I = \{\alpha_3\}$  or  $\{\alpha_{\ell-2}\}$ .
- $\Phi = A_3$ ,  $I = \{\alpha_1, \alpha_2\}$  or  $\{\alpha_2, \alpha_3\}$ .
- $\Phi = B_\ell$  ( $\ell \geq 3$ ),  $C_\ell$  ( $\ell \geq 2$ ), or  $D_\ell$  ( $\ell \geq 4$ ),  $I = \{\alpha_1\}$  or  $\{\alpha_2\}$ .
- $\Phi = B_\ell$  or  $C_\ell$  and  $\ell = 3, 4, 5$ ,  $I = \{\alpha_\ell\}$ .
- $\Phi = D_\ell$  and  $\ell = 4, 5, 6, 7$ ,  $I = \{\alpha_{\ell-1}\}$  or  $\{\alpha_\ell\}$ .

*Proof.* Suppose that  $\Phi = A_\ell$ . If  $\alpha_i \in I$  for  $3 \leq i \leq \ell - 2$ , one applies (5.4.b) to get

$$|\mathcal{W}\lambda| \geq |\mathcal{W}\omega_i| \geq |\mathcal{W}\omega_3| = \binom{\ell+1}{3} = \frac{\ell-1}{3}\mathcal{C} > 2\mathcal{C} \quad (\ell \geq 7).$$

Thus, we must have  $I \subseteq \{\alpha_1, \alpha_2, \alpha_{\ell-1}, \alpha_\ell\}$  for  $\ell \geq 7$ . One now verifies by hand that  $|\mathcal{W}\lambda| > 2\mathcal{C}$  in the following situations: when  $\ell = 3$  and  $I = \Delta$ ; when  $\ell = 5, 6, 7$  and  $\{\alpha_3\}$  is a proper subset of  $I$ ; and when  $\ell \geq 4$  and  $I = \{\alpha_2, \alpha_\ell\}$ ,  $I = \{\alpha_1, \alpha_{\ell-1}\}$ , or  $I = \{\alpha_2, \alpha_{\ell-1}\}$ . This verifies the claim for  $A_\ell$ .

Consider now  $\Phi = B_\ell, C_\ell$ , or  $D_\ell$ . Notice that if  $\alpha_i \in I$  for  $3 \leq i \leq \ell - 2$ , we have

$$|\mathcal{W}\lambda| \geq |\mathcal{W}\omega_i| = 2^i \binom{\ell}{i} > 2^{i-1} \binom{\ell}{2} = 2^{i-2}\mathcal{C}.$$

Thus, we must have  $I \subset \{\alpha_1, \alpha_2, \alpha_{\ell-1}, \alpha_\ell\}$ . We rule out  $\omega_1 + \omega_2$  by noting that

$$|\mathcal{W}(\omega_1 + \omega_2)| = 2^3 \binom{\ell}{2} > 2\mathcal{C}.$$

Suppose that  $\Phi = B_\ell$  or  $C_\ell$ . We observe that when  $\ell \geq 4$  and  $\alpha_{\ell-1} \in I$ ,  $|\mathcal{W}\lambda| \geq \ell 2^{\ell-1} > 2\mathcal{C}$ . When  $\ell \geq 6$  and  $\alpha_\ell \in I$ ,  $|\mathcal{W}\lambda| \geq 2^\ell > 2\mathcal{C}$ . One checks directly that for  $\ell = 3, 4, 5$ ,  $I = \{\alpha_1, \alpha_\ell\}$  and  $I = \{\alpha_2, \alpha_\ell\}$  do not satisfy the hypothesis; this completes the claim for these root systems.

Finally, suppose that  $\Phi = D_\ell$ . When  $\ell \geq 6$  and  $\alpha_\ell \in I$  or  $\alpha_{\ell-1} \in I$ ,  $|\mathcal{W}\lambda| \geq 2^{\ell-1} > 2\mathcal{C}$ . When  $\ell = 4, 5$ , one checks directly that  $I = \{\alpha_1, \alpha_\ell\}$ ,  $I = \{\alpha_1, \alpha_{\ell-1}\}$ ,  $I = \{\alpha_2, \alpha_\ell\}$ , and  $I = \{\alpha_2, \alpha_{\ell-1}\}$  do not satisfy the hypothesis. This completes the proof for  $D_\ell$ .  $\square$

*Definition 5.4.5.* . Let  $\lambda \in X_+ \setminus pX_+$  and assume that  $\lambda$  is not a restricted weight. Then  $\lambda$  may be written in the form  $\lambda = \lambda_0 + p^s \lambda_1$  with  $0 \neq \lambda_0 \in X_1$ ,  $0 \neq \lambda_1 \in X_+ \setminus pX_+$ , and  $s \in \mathbb{N}_{>0}$ . We shall say that  $\lambda$  is a *pnr weight* (possible non-restricted weight) if the set  $\{\lambda_0, \lambda_1\}$  is on the following list.

- $\Phi = A_\ell, \ell \geq 2$ :  $\{\omega_1, \omega_\ell\}, \{\omega_1\}, \{\omega_\ell\}$ .
- $\Phi = A_\ell, 2 \leq \ell < p - 3$ :  $\{2\omega_1, \omega_\ell\}, \{2\omega_1, \omega_1\}, \{\omega_1, 2\omega_\ell\}, \{\omega_\ell, 2\omega_\ell\}$ .
- $\Phi = A_\ell, 2 \leq \ell < p - 1$ :  $\{\omega_1, \omega_2\}, \{\omega_1, \omega_{\ell-1}\}, \{\omega_2, \omega_\ell\}, \{\omega_{\ell-1}, \omega_\ell\}$ .
- $\Phi = B_\ell$  ( $\ell \geq 3$ ),  $\Phi = C_\ell$  ( $\ell \geq 2$ ),  $\Phi = D_\ell$  ( $\ell \geq 5$ ):  $\{\omega_1, \omega_1\}$
- $\Phi = B_3, p = 11, \{\omega_1, \omega_3\}$ .
- $\Phi = B_4, p = 13, \{\omega_1, \omega_4\}$ .
- $\Phi = D_4, \{\omega_i, \omega_j\}$  where  $i, j \in \{1, 3, 4\}$

**Lemma 5.4.6.** *Let  $\Phi$  be an irreducible root system of rank  $\ell \geq 2$ , suppose that  $p$  satisfies condition (5.4.a) and that  $p \neq 2$ . Let  $\lambda \in X_+ \setminus pX$  and assume that  $\lambda$  is not a restricted weight. Let  $M$  denote the minimal dimension of a non-trivial  $G$  module (see Lemma 5.4.1). If  $\dim_k L(\lambda) \leq \mathcal{C}p - M$ , one can conclude that  $\lambda$  is a pnr weight (see definition 5.4.5).*

*Proof.* For  $i = 1, 2$ , Lemma 5.4.3 shows that  $\lambda_i$  is restricted and has  $\dim_k L(\lambda_i) \leq 2\mathcal{C}$ . Since  $|\mathcal{W}\lambda_i| \leq \dim_k L(\lambda_i)$ , we may apply Lemma 5.4.4 to learn that  $\lambda_i$  has support  $I_{\lambda_i}$  as specified in that lemma. We claim also that  $\lambda_i \in \mathcal{I}$ . Indeed, suppose that  $\lambda_i \notin \mathcal{I}$ ; Proposition 3.2.1 shows that  $\lambda_i$  is allowable. We therefore have

$$\mathcal{C}\langle \lambda_i + \rho, \alpha_0 \check{\rangle} \leq \dim_k L(\lambda_i) \leq 2\mathcal{C}$$

so that  $\langle \lambda_i + \rho, \alpha_0 \check{\rangle} \leq 2$ . Since  $\ell \geq 2$  and  $\lambda_i \neq 0$ , this is impossible.

Let  $\Phi = A_\ell$ . Fix  $i \in \{0, 1\}$ . Assume that  $I_{\lambda_i} = \{\alpha_3\}$  when  $\ell = 5, 6, 7$ ,  $\{\alpha_4\}$  when  $\ell = 6$ , or  $\{\alpha_5\}$  when  $\ell = 7$ . Inspecting table 3.1.1, we observe that  $\lambda_i = \mathfrak{w}_3$  or  $\mathfrak{w}_3^*$ . It follows from Proposition 4.2.2 that  $\dim_k L(\lambda_i) = \binom{\ell+1}{3}$ . Note that “rounding down to the nearest prime”, (5.4.a) shows that  $p \leq N' = 7, 7, 11$  for  $\ell = 5, 6, 7$  respectively. Using the minimal module dimension  $M = \ell + 1$ , we have  $(\ell + 1) \binom{\ell+1}{3} \leq \dim_k L(\lambda) \leq \mathcal{C}p \leq \mathcal{C}N'$ . For  $\ell = 5, 6, 7$  we get  $(\ell + 1) \binom{\ell+1}{3} = 120, 245, 448$  and  $\mathcal{C}N' = 105, 147, 308$ ; these numbers are incompatible with the inequality. It follows that  $\lambda_i$  can be neither  $\mathfrak{w}_3$  nor  $\mathfrak{w}_3^*$ .

Next, suppose that  $I_{\lambda_i} = \{\alpha_1, \alpha_\ell\}$ . Inspecting table 3.1.1, we deduce that  $\lambda_i = \mathfrak{w}_1 + \mathfrak{w}_\ell$ ,  $\lambda_i = 2\mathfrak{w}_1 + \mathfrak{w}_\ell$ , or  $\lambda_i = \mathfrak{w}_1 + 2\mathfrak{w}_\ell$ . One has the estimate  $\dim_k L(\lambda_i) \geq |\mathcal{W}\lambda_i| = 2\mathcal{C}$ ; thus, we have  $(\ell + 1)2\mathcal{C} \leq \dim_k L(\lambda) \leq \mathcal{C}p$ . It follows that  $2\ell + 2 \leq p$  which is incompatible with (5.4.a) for  $\ell \geq 3$ . When  $\ell = 2$ , Proposition 4.6.10 may be used to verify the explicit bound  $\dim_k L(\lambda_i) \geq 8$ . It follows that  $3 \cdot 8 = 24 \leq 2p$  so that  $12 \leq p$ ; in particular,  $p$  fails to satisfy (5.4.a). We deduce in all cases that  $I_{\lambda_i} \neq \{\alpha_1, \alpha_\ell\}$ .

When  $\ell = 3$ , we must consider  $I_{\lambda_i} = \{\alpha_1, \alpha_2\}$  or  $I_{\lambda_i} = \{\alpha_2, \alpha_3\}$ . In this case, one has  $\lambda_i = \mathfrak{w}_1 + \mathfrak{w}_2$ ,  $2\mathfrak{w}_1 + \mathfrak{w}_2$ ,  $\mathfrak{w}_2 + 2\mathfrak{w}_3$  or  $\mathfrak{w}_2 + \mathfrak{w}_3$ . We have in all cases the estimate  $\dim_k L(\lambda_i) \geq |\mathcal{W}\lambda_i| = 12$ . Using the minimal non-trivial module dimension  $M = 4$ , we obtain  $4 \cdot 12 \leq \dim_k L(\lambda) \leq 6 \cdot p$ ; this is incompatible with (5.4.a).

Next, assume that  $I_{\lambda_i} = \{\alpha_1\}$  or  $\{\alpha_\ell\}$ . Inspection of table 3.1.1 shows that  $\lambda_i = r\mathfrak{w}_1$  or  $r\mathfrak{w}_\ell$  for  $r = 1, 2, 3, 4$  (with some conditions on  $\ell$ ). We claim that we must have  $\lambda_i = \mathfrak{w}_1, \mathfrak{w}_\ell, 2\mathfrak{w}_1$  or  $2\mathfrak{w}_\ell$ . Indeed, since  $\lambda_i$  is restricted we know by Proposition 4.2.2 that  $L(r\mathfrak{w}_1) \simeq S^r V$ ; it follows that  $\dim_k L(r\mathfrak{w}_1) = \binom{\ell+r}{r} \geq \binom{\ell+3}{3}$  for  $r \geq 3$ . Using the minimal non-trivial module dimension  $\ell + 1$ , we obtain  $(\ell + 1) \binom{\ell+3}{3} \leq \dim_k L(\lambda) \leq \mathcal{C}N = \mathcal{C}(\ell + 4)$ . One checks that this condition is impossible.

Finally, notice that if  $I_{\lambda_i} = \{\alpha_2\}$ , then inspection of table 3.1.1 leads to  $\lambda_i = \mathfrak{w}_2$  for  $\ell \geq 2$  or  $\lambda_i = 2\mathfrak{w}_2$  for  $3 \leq \ell \leq 7$ . If  $\lambda_i = 2\mathfrak{w}_2$ , one notes that  $\mathfrak{w}_1 + \mathfrak{w}_3$  is a subdominant weight so that  $\dim_k L(\lambda_i) \geq \binom{\ell+1}{3}$ ; we rule this out just as in the case  $I_{\lambda_i} = \{\alpha_3\}$  above.

To prove the result for  $A_\ell$ , three tasks remain: (i) we must rule out the configurations  $\{\lambda_0, \lambda_1\} = \{2\mathfrak{w}_1\}, \{2\mathfrak{w}_\ell\}, \{2\mathfrak{w}_1, 2\mathfrak{w}_\ell\}, \{\mathfrak{w}_2, \mathfrak{w}_{\ell-1}\}, \{\mathfrak{w}_2, 2\mathfrak{w}_1\}, \{\mathfrak{w}_{\ell-1}, 2\mathfrak{w}_1\}, \{\mathfrak{w}_{\ell-1}, 2\mathfrak{w}_\ell\}$  and  $\{\mathfrak{w}_2, 2\mathfrak{w}_\ell\}$ , (ii) we must establish the condition  $\ell + 3 < p$  when some  $\lambda_i = 2\mathfrak{w}_1$  or  $2\mathfrak{w}_\ell$ , and (iii) we must establish the condition  $\ell + 1 < p$  when some  $\lambda_i = \mathfrak{w}_2$  or  $\mathfrak{w}_{\ell-1}$ .

Since  $\dim_k L(\mathfrak{w}_2) = \dim_k L(\mathfrak{w}_{\ell-1}) = \mathcal{C}$  and  $\dim_k L(2\mathfrak{w}_1) = \dim_k L(2\mathfrak{w}_\ell) = \binom{\ell+2}{2} > \mathcal{C}$ , it suffices for (i) to point out that  $\mathcal{C}^2 \leq \dim_k L(\lambda) \leq \mathcal{C}p$  leads to  $\mathcal{C} \leq p$ , which is incompatible with (5.4.a).

If  $\lambda_i$  is as indicated in (ii), one obtains the condition

$$(\ell + 1) \binom{\ell+2}{2} = (\ell + 1) \dim_k L(\lambda_i) \leq \dim_k L(\lambda) \leq \mathcal{C}p - M;$$



if  $\lambda_i$  is as indicated in (iii), one obtains

$$(\ell + 1)\mathcal{C} = (\ell + 1) \cdot \dim_k L(\lambda_i) \leq \dim_k L(\lambda) \leq \mathcal{C}p - M.$$

In each case, the result follows after division by  $\mathcal{C}$ .

Now suppose that  $\Phi = B_\ell, C_\ell,$  or  $D_\ell$ . If  $\{\lambda_0, \lambda_1\} = \{\omega_1\}$ , then evidently  $L(\lambda)$  has sufficiently small dimension. We are thus led to the situation  $I_{\lambda_i} \neq \{\alpha_1\}$ .

First, suppose that  $I_{\lambda_i} = \{\alpha_2\}$ . According to table 3.1.1, we have  $\lambda_i = \omega_2$ . According to Propositions 4.2.2 and 4.8.2, one has  $\dim_k L(\lambda_i) \geq \binom{2\ell}{2} - 1$ . Since  $\binom{2\ell}{2} - 1 > 2\mathcal{C}$ , (a) shows that this configuration is impossible.

Next, suppose that  $\Phi = B_\ell$  or  $C_\ell$ ,  $\ell = 3, 4, 5$  and  $I_{\lambda_i} = \{\alpha_\ell\}$ . If  $\Phi = B_\ell$ , table 3.1.1 shows that we must have  $\lambda_i = \omega_\ell$  or  $2\omega_\ell$ . Since  $\dim_k L(\lambda_i) \geq 2^\ell$ , we obtain

$$(2\ell + 1)2^\ell \leq \dim_k L(\lambda) \leq \mathcal{C}p$$

using the minimal module dimension for the other tensor factor. The left hand side of this expression is 56, 144, 352 when  $\ell = 3, 4, 5$ ; division by  $\mathcal{C}$  gives respectively  $9 < p$ ,  $12 < p$ , and  $18 < p$ . When  $\ell = 5$ , this is incompatible with (5.4.a). When  $\ell = 3, 4$ , one notes that the remaining tensor factor must be a twist of  $L(\omega_1)$ ; this gives the weights described in the statement. When  $\Phi = C_\ell$ , table 3.1.1 shows that  $\lambda_i = \omega_\ell$ . Since  $\dim_k L(\lambda_i) \geq 2^\ell$  again, we have  $2\ell \cdot 2^\ell \leq \dim_k L(\lambda) \leq \mathcal{C}p$  (again using the minimal module dimension for the other tensor factor). The left hand side is 48, 128, 320 when  $\ell = 3, 4, 5$ ; dividing by  $\mathcal{C}$  we get  $8 < p$ ,  $10 < p$ ,  $16 < p$  respectively. These inequalities are not compatible with (5.4.a).

Finally, if  $\Phi = D_\ell$ ,  $\ell = 4, 5, 6, 7$  and  $I_\lambda = \{\alpha_\ell\}$  or  $\{\alpha_{\ell-1}\}$ , one checks table 3.1.1 to see that  $\lambda_i = \omega_\ell$  or  $\omega_{\ell-1}$ . The claim when  $\ell = 4$  now follows; so assume  $\ell = 5, 6, 7$ . We have  $\ell \cdot 2^\ell \leq \dim_k L(\lambda) \leq \mathcal{C}p$ . Computing  $\ell \cdot 2^\ell / \mathcal{C}$ , we deduce that  $11 \leq p$ ,  $13 \leq p$ , and  $23 \leq p$  for  $\ell = 5, 6, 7$ . These conditions are incompatible with (5.4.a); the lemma now follows.  $\square$

We now present some technical results on extensions between simple modules where at least one of the high weights is a *pnr weight*.

**Lemma 5.4.7.** *Let  $\lambda = \lambda_0 + p^r \lambda_1$  be a pnr weight. If  $\mu \in pX$ , then  $\text{Ext}_G^1(L(\lambda), L(\mu)) = 0$ .*

*Proof.* We argue as in Lemma 5.3.4; in particular, it suffices to show that  $H^1(G_1, L(\lambda_0)) = 0$ . To deduce this, one compares the list of Lemma 5.3.3 with the possible  $\lambda_0$  from (5.4.5).  $\square$

The extension arguments we shall now give rely on Lemma 2.3.3 (a); this result describes extensions of simple modules  $L$  and  $L'$  where the highest weights  $\lambda$  and  $\lambda'$  have distinct leading  $p$ -adic terms. Let us make the following definition.

*Definition 5.4.8.* For  $0 \neq \mu \in X$ , define  $h_p(\mu) = \max\{i \geq 0 \mid \mu \in p^i X\}$ ; put  $h_p(0) = \infty$ .

It is clear that  $\lambda$  and  $\lambda'$  will have satisfy the hypothesis of Lemma 2.3.3 (a) just when  $h_p(\lambda - \lambda') = 0$ .

**Lemma 5.4.9.** *Let  $\lambda = \lambda_0 + p^s \lambda_1$  where  $\lambda_0, \lambda_1 \in X_1$ ,  $s \in \mathbb{N}_{>0}$ , and let  $\mu \in X_1$ . Suppose that  $h_p(\lambda - \mu) \neq 0$ ,  $H^0(\lambda_0) = L(\lambda_0)$  and  $H^0(\mu) = L(\mu)$ . If  $\text{Ext}_G^1(L(\lambda), L(\mu)) \neq 0$  then  $\xi_i \leq \mu + \lambda_0^*$  for some  $1 \leq i \leq \ell$ , where  $\xi_i = p\omega_i - \alpha_i$ .*

*Proof.* According to (a) of Lemma 2.3.3, we have

$$(5.4.c) \quad \text{Ext}_G^1(L(\lambda), L(\mu)) \simeq \text{Hom}_G(L(p^{s-1}\lambda_1), \text{Ext}_{G_1}^1(L(\lambda_0), L(\mu))^{[-1]}).$$

There is a natural isomorphism

$$(5.4.d) \quad \text{Ext}_{G_1}^1(L(\lambda_0), L(\mu)) \simeq H^1(G_1, L(\lambda_0)^* \otimes L(\mu)).$$

Since  $L(\lambda_0) = H^0(\lambda_0)$ , the dual of this module is again an induced module; indeed,  $L(\lambda_0)^* = H^0(\lambda_0^*)$ . Thus,  $M = L(\lambda_0)^* \otimes L(\mu) = H^0(\lambda_0^*) \otimes H^0(\mu)$ . According to Lemma 4.6.2,  $M$  has an  $H$ -filtration.

In Jantzen's paper [11] Proposition 4.1, it is shown that

$$H^1(G_1, H^0(\xi)) = 0 \text{ unless } \xi = p\varpi_i - \alpha_i \text{ for some } 1 \leq i \leq \ell;$$

furthermore, for  $p > 3$  we have always  $H^1(G_1, H^0(p\varpi_i - \alpha_i))^{[-1]} = L(\varpi_i)$ .

Applying (a) of Lemma 4.6.4 to the group scheme  $G_1$ , we deduce that  $H^1(G_1, M) \neq 0$  implies  $H^0(\xi_i)$  is a filtration factor of  $M$  for some  $1 \leq i \leq \ell$ ; in particular,  $\xi_i \leq \lambda_0^* + \mu$ .  $\square$

**Lemma 5.4.10.** *Let  $\lambda$  be a pnr weight in the sense of definition 5.4.5. If  $\mu \in X_+$ ,  $h_p(\lambda - \mu) \neq 0$ , and furthermore*

$$(5.4.e) \quad \dim_k L(\lambda) + \dim_k L(\mu) \leq \mathcal{C}p,$$

then we can conclude that  $\text{Ext}_G^1(L(\lambda), L(\mu)) = 0$  provided that  $\mu$  satisfies one of the following:

- (a)  $\mu$  is a restricted weight,  $\mu \in \mathcal{I}$ , and  $L(\mu) = H^0(\mu)$ .
- (b)  $\mu$  is a pnr weight.

*Proof.* Suppose that  $\lambda$  satisfies the hypothesis (5.4.e) for some  $\mu$ . Steinberg's tensor product Theorem 2.2.1 permits us to deduce the following: when  $\Phi = A_\ell$ ,

$$(5.4.f) \quad 2\mathcal{C} < (\ell + 1)^2 \leq \dim_k L(\lambda) \leq p\mathcal{C}$$

so that  $p > 2$ ; when  $\Phi = B_\ell, C_\ell$  or  $D_\ell$ ,

$$(5.4.g) \quad 4\ell^2 \leq \dim_k L(\lambda)$$

so that  $p > 3$ .

To handle (a), we suppose that there is a nontrivial extension of  $L(\lambda)$  by  $L(\mu)$  and derive a contradiction. Observe that the condition in (a) insures that we may apply Lemma 5.4.9. We may thus find  $1 \leq i \leq \ell$  so that  $\xi_i \leq \mu + \lambda_0^*$ . It follows that  $\langle \xi_i, \alpha_0^\vee \rangle \leq \langle \mu + \lambda_0^*, \alpha_0^\vee \rangle$ .

Let us first treat the case  $\Phi = A_\ell$ . In this case, we have the estimate  $\langle \xi_i, \alpha_0^\vee \rangle \geq p - 1$  for each  $i$ . Also, studying table 3.1.1 of possible  $\mu \in \mathcal{I}$ , shows that  $\langle \mu, \alpha_0^\vee \rangle \leq 4$

Assume first that  $\lambda_0 = 2\varpi_1$  or  $2\varpi_\ell$ . For this weight to occur in  $\lambda$ , Lemma 5.4.3 shows that  $\ell < p - 3$ . Applying the above estimate, one has  $\langle \mu + \lambda_0^*, \alpha_0^\vee \rangle \leq 6$ . We deduce that  $p - 1 \leq 6$  so that  $p = 3, 5, 7$ . In particular, we have  $\ell \leq 6$ . The condition  $\ell < p - 3$  rules out  $p = 3, 5$ . When  $p = 7$  and  $\ell = 2, 3$  one determines that  $\dim_k L(\lambda) > 7\mathcal{C}$  which yields the desired contradiction in this case.

For the remaining possibilities for  $\lambda_0$ , we have  $\langle \mu + \lambda_0^*, \alpha_0^\vee \rangle \leq 5$ ; in particular,  $p - 1 \leq 5$  so that  $p = 3, 5$ . On the other hand, we have  $p - 1 \leq \langle \mu + \lambda_0^*, \alpha_0^\vee \rangle = \langle \mu, \alpha_0^\vee \rangle + 1$  so that  $p - 2 \leq \langle \mu, \alpha_0^\vee \rangle$ .

Suppose that  $p = 5$ . The above discussion shows that  $3 \leq \langle \mu, \alpha_0^\vee \rangle$ . Inspecting table 3.1.1, one sees that (up to diagram automorphism) the only  $\mu \in \mathcal{I}$  with this property are as follows:

$3\mathfrak{w}_1$ ;  $2\mathfrak{w}_1 + \mathfrak{w}_\ell$ ; when  $\ell = 5$ ,  $4\mathfrak{w}_1$ ; and when  $\ell = 2$ ,  $3\mathfrak{w}_1 + \mathfrak{w}_2$ . Using Proposition 4.2.2, Proposition 4.6.10, and table 4.5.2, can check for each  $\mu$  that  $\dim_k L(\lambda) + \dim_k L(\mu) > 5\mathfrak{C}$ , contrary to (5.4.e).

Now let  $p = 3$ . Conditions (5.4.e) and (5.4.f) lead to  $\dim_k L(\mu) \leq 3\mathfrak{C} - \dim_k L(\lambda) < \mathfrak{C}$ ; in particular, we must have  $|\mathcal{W}\mu| < \mathfrak{C}$ . One should observe that the only weights  $\mu \in \mathcal{I}$  satisfying this condition are  $\mu = \mathfrak{w}_1$  and  $\mathfrak{w}_\ell$ .

If  $\mathfrak{w}_2$  or  $\mathfrak{w}_{\ell-1}$  occurs as a  $p$ -adic term in  $\lambda$ , then  $\dim_k L(\lambda) = (\ell + 1) \binom{\ell + 1}{2}$ ; since  $\ell \geq 2$ , this will contradict (5.4.e). We similarly rule out  $\mathfrak{w}_3$  when  $\ell = 5$ . We may thus assume  $\{\lambda_0, \lambda_1\} = \{\mathfrak{w}_1\}$ ,  $\{\mathfrak{w}_\ell\}$  or  $\{\mathfrak{w}_1, \mathfrak{w}_\ell\}$ . It follows that  $\dim_k L(\lambda) + \dim_k L(\mu) = (\ell + 1)^2 + (\ell + 1)$  which can be seen to exceed  $3\mathfrak{C}$  when  $\ell = 2, 3$ . When  $\ell > 3$ ,  $\lambda_0^* + \mu$  is one of  $2\mathfrak{w}_1$ ,  $2\mathfrak{w}_\ell$ , or  $\mathfrak{w}_1 + \mathfrak{w}_\ell$ . It is straightforward to see that  $\xi_i \not\leq \lambda_0^* + \mu$  for all  $1 \leq i \leq \ell$ .

We next consider the case  $\Phi = B_\ell$ . In this case, we have the estimate  $\langle \xi_i, \alpha_0^\vee \rangle \geq 2(p - 1)$  for  $1 \leq i \leq \ell - 1$  and  $\langle \xi_\ell, \alpha_0^\vee \rangle = p$ . Studying the possible  $\mu \in \mathcal{I}$  listed in table 3.1.1, one may check that  $\langle \mu, \alpha_0^\vee \rangle \leq 6$ .

Let us initially suppose that  $\{\lambda_0, \lambda_1\} = \{\mathfrak{w}_1, \mathfrak{w}_\ell\}$  when  $\ell = 3, 4$  with  $p = 11, 13$  respectively. We have  $\langle \xi_i, \alpha_0^\vee \rangle \geq 10$  in all cases; clearly  $10 \leq \langle \mu + \lambda_0^*, \alpha_0^\vee \rangle \leq 6 + \langle \lambda_0^*, \alpha_0^\vee \rangle$  is impossible.

We may now suppose that  $\{\lambda_0, \lambda_1\} = \{\mathfrak{w}_1\}$ . If we suppose that  $\xi_i \leq \mu + \mathfrak{w}_1$  for  $1 \leq i \leq \ell - 1$ , then  $2(p - 1) \leq \langle \mu + \mathfrak{w}_1, \alpha_0^\vee \rangle \leq 8$  so that  $p = 5$ . Assume now that this is the case. Since  $\langle \mu + \mathfrak{w}_1, \alpha_0^\vee \rangle = \langle \mu, \alpha_0^\vee \rangle + 2$ , we deduce that  $6 = \langle \mu, \alpha_0^\vee \rangle$ . Table 3.1.1 shows that we must assume  $\mu = 3\mathfrak{w}_1$ . Since by assumption  $V(3\mathfrak{w}_1) = L(3\mathfrak{w}_1)$ , Proposition 4.7.4 yields  $\dim_k L(3\mathfrak{w}_1) = \binom{2\ell + 3}{3} - (2\ell + 1)$ . On the other hand, (5.4.e) implies

$$\dim_k L(3\mathfrak{w}_1) \leq 5\mathfrak{C} - (2\ell + 1)^2 = \ell^2 - 9\ell - 1.$$

One checks that these conditions are incompatible; this yields the desired contradiction in this case.

To complete the argument for type  $B_\ell$ , we suppose that  $\xi_i \not\leq \mu + \mathfrak{w}_1$  for  $i \neq \ell$ . It is then clear from the computations in [11] that  $H^1(G_1, L(\mu) \otimes L(\mathfrak{w}_1))$  can have only composition factors of type  $L(\mathfrak{w}_\ell)$ ; using this fact, it is easy to argue that the right hand side of (5.4.c) vanishes. It follows that there is no non-trivial extension between  $L(\lambda)$  and  $L(\mu)$  in this case, contrary to our assumption.

We now turn to  $\Phi = C_\ell$ . The only possibility for  $\{\lambda_0, \lambda_1\}$  is  $\{\mathfrak{w}_1\}$ . We have  $\langle \xi_i, \alpha_0^\vee \rangle \geq p$  for  $1 \leq i \leq \ell$ . On the other hand, inspection of table 3.1.1 yields  $\langle \mu + \mathfrak{w}_1, \alpha_0^\vee \rangle \leq 5$ . The condition  $\xi_i \leq \mu + \mathfrak{w}_1$  leads to  $p \leq \langle \xi_i, \alpha_0^\vee \rangle \leq \langle \mu + \mathfrak{w}_1, \alpha_0^\vee \rangle \leq 5$ . Thus, the lemma holds for  $p \neq 5$ .

When  $p = 5$ , we must have  $\langle \mu, \alpha_0^\vee \rangle \geq 4$ . Inspecting table 3.1.1, the only such  $\mu$  are  $\mu = 2\mathfrak{w}_2$  when  $\ell = 3, 4$  and  $\mu = \mathfrak{w}_2 + \mathfrak{w}_4$  when  $\ell = 4$ . Note that when  $\ell = 4$ ,  $5\mathfrak{C} = 60$  and  $\dim_k L(\lambda) = 64$ , so that (5.4.e) is violated. When  $\ell = 3$ ,  $5\mathfrak{C} = 30$  and  $\dim_k L(\lambda) = 36$ ; this again violates (5.4.e).

Next, we consider  $\Phi = D_\ell$ ,  $\ell > 4$ . For these  $\ell$ ,  $\{\lambda_0, \lambda_1\} = \{\mathfrak{w}_1\}$ . We have  $\langle \xi_i, \alpha_0^\vee \rangle \geq p$  for each  $1 \leq i \leq \ell$ . On the other hand, inspection of table 3.1.1 yields the condition  $\langle \mu + \mathfrak{w}_1, \alpha_0^\vee \rangle \leq 4$ . The condition  $\xi_i \leq \mu + \mathfrak{w}_1$  therefore leads to

$$p \leq \langle \xi_i, \alpha_0^\vee \rangle \leq \langle \mu + \mathfrak{w}_1, \alpha_0^\vee \rangle \leq 4.$$

This contradicts the condition  $p > 3$ .

Finally, we consider  $\Phi = D_4$ . Here,  $\{\lambda_0, \lambda_1\} = \{\omega_i, \omega_j\}$  for  $i, j \in \{1, 3, 4\}$ ; up to diagram automorphism we may assume that  $\lambda_0 = \omega_1$ . The argument given for  $\ell > 4$  may now be applied in this case.

We now consider part (b). When  $\Phi = A_\ell$ , recall that we ruled out in the proof of (a) any occurrence of  $2\omega_1$  or  $2\omega_\ell$  as a tensor factor. We now have

$$\text{Ext}_G^1(L(\lambda), L(\mu)) = \text{Hom}_G(L(p^{s-1}L(\lambda_1), H^1(G_1, L(\lambda_0)^* \otimes L(\mu_0))^{[-1]} \otimes L(\mu_1));$$

in particular, (b) will follow provided that (\*)  $H^1(G_1, L(\lambda_0)^* \otimes L(\mu_0)) = 0$ . Since  $p > 2$ , one can check that  $\xi_i \not\leq \lambda_0^* + \mu_0$  for any choices of  $\lambda_0$  and  $\mu_0$ . Since  $L(\lambda_0)^*$  and  $L(\mu_0)$  are both induced modules, this tensor product has an  $H$ -filtration. Condition (\*) now follows from (5.4.c) by arguing as in Lemma 5.4.9. Essentially the same argument settles (b) for types  $B_\ell$ ,  $C_\ell$ , and  $D_\ell$ .  $\square$

**5.5. The proof of Theorem 2.** Let  $L = L(\lambda)$  and  $L' = L(\lambda')$  be simple modules, and suppose that  $\text{Ext}_G^1(L, L') \neq 0$ . We must show that  $\lambda$  and  $\lambda'$  have the correct form. We do this first when the prime  $p$  is not special; we assume this until step 5.

*Step 1. Untwisting.* We claim that it suffices to prove the following:

$$(5.5.a) \quad \text{Let } L, L' \text{ and } \lambda, \lambda' \text{ be as above. If } h_p(\lambda - \lambda') = 0, \text{ then } \{\lambda, \lambda'\} \text{ is on the list of Lemma 5.1.1.}$$

We claim that the general result follows from (5.5.a). Indeed, if  $h_p(\lambda - \lambda') = \infty$ , then  $\lambda = \lambda'$  and  $\text{Ext}_G^1(L, L') = 0$  by [10], II.2.12 (1). If  $h_p(\lambda - \lambda') = j > 0$ , let

$$\mu = \sum_{i=j}^r p^{i-j} \lambda_i \quad \text{and} \quad \mu' = \sum_{i=j}^r p^{i-j} \lambda'_i$$

where  $\lambda = \sum_{i=0}^r p^i \lambda_i$  and  $\lambda' = \sum_{i=0}^r p^i \lambda'_i$ . Then  $\text{Ext}_G^1(L, L') \simeq \text{Ext}_G^1(L(\mu), L(\mu'))$  by part (b) of Lemma 2.3.3. Furthermore,  $h_p(\mu - \mu') = 0$ .

Now, Steinberg's tensor product theorem shows that  $\dim_k L(\mu) \leq \dim_k L$  (and similarly for  $\mu'$ ). Thus, we are in the situation where (5.5.a) applies; we deduce that any non-trivial extension  $F$  between  $L(\mu)$  and  $L(\mu')$  is one of the modules from Lemma 5.1.1. The theorem will follow provided we show that  $E$  is a Frobenius twist of  $F$ . For this, it is enough if  $\mu = p^j \lambda$  and  $\mu' = p^j \lambda'$ . To prove this, we suppose otherwise. Steinberg's tensor product theorem applied to  $L$  and  $L'$  then shows that  $\dim_k E \geq M \cdot \dim_k F$  where  $M$  is the minimal dimension for a non-trivial  $G$  module (see Lemma 5.4.1). Inspecting the possibilities for  $F$ , one sees that  $E$  must fail to satisfy  $\dim_k E \leq \mathcal{C}p$ . This contradiction yields the needed result.

We suppose from now on that  $h_p(\lambda - \lambda') = 0$ .

*Step 2. Conditions on initial  $p$ -adic terms.* Let  $\lambda_i, \lambda'_i$  ( $i \geq 0$ ) be the coefficients of  $\lambda$  and  $\lambda'$  written in their  $p$ -adic expansions; the condition  $h_p(\lambda - \lambda') = 0$  shows that at least one of  $\{\lambda_0, \lambda'_0\}$  is non-zero. We may further restrict these initial terms; namely we may suppose that

$$(5.5.b) \quad \langle \gamma + \rho, \alpha_0^\vee \rangle \geq p \text{ where either } \gamma = \lambda_0 \text{ or } \gamma = \lambda'_0.$$

Indeed, according to Lemma 2.3.2, the group  $\text{Ext}_G^1(L, L')$  will vanish if this condition fails to hold. We suppose from now on that (5.5.b) holds.

*Step 3. Further restrictions on  $\lambda$ .*

According to Proposition 5.2.1, Theorem 2 is true when both  $\lambda$  and  $\lambda'$  are restricted. According to Proposition 5.3.4, the theorem is true provided (say)  $\lambda$  is restricted and  $\lambda' \in pX$ . Thus, we can restrict our attention to the situation where, without loss of generality,  $\lambda$  is not in  $pX$  and has more than one  $p$ -adic term.

The previous step shows that we may assume (5.5.b); combining this with Lemma 5.4.2, we may now suppose that (5.4.a) is valid. We now apply Lemma 5.4.6 to learn that  $\lambda$  is a *pnr weight*.

*Step 4. Handling the possibilities for  $\lambda'$ .* We now consider the possibilities for  $\lambda'$ . First of all, if  $\lambda' \in pX$ , we get the vanishing of Ext by Lemma 5.4.7. We now suppose that  $\lambda' \notin pX$ .

Let us first deal with the case where  $\lambda'$  is not restricted. If this is the case, we may now apply Lemma 5.4.6 to learn that  $\lambda'$  is a *pnr weight* as well. In this case, the vanishing of Ext follows from (b) of Lemma 5.4.10.

This reduces us to the setting where  $\lambda'$  is restricted. The possibilities for  $\lambda'$  are then determined by Lemma 4.1.1; in particular, we have either  $\lambda' \in \mathcal{I}$  or  $\lambda' \in \mathcal{C}$ .

If  $\lambda' \in \mathcal{I}$ , the vanishing of Ext follows from (a) of Lemma 5.4.10. We now suppose that  $\lambda' \in \mathcal{C}$ ; in particular, this means that  $\langle \lambda' + \rho, \alpha_0^\vee \rangle < p$ .

Put  $h = \langle \rho, \alpha_0^\vee \rangle + 1$  (so  $h$  is the Coxeter number of  $\Phi$ ). Evidently, we have  $h < p$ . Since  $\lambda$  is a *pnr weight*, one can observe that  $\langle \lambda_0 + \rho, \alpha_0^\vee \rangle \leq h + 1$  for every possible  $\lambda$ . The condition (5.5.b) guarantees that  $\langle \lambda_0 + \rho, \alpha_0^\vee \rangle \geq p$ ; in particular, we must have  $p \leq \langle \lambda_0 + \rho, \alpha_0^\vee \rangle \leq h + 1$ . Observe that this can only happen if  $h + 1 = p = \langle \lambda_0 + \rho, \alpha_0^\vee \rangle$ . When this occurs, we have  $\langle \lambda + \rho, \alpha_0^\vee \rangle \in p\mathbb{Z}$ ; it is then clear that  $\lambda$  and  $\lambda'$  are not conjugate under  $\mathcal{W}_p$ . The vanishing of Ext now follows from the linkage principle (Proposition 4.4.2).

This completes the proof of Theorem 2 when  $p$  is not special.

*Step 5. Special primes.*

Now suppose that  $p$  is a special prime. There are numerous difficulties in this situation; in particular, the arguments in step 1 fail for type  $C_\ell$ . Furthermore, Propositions 5.2.1 and 5.3.4 were only proved for non-special primes. To handle this case, we give here essentially an independent proof of theorem 2.

To discuss the representation theory of a group of type  $C_\ell$  for the special prime  $p = 2$ , we must discuss the relationship between groups of type  $B_\ell$  and  $C_\ell$  in characteristic 2. Let  $C_\ell(k)$  and  $B_\ell(k)$  denote simply connected groups with the appropriate root systems and suppose  $p = 2$ . According to [24] §10 theorem 28 (p.146), there is an algebraic group homomorphism  $\phi : B_\ell(k) \rightarrow C_\ell(k)$  which is an isomorphism of *abstract groups*. ( $\phi$  is *not* an isomorphism of algebraic groups.) If  $L = L(\lambda; C_\ell)$  is a simple module for  $C_\ell(k)$  where  $\lambda = \sum_{i=1}^{\ell} n_i \omega_i$ , the description of  $\phi$  in [24] makes it clear that when  $L$  is regarded as a module for  $B_\ell$ , its highest weight is  $\sum_{i=1}^{\ell-1} n_i \omega_i + 2n_\ell \omega_\ell$ . (This terminology is mildly abusive since the symbols  $\omega_i$  are playing two roles here; however, no serious confusion should arise.)

The preliminary reductions made above show that when  $p$  is special, Theorem 2 will follow from the following lemma.

**Lemma 5.5.1.** *Assume that  $p$  is special. Let  $\lambda, \mu \in X_+$  satisfy*

$$\dim_k L(\lambda) + \dim_k L(\mu) \leq p\mathcal{C}.$$

If  $\Phi = B_\ell$  and  $p = 2$ , assume that  $\{\lambda, \mu\} \neq \{2^s \varpi_1, 0\}$  for  $s \in \mathbb{N}_{\geq 0}$ , and if  $\Phi = C_\ell$  and  $p = 2$ , assume that  $\{\lambda, \mu\} \neq \{2^s \varpi_1, 0\}$  for  $s \in \mathbb{N}_{> 0}$ . Then  $\text{Ext}_G^1(L(\lambda), L(\mu)) = 0$ .

*Proof.* Since  $\text{Ext}_G^1(L(0), L(0)) = 0$  for any  $G$ , we may assume without loss of generality that  $\lambda \neq 0$ . We will use the minimal module dimensions recorded in Lemma 5.4.1.

Let  $\Phi = G_2$ . Then  $p = 2$  is a special prime. For this  $p$ , the minimal non-trivial module dimension is 6, and  $2\mathcal{C} = 6$ . This implies that  $0 = \lambda = \mu$ , contrary to our supposition above. The prime  $p = 3$  is also special. For this prime, the minimal non-trivial module dimension is 7, and  $3\mathcal{C} = 9$ . We may evidently suppose that  $\mu = 0$ . Since the Weyl group has order 12, we may suppose that  $I_\lambda$  is a singleton. Furthermore, we must have  $\lambda = p^r \lambda'$  with  $\lambda' \in X_1$ ; otherwise Steinberg's tensor product Theorem 2.2.1 yields  $\dim_k L(\lambda) \geq 49$  which exceeds the bound 9. Since  $\mu = 0$ , we have by Lemma 2.3.3 (b) the isomorphism  $\text{Ext}_G^1(L(\lambda), L(0)) \simeq \text{Ext}_G^1(L(\lambda'), L(0))$ . Thus, we are led to consider the following possibilities:  $\lambda' = \varpi_1, 2\varpi_1, \varpi_2$  and  $2\varpi_2$ . In [23], Table 2 on page 103 shows that  $L(\lambda')$  does not extend the trivial module when  $p = 3$ . The result for  $G_2$  now holds.

Let  $\Phi = F_4$ . For this root system,  $p = 2$  is a special prime. The minimal non-trivial module dimension is 26 when  $p = 2$ , whereas  $2\mathcal{C} = 24$ . It follows that  $0 = \lambda = \mu$ ; the lemma therefore holds for type  $F_4$ .

Let  $\Phi = B_\ell$  or  $C_\ell$ ; then  $p = 2$  is a special prime. In each case, the minimal non-trivial module dimension is  $2\ell$  when  $p = 2$ . Notice that  $(2\ell)^2$  exceeds  $2\mathcal{C}$ ; it follows from Steinberg's tensor product Theorem 2.2.1 that  $\lambda = 2^r \lambda'$  and  $\mu = 2^s \mu'$  for  $\lambda', \mu' \in X_1$ . The possibilities for  $\lambda'$  are given by Lemma 5.4.4. They show that (since  $\lambda'$  and  $\mu'$  is restricted)  $\lambda'$  and  $\mu'$  are each one of  $\varpi_1, \varpi_2$ , or  $\varpi_\ell$ ; furthermore  $\varpi_\ell$  may occur only when  $\ell = 3, 4, 5$ . When  $\lambda' = \varpi_2$ , note that  $\dim_k L(\varpi_2) \geq |\mathcal{W}\varpi_2| = 2\mathcal{C}$ . Thus, there is no room for the module  $L(\mu)$ , so we rule out this situation.

Now, suppose that  $\Phi = B_\ell$ . We may assume that  $r \leq s$ . An application of Lemma 2.3.3 (b) shows that

$$\text{Ext}_G^1(L(\lambda), L(\mu)) \simeq \text{Ext}_G^1(L(\lambda'), L(2^{s-r} \mu'));$$

thus, we may assume that  $r = 0$ .

According to Proposition 5.3.2, we have  $H^1(G_1, H^0(\varpi_1)) = 0$  and  $H^1(G_1, H^0(\varpi_\ell)) = 0$ . We claim that  $H^1(G_1, L(\varpi_1)) \simeq k$ . Indeed, this follows by considering the exact sequence  $0 \rightarrow L(\varpi_1) \rightarrow H^0(\varpi_1) \rightarrow k \rightarrow 0$  whose existence is remarked in Proposition 5.1.1. One now considers the corresponding long exact sequence of cohomology; the relevant part of the sequence is

$$\cdots \rightarrow H^0(\varpi_1)^{G_1} \rightarrow k^{G_1} \rightarrow H^1(G_1, L(\varpi_1)) \rightarrow H^1(G_1, H^0(\varpi_1)) = 0$$

The module  $H^0(\varpi_1)^{G_1}$  is by definition the  $G_1$  invariants of  $H^0(\varpi_1)$ ; the group  $G$  acts on these invariants. Since  $L(\varpi_1)$  is the only  $G$  submodule of  $H^0(\varpi_1)$ , and since  $L(\varpi_1)$  is a simple module for  $G_1$ , one deduces that  $H^0(\varpi_1)^{G_1} = 0$ . One gets therefore  $H^1(G_1, L(\varpi_1)) \simeq k^{G_1} \simeq k$ .

Suppose that  $s > 0$  and that  $\mu' \neq 0$ . According to (a) of Lemma 2.3.3 we have

$$(*) \text{Ext}_G(L(\lambda), L(2^s \mu')) = \text{Hom}_G(L(0), H^1(G_1, L(\lambda)) \otimes L(2^{s-1} \mu')).$$

If  $\lambda = \varpi_\ell$ , then  $H^1(G_1, L(\lambda)) = 0$  so  $(*)$  vanishes in this case. If  $\lambda = \varpi_1$ , the right hand side becomes  $\text{Hom}_G(L(0), L(2^{s-1} \mu'))$ . Recall that any  $G$  homomorphism must preserve weight spaces. The module  $L(2^{s-1} \mu')$  has no 0 weight space for  $\mu' = \varpi_1$  or  $\varpi_\ell$ ; thus  $(*)$  vanishes.

We may now suppose that  $s = 0$ . We have the possibilities  $\mu = 0$ ,  $\mu = \varpi_1$  or  $\mu = \varpi_\ell$ . The possibility  $\mu = 0$  was excluded in the statement of the result (and leads to a non-trivial extension; see Proposition 5.1.1). For the remaining possibilities, one knows the radicals of all Weyl modules involved. One applies (c) of lemma 2.3.1 to deduce the vanishing of Ext required for the result.

Finally, let  $\Phi = C_\ell$ . Suppose that  $0 \rightarrow L(\lambda) \rightarrow E \rightarrow L(\mu) \rightarrow 0$  is an exact sequence. We must show that this sequence splits provided  $\{\lambda, \mu\} \neq \{2^s \varpi_1, 0\}$  for  $s = 1, 2, \dots$ . Let us regard  $L(\lambda)$  and  $L(\mu)$  as modules for  $B_\ell(k)$ , and let us write  $\tilde{\lambda}$  and  $\tilde{\mu}$  for the highest weights of these modules with respect to the group  $B_\ell(k)$ . Assume that  $\{\tilde{\lambda}, \tilde{\mu}\} \neq \{2^s \varpi_1, \varpi_0\}$  for  $s = 1, 2, \dots$ . We have just shown that the sequence must split for the group  $B_\ell(k)$ . In particular, there is a section  $s : L(\mu) \rightarrow E$  which commutes with the action of  $B_\ell(k)$ . Since  $\phi$  is a surjection,  $s$  must necessarily commute with the action of  $C_\ell(k)$  so that the sequence is split.

If  $\{\tilde{\lambda}, \tilde{\mu}\} = \{2^s \varpi_1, \varpi_0\}$ , then according to the description of  $\phi$ , we have  $\{\lambda, \mu\} = \{2^s \varpi_1, \varpi_0\}$ . We have excluded these weights provided that  $s \geq 1$ . When  $s = 0$ , one knows that the Weyl module  $V(\varpi_1)$  for type  $C_\ell$  is simple. It follows that  $\text{Ext}_{C_\ell(k)}^1(L(\varpi_1), L(0)) = 0$  in this case.  $\square$

This completes the proof of Theorem 2 for special primes.

## 6. APPENDIX: ALLOWABLE WEIGHT VERIFICATION

In the tables below, we describe the outcome of the procedure required for the proof of Proposition 3.2.8. We remind the reader that a weight marked with  $(\dagger)$  is asserted to be allowable whereas a weight marked with  $(*)$  is not (and is contained in the set  $\mathcal{I}$ ).

For each of the irreducible root system types, we must consider those minimal weights (see Lemma 3.2.2) which fail to be allowable. One may refer to Lemma 3.2.2 for a complete list of the non-zero minimal weights.

Table App.1.  $B_\ell$  results.

The minimal weight  $\mathfrak{w}_\ell$  for  $\Phi = B_\ell$  is allowable for  $\ell \geq 12$ ; one obtains this condition by comparing  $|\Pi(\mathfrak{w}_\ell)| = |\mathcal{W}\mathfrak{w}_\ell| = 2^\ell$  with  $\mathcal{C}\langle \mathfrak{w}_\ell + \rho, \alpha_0 \rangle = \ell(\ell - 1)(2\ell + 1)$ .

$\Phi = B_\ell : 0 \quad (\ell \geq 3)$				$\Phi = B_\ell : \mathfrak{w}_\ell \quad (3 \leq \ell \leq 11)$			
$\lambda_1$	$\ell \geq 3$	(*)	$\mathfrak{w}_1$	$\lambda_1$	$6 \leq \ell \leq 11$	(†)	$\mathfrak{w}_1 + \mathfrak{w}_\ell$
$\lambda_2$	$\ell \geq 3$	(*)	$2\mathfrak{w}_1$		$3 \leq \ell \leq 5$	(*)	$\mathfrak{w}_1 + \mathfrak{w}_\ell$
		(*)	$\mathfrak{w}_2$	$\lambda_2$	$3 \leq \ell \leq 5$	(†)	$2\mathfrak{w}_1 + \mathfrak{w}_\ell$
$\lambda_3$	$\ell \geq 3$	(*)	$3\mathfrak{w}_1$		$\ell = 5$	(†)	$\mathfrak{w}_2 + \mathfrak{w}_\ell$
		(*)	$\mathfrak{w}_1 + \mathfrak{w}_2$		$\ell = 3, 4$	(*)	$\mathfrak{w}_2 + \mathfrak{w}_\ell$
		(*)	$\mathfrak{w}_3$	$\lambda_3$	$\ell = 4$	(†)	$\mathfrak{w}_1 + \mathfrak{w}_2 + \mathfrak{w}_4$
$\lambda_4$	$\ell \geq 3$	(†)	$4\mathfrak{w}_1$			(†)	$\mathfrak{w}_3 + \mathfrak{w}_4$
		(†)	$2\mathfrak{w}_1 + \mathfrak{w}_2$		$\ell = 3$	(†)	$\mathfrak{w}_1 + \mathfrak{w}_2 + \mathfrak{w}_3$
		(†)	$2\mathfrak{w}_2$			(†)	$3\mathfrak{w}_3$
		(†)	$2\mathfrak{w}_1 + \mathfrak{w}_2$				
	$\ell \geq 4$	(†)	$\mathfrak{w}_1 + \mathfrak{w}_3$				
	$\ell = 3$	(†)	$\mathfrak{w}_1 + 2\mathfrak{w}_3$				
	$\ell \geq 5$	(†)	$\mathfrak{w}_4$				
	$\ell = 4$	(*)	$2\mathfrak{w}_4$				
$\lambda_5$	$\ell = 4$	(†)	$\mathfrak{w}_1 + 2\mathfrak{w}_4$				



Table App.2.  $D_\ell$  results.

When  $\Phi = D_\ell$ , and  $\ell \geq 5$ , the minimal weights  $\varpi_{\ell-1}$  and  $\varpi_\ell$  are exchanged by the non-trivial graph automorphism; thus, we need only consider  $\varpi_\ell$  here. Using a calculation similar to that for  $B_\ell$ , one finds that  $\varpi_\ell$  is allowable provided that  $\ell \geq 13$ . When  $\ell = 4$ , the group of diagram automorphisms acts as the full permutation group of the non-0 minimal weights; we therefore consider only the weight  $\varpi_1$  in this case.

$\Phi = D_\ell : 0 \quad (\ell \geq 4)$				$\Phi = D_\ell : \varpi_1 \quad (\ell \geq 5)$			
$\lambda_1$	$\ell \geq 4$	(*)	$\varpi_2$	$\lambda_1$	$\ell \geq 5$	(*)	$\varpi_1 + \varpi_2$
$\lambda_2$	$\ell \geq 4$	(†)	$2\varpi_2$			(*)	$\varpi_3$
		(*)	$2\varpi_1$	$\lambda_2$	$\ell \geq 5$	(†)	$\varpi_1 + 2\varpi_2$
	$\ell \geq 6$	(†)	$\varpi_4$			(†)	$\varpi_2 + \varpi_3$
	$\ell = 5$	(*)	$\varpi_4 + \varpi_5$		$\ell \geq 6$	(†)	$\varpi_1 + \varpi_4$
	$\ell = 4$	(*)	$2\varpi_3$		$\ell \geq 7$	(†)	$\varpi_5$
	$\ell = 4$	(*)	$2\varpi_4$		$\ell = 6$	(†)	$\varpi_5 + \varpi_6$
$\lambda_3$	$\ell \geq 5$	(†)	$\varpi_1 + \varpi_3$		$\ell = 5$	(†)	$\varpi_1 + \varpi_4 + \varpi_5$
	$\ell = 5$	(†)	$\varpi_2 + \varpi_4 + \varpi_5$			(*)	$2\varpi_4$
	$\ell \geq 4$	(†)	$2\varpi_1 + \varpi_2$			(*)	$2\varpi_5$
		(†)	$2\varpi_3 + \varpi_2$	$\lambda_3$	$\ell = 5$	(†)	$\varpi_2 + 2\varpi_4$
		(†)	$2\varpi_4 + \varpi_2$			(†)	$\varpi_2 + 2\varpi_5$
		(†)	$\varpi_1 + \varpi_3 + \varpi_4$	$\Phi = D_4 : \varpi_1$			
$\Phi = D_\ell : \varpi_\ell \quad (12 \geq \ell \geq 5)$				$\lambda_1$		(*)	$\varpi_3 + \varpi_4$
$\lambda_1$	$12 \geq \ell \geq 5$	(†)	$\varpi_2 + \varpi_\ell$			(*)	$\varpi_1 + \varpi_2$
	$\ell = 5, 6, 7$	(*)	$\varpi_1 + \varpi_{\ell-1}$	$\lambda_2$		(†)	$\varpi_2 + \varpi_3 + \varpi_4$
	$\ell \geq 8$	(†)	$\varpi_1 + \varpi_{\ell-1}$			(†)	$3\varpi_1$
$\lambda_2$	$\ell = 5, 6, 7$	(†)	$\varpi_1 + \varpi_2 + \varpi_{\ell-1}$			(†)	$\varpi_1 + 2\varpi_3$
		(†)	$\varpi_3 + \varpi_{\ell-1}$			(†)	$\varpi_1 + 2\varpi_4$
		(†)	$2\varpi_1 + \varpi_\ell$				

Table App.3.  $A_\ell$  results.

For type  $A_\ell$ , not every minimal weight fails to be allowable. One can verify that the minimal weights which are not allowable are precisely 0 when  $\ell \geq 2$ ,  $\varpi_1$  when  $\ell \geq 2$ ,  $\varpi_2$  when  $\ell \geq 3$ ,  $\varpi_3$  when  $\ell \geq 5$ ,  $\varpi_4$  when  $11 \geq \ell \geq 7$ , and  $\varpi_5$  when  $11 \geq \ell \geq 9$ .

$\Phi = A_\ell : 0 \quad (\ell \geq 2)$				$\Phi = A_\ell : \varpi_1 \quad (\ell \geq 2)$			
$\lambda_1$	$\ell \geq 2$	(*)	$\varpi_1 + \varpi_\ell$	$\lambda_1$	$\ell \geq 2$	(*)	$2\varpi_1 + \varpi_\ell$
$\lambda_2$	$\ell \geq 2$	(†)	$2\varpi_1 + 2\varpi_\ell$			(*)	$\varpi_2 + \varpi_\ell$
	$\ell \geq 4$	(†)	$2\varpi_1 + \varpi_{\ell-1}$	$\lambda_2$	$\ell \geq 3$	(†)	$\varpi_1 + \varpi_2 + \varpi_{\ell-1}$
	$\ell \geq 4$	(†)	$\varpi_2 + 2\varpi_\ell$			(†)	$3\varpi_1 + 2\varpi_\ell$
	$\ell \geq 5$	(†)	$\varpi_2 + \varpi_{\ell-1}$			(†)	$3\varpi_1 + \varpi_{\ell-1}$
	$\ell = 4$	(*)	$\varpi_2 + \varpi_3$			(†)	$\varpi_1 + \varpi_2 + 2\varpi_\ell$
	$\ell = 3$	(*)	$2\varpi_1 + \varpi_2$		$\ell \geq 5$	(†)	$\varpi_3 + 2\varpi_\ell$
		(*)	$\varpi_2 + 2\varpi_3$			(†)	$\varpi_3 + \varpi_{\ell-1}$
	$\ell = 2$	(*)	$3\varpi_1$		$\ell = 4$	(*)	$\varpi_3 + 2\varpi_4$
		(*)	$3\varpi_3$			(*)	$2\varpi_3$
$\lambda_3$	$\ell = 4$	(†)	$\varpi_1 + \varpi_2 + \varpi_3 + \varpi_4$		$\ell = 3$	(*)	$3\varpi_3$
		(†)	$\varpi_1 + 2\varpi_2$			(†)	$\varpi_1 + 2\varpi_2$
		(†)	$2\varpi_3 + \varpi_4$		$\ell = 2$	(*)	$4\varpi_1$
	$\ell = 3$	(†)	$3\varpi_1 + \varpi_2 + \varpi_3$			(*)	$\varpi_1 + 3\varpi_2$
		(†)	$\varpi_1 + \varpi_2 + 3\varpi_3$			(†)	$3\varpi_1 + 2\varpi_2$
		(†)	$\varpi_1 + 2\varpi_2 + \varpi_3$	$\lambda_3$	$\ell = 4$	(*)	$4\varpi_4$
	$\ell = 2$	(†)	$4\varpi_1 + \varpi_2$			(†)	$\varpi_1 + \varpi_3 + 3\varpi_4$
		(†)	$\varpi_1 + 4\varpi_2$			(†)	$\varpi_1 + 2\varpi_3 + \varpi_4$
$\Phi = A_\ell : \varpi_2 \quad (\ell \geq 3)$					$\ell = 3$	(†)	$\varpi_1 + 4\varpi_3$
$\lambda_1$	$\ell \geq 3$	(*)	$2\varpi_1$		$\ell = 2$	(†)	$5\varpi_1 + \varpi_2$
		(†)	$\varpi_1 + \varpi_2 + \varpi_\ell$			(†)	$3\varpi_1 + \varpi_2$
		(*)	$\varpi_3 + \varpi_\ell$			(†)	$2\varpi_1 + 4\varpi_2$
	$\ell = 3$	(*)	$2\varpi_3$			(†)	$5\varpi_2$
$\lambda_2$	$\ell \geq 3$	(†)	$3\varpi_1 + \varpi_\ell$	$\lambda_4$	$\ell = 4$	(†)	$\varpi_1 + 5\varpi_4$
	$\ell > 3$	(†)	$\varpi_1 + \varpi_3 + 2\varpi_\ell$			(†)	$\varpi_1 + \varpi_3 + 3\varpi_4$
	$\ell > 3$	(†)	$\varpi_1 + \varpi_3 + \varpi_{\ell-1}$				
	$\ell > 5$	(†)	$\varpi_4 + 2\varpi_\ell$				
	$\ell > 5$	(†)	$\varpi_4 + \varpi_{\ell-1}$				
	$\ell = 5$	(*)	$\varpi_4 + 2\varpi_5$				
	$\ell = 5$	(*)	$2\varpi_4$				
	$\ell = 4$	(*)	$3\varpi_4$				
$\lambda_3$	$\ell = 5$	(†)	$\varpi_1 + \varpi_4 + 3\varpi_5$				
		(*)	$4\varpi_5$				
		(†)	$\varpi_1 + 2\varpi_4 + \varpi_5$				
	$\ell = 4$	(†)	$\varpi_1 + 4\varpi_4$				
$\lambda_4$	$\ell = 5$	(†)	$\varpi_1 + 5\varpi_5$				
	$\ell = 5$	(†)	$\varpi_1 + \varpi_4 + 3\varpi_5$				

Table App.4. Type  $A_\ell$  continued.

$\Phi = A_\ell : \varpi_3 \quad (\ell \geq 5)$		
$\lambda_1$	$\ell \geq 5$ (*) $\varpi_1 + \varpi_2$	
	(†) $\varpi_1 + \varpi_3 + \varpi_\ell$	
	$\ell = 5, 6$ (*) $\varpi_4 + \varpi_\ell$	
	$\ell \geq 7$ (†) $\varpi_4 + \varpi_\ell$	
$\lambda_2$	$\ell \geq 5$ (*) $3\varpi_1$	
	(†) $2\varpi_1 + \varpi_2 + \varpi_\ell$	
	(†) $2\varpi_2 + \varpi_\ell$	
	$\ell = 5, 6$ (†) $\varpi_1 + \varpi_4 + 2\varpi_\ell$	
	$\ell = 5, 6$ (†) $\varpi_1 + \varpi_3 + \varpi_\ell$	
	$\ell = 5, 6$ (†) $\varpi_1 + \varpi_4 + \varpi_{\ell-1}$	
	$\ell = 6$ (*) $2\varpi_5$	
	$\lambda_3$	$\ell \geq 5$ (†) $4\varpi_1 + \varpi_\ell$
		$\ell = 6$ (†) $\varpi_1 + 2\varpi_5 + \varpi_6$
		(†) $\varpi_5 + 2\varpi_6$

$\Phi = A_\ell : \varpi_4 \quad (16 \geq \ell \geq 7)$	
$\lambda_1$	$16 \geq \ell \geq 7$ (†) $\varpi_1 + \varpi_4 + \varpi_\ell$
	(†) $\varpi_1 + \varpi_3$
	(†) $\varpi_5 + \varpi_\ell$
$\Phi = A_\ell : \varpi_5 \quad (11 \geq \ell \geq 9)$	
$\lambda_1$	$11 \geq \ell \geq 9$ (†) $\varpi_1 + \varpi_5 + \varpi_\ell$
	(†) $\varpi_1 + \varpi_4$
	(†) $\varpi_6 + \varpi_\ell$

 Table App.5.  $C_\ell$  results.

$\Phi = C_\ell : 0 \quad (\ell \geq 4)$	
$\lambda_1$	$\ell \geq 4$ (*) $\varpi_2$
$\lambda_2$	$\ell \geq 4$ (*) $2\varpi_1$
	(†) $2\varpi_2$
	$\ell \geq 7$ (†) $\varpi_4$
	$4 \leq \ell \leq 6$ (*) $\varpi_4$
$\lambda_3$	$\ell \geq 4$ (†) $\varpi_1 + \varpi_3$
	(†) $2\varpi_1 + \varpi_2$
	$\ell = 4, 5, 6$ (†) $\varpi_2 + \varpi_4$
	$\ell = 6$ (*) $\varpi_6$
$\lambda_4$	$\ell = 6$ (†) $\varpi_2 + \varpi_6$
$\Phi = C_\ell : 0 \quad (\ell = 2, 3)$	
$\lambda_1$	$\ell = 2, 3$ (*) $\varpi_2$
$\lambda_2$	$\ell = 2, 3$ (*) $2\varpi_1$
	(*) $2\varpi_2$
$\lambda_3$	$\ell = 2, 3$ (†) $2\varpi_1 + \varpi_2$
	(†) $3\varpi_2$
	$\ell = 3$ (*) $\varpi_1 + \varpi_3$
$\lambda_4$	$\ell = 3$ (†) $\varpi_1 + \varpi_2 + \varpi_3$
	$\ell = 3$ (†) $2\varpi_3$

$\Phi = C_\ell : \varpi_1 \quad (\ell \geq 2)$	
$\lambda_1$	$\ell \geq 2$ (*) $\varpi_1 + \varpi_2$
	(*) $\varpi_3$
$\lambda_2$	$\ell \geq 2$ (*) $3\varpi_1$
	(†) $\varpi_1 + 2\varpi_2$
	(†) $\varpi_2 + \varpi_3$
	$\ell \geq 7$ (†) $\varpi_5$
	$\ell = 5, 6$ (*) $\varpi_5$
$\lambda_3$	$\ell \geq 2$ (†) $3\varpi_1 + \varpi_2$
	(†) $2\varpi_1 + \varpi_3$
	$\ell = 5, 6$ (†) $\varpi_2 + \varpi_5$

Table App.6. Exceptional type results

When  $\Phi = E_6$ , the minimal weights  $\omega_1$  and  $\omega_6$  are exchanged by the non-trivial graph automorphism; we therefore only describe the outcome of the procedure for  $\omega_1$ .

$\Phi = E_6 : 0$			$\Phi = E_7 : 0$			$\Phi = E_8 : 0$		
$\lambda_1$	(*)	$\omega_2$	$\lambda_1$	(*)	$\omega_1$	$\lambda_1$	(†)	$\omega_i (2 \leq i \leq 7)$
	(*)	$\omega_1 + \omega_6$		(*)	$\omega_6$		(*)	$\omega_1$
	(†)	$\omega_3 + \omega_5$		(†)	$\omega_3$		(*)	$\omega_8$
	(†)	$\omega_4$		(†)	$\omega_4$	$\lambda_2$	(†)	$\omega_1 + \omega_i$
$\lambda_2$	(†)	$2\omega_2$	$\lambda_2$	(†)	$2\omega_1$		(†)	$\omega_i + \omega_8 (1 \leq i \leq 8)$
	(†)	$\omega_2 + \omega_4$		(†)	$\omega_1 + \omega_6$	$\Phi = F_4, \tau = 0$		
	(†)	$\omega_2 + \omega_3 + \omega_5$		(†)	$\omega_1 + \omega_3$	$\lambda_1$	(*)	$\omega_1$
	(†)	$\omega_1 + \omega_2 + \omega_6$		(†)	$\omega_1 + \omega_4$		(*)	$\omega_3$
	(†)	$\omega_1 + \omega_4 + \omega_6$		(†)	$2\omega_6$		(*)	$\omega_4$
	(†)	$\omega_1 + \omega_3 + \omega_5 + \omega_6$		(†)	$\omega_6 + \omega_3$		(†)	$\omega_2$
	(†)	$2\omega_1 + 2\omega_6$		(†)	$\omega_6 + \omega_4$		(†)	$2\omega_1$
	(†)	$\omega_1 + \omega_3$		(*)	$2\omega_7$	$\lambda_2$	(*)	$2\omega_4$
	(†)	$\omega_5 + \omega_6$	$\lambda_3$	(†)	$\omega_1 + 2\omega_7$		(†)	$\omega_1 + \omega_2$
$\Phi = E_6 : \omega_1$				(†)	$\omega_6 + 2\omega_7$		(†)	$\omega_1 + \omega_3$
$\lambda_1$	(*)	$\omega_5$		(†)	$\omega_3 + 2\omega_7$		(†)	$\omega_1 + \omega_4$
	(†)	$\omega_1 + \omega_2$		(†)	$\omega_4 + 2\omega_7$		(†)	$\omega_2 + \omega_3$
	(†)	$\omega_1 + \omega_4$		(†)	$\omega_2 + \omega_7$		(†)	$2\omega_3$
	(†)	$\omega_1 + \omega_3 + \omega_5$	$\Phi = E_7 : \omega_7$				(†)	$\omega_3 + \omega_4$
	(†)	$2\omega_1 + \omega_6$	$\lambda_1$	(†)	$\omega_1 + \omega_7$		(†)	$\omega_1 + 2\omega_4$
$\lambda_2$	(*)	$2\omega_6$		(†)	$\omega_6 + \omega_7$		(†)	$\omega_2 + 2\omega_4$
	(†)	$\omega_2 + \omega_5$		(†)	$\omega_3 + \omega_7$		(†)	$\omega_3 + 2\omega_4$
	(†)	$\omega_4 + \omega_5$		(†)	$\omega_4 + \omega_7$		(†)	$3\omega_4$
	(†)	$\omega_3 + 2\omega_5$		(*)	$\omega_3$	$\Phi = G_2, \tau = 0$		
	(†)	$\omega_1 + \omega_5 + \omega_6$	$\lambda_2$	(†)	$\omega_1 + \omega_2$	$\lambda_1$	(*)	$\omega_2$
$\lambda_3$	(†)	$\omega_2 + 2\omega_6$		(†)	$\omega_2 + \omega_6$	$\lambda_2$	(*)	$2\omega_2$
	(†)	$\omega_4 + 2\omega_6$		(†)	$\omega_2 + \omega_3$		(*)	$\omega_1$
	(†)	$\omega_3 + \omega_5 + 2\omega_6$		(†)	$\omega_2 + \omega_4$	$\lambda_3$	(*)	$3\omega_2$
	(†)	$\omega_1 + 3\omega_6$		(†)	$\omega_5$		(†)	$\omega_1 + \omega_2$
	(†)	$\omega_3 + \omega_6$				$\lambda_4$	(†)	$4\omega_2$
							(†)	$\omega_1 + 2\omega_2$

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