COMPLETELY REDUCIBLE LIE SUBALGEBRAS

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Abstract. Let G be a connected and reductive group over the algebraically closed field K. J-P. Serre has introduced the notion of a G-completely reducible subgroup $H \subset G$. In this note, we give a notion of G-complete reducibility – G-cr for short – for Lie subalgebras of $\mathrm{Lie}(G)$, and we show that if the closed subgroup $H \subset G$ is G-cr, then $\mathrm{Lie}(H)$ is G-cr as well.

1. Introduction

Let G be a connected and reductive group over the algebraically field K, and write $\mathfrak{g}$ for the Lie algebra of G. J-P. Serre has introduced the notion of a G-completely reducible subgroup; we state the definition here only for a closed subgroup $H \subset G$. We say H is G-cr provided that whenever $H \subset P$ for a parabolic subgroup of G, there is a Levi factor $L \subset P$ such that $H \subset L$; cf. [Ser 05]. When $G = \mathrm{GL}(V)$, the subgroup H is G-cr if and only if V is a semisimple H-module. Similarly, if the characteristic of K is not 2 and G is either the symplectic group $\mathrm{Sp}(V)$ or the orthogonal group $\mathrm{SO}(V)$, a subgroup H of G is G-cr if and only V is a semisimple H-module.

B. Martin [Ma 03] used some techniques from "geometric invariant theory" – due to G. Kempf and to G. Rousseau – to prove that if $H \subset G$ is G-cr, and if N is a normal subgroup of H, then N is G-cr as well; cf. [Ser 05, Théorème 3.6]. Martin's result was obtained first for *strongly reductive* subgroups in the sense of Richardson; it follows from [BMR 05] that the strongly reductive subgroups of G are precisely the G-cr subgroups. See also [Ser 05, §3.3] for an overview of these matters.

We are going to prove in this note a result related to that of Martin. If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, say that $\mathfrak{h}$ is G-cr provided that whenever $\mathfrak{h} \subset \mathrm{Lie}(P)$ for a parabolic subgroup P of G, there is a Levi factor $L \subset P$ such that $\mathfrak{h} \subset \mathrm{Lie}(L)$.

We will prove:

Theorem 1. *Suppose that G is a reductive group over the algebraically closed field K.*

1. *Let $X_1, \dots, X_d$ be a basis for the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then $\mathfrak{h}$ is G-cr if and only if the $\mathrm{Ad}(G)$-orbit of $(X_1, \dots, X_d)$ is closed in $\bigoplus^d \mathfrak{g}$.*
2. *If the closed subgroup $H \subset G$ is G-cr, then $\mathrm{Lie}(H)$ is G-cr as well.*

Our result – and our techniques – are related to those used by Richardson in [Ri 88], though he treats mainly the case of characteristic 0. See e.g. *loc. cit.* Theorem 3.6.

The converse to Theorem 1(2) is not true. Indeed, suppose the characteristic p of K is positive, and consider a finite subgroup $H \subset G$ whose order is a power of p. Then $\mathrm{Lie}(H) = 0$ is clearly G-cr; however, if $G = \mathrm{SL}(V)$ and if H is non-trivial, then V is not semisimple as an H-module, thus H is not G-cr. The converse to Theorem 1(2) is even false for connected H; I thank Ben Martin for pointing out the following example. Take for H any semisimple group

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in characteristic $p > 0$, let $\rho_i : H \to \mathrm{SL}(V_i)$ be representations for $i = 1, 2$ with ρ_1 semisimple and ρ_2 not semisimple, and consider the representation $\rho : H \to G = \mathrm{SL}(V_1 \oplus V_2)$ given by $h \mapsto \rho_1(h) \oplus \rho_2(F(h))$ where $F : H \to H$ is the Frobenius endomorphism. If J denotes the image of ρ, then J is not G-cr since $V_1 \oplus V_2$ is not semisimple as a J-module. However, $\mathrm{Lie}(J) = \mathrm{im}\, d\rho$ lies in the the Lie algebra of the subgroup $M = \mathrm{SL}(V_1) \times \mathrm{SL}(V_2)$; moreover, M is a Levi factor of a parabolic subgroup of G, and $\mathrm{Lie}(J) = \mathrm{im}\, d\rho_1 \oplus 0 \subset \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2) = \mathrm{Lie}(M)$. Since the image of $\rho_1 \times 1 : H \to M$ is M-cr (use [BMR 05, Lemma 2.12(i)]), the main result of this paper implies $\mathrm{Lie}(J)$ to be M-cr[1], hence Lemma 4 below shows that $\mathrm{Lie}(J)$ is G-cr as well.

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2. Preliminaries

We work throughout in the *geometric* setting; thus, K is an algebraically closed field. A variety will mean a separated and reduced scheme of finite type over K. The group G will be a connected and reductive algebraic group (over K). A closed subgroup $H \subset G$ is in particular a subvariety of G and so H is necessarily *reduced* – e.g. if G acts on a variety X and if $x \in X$, then $\mathrm{Stab}_G(x)$ will mean the reduced subgroup determined by the "abstract group theoretic" stabilizer (even if the G-orbit of x is not separable).

2.1. **Closed orbits.** Let X be an affine G-variety, let $x \in X$ and choose a maximal torus $S \subset \mathrm{Stab}_G(x)$ of the stabilizer in G of x. Let $L = C_G(S)$; thus L is a Levi factor of a parabolic subgroup of G.

Proposition 2. *If the G-orbit $G \cdot x$ is closed in X, then the L-orbit $L \cdot x$ is closed in X.*

Proof. The fixed point set X^S is closed in X. Since by assumption $G \cdot x$ is closed in X, it follows that
$$(G \cdot x)^S = X^S \cap G \cdot x$$
is closed in X.

Let now $N = N_G(S)$ be the normalizer in G of S. We claim that $(G \cdot x)^S = N \cdot x$. Indeed, let $g \in G$ and suppose $g \cdot x$ is fixed by S. The claim follows once we prove that $g \cdot x \in N \cdot x$. Well, for each $s \in S$ we have $sg \cdot x = g \cdot x$ so that $g^{-1}sg \in \mathrm{Stab}_G(x)$. Thus $g^{-1}Sg$ is a maximal torus of $\mathrm{Stab}_G(x)$. Since maximal tori are conjugate [Spr 98, Theorem 6.4.1], there is an element $h \in \mathrm{Stab}_G(x)$ such that $g^{-1}Sg = hSh^{-1}$. But then $gh \in N$, and moreover, $g \cdot x = gh \cdot x$.

Note that N contains L as a normal subgroup. We now observe that the stabilizer in L of a point y of the orbit $N \cdot x$ is conjugate to $\mathrm{Stab}_L(x)$ by an element of N. Indeed, choosing $h \in N$ such that $h \cdot x = y$, one knows that $h \cdot \mathrm{Stab}_L(y) \cdot h^{-1} = \mathrm{Stab}_L(x)$. It follows that all L-orbits in $N \cdot x$ have the same dimension.

Since the closure of any L-orbit must be the union of orbits of strictly smaller dimension, it follows that the L-orbits in $N \cdot x$ are closed.

Since $N \cdot x = X^S \cap G \cdot x$ is closed in X, it follows at once that $L \cdot x$ is closed in X, as required. □

[1]This can be seen more easily: it is straightforward to check that a Lie subalgebra $\mathfrak{h} \subset \mathfrak{sl}(V)$ is $\mathrm{SL}(V)$-cr if and only if V is a semisimple $\mathfrak{h}$-module.

Remark 3. With notation as in the previous Proposition, if $N = N_G(S)$, it follows from the rigidity of tori [Spr 98, Corollary 3.2.9] that L has finite index in N. In particular, $(G \cdot x)^S$ is a finite union of L-orbits which are permuted transitively by N; moreover, these L-orbits are precisely the connected components of $(G \cdot x)^S$.

2.2. **Complete reducibility.** The interpretation of complete reducibility using the spherical building of G permits one to prove the following:

Lemma 4. *Let G be reductive and let $M \subset G$ be a Levi factor of a parabolic subgroup of G. Suppose that $J \subset M$ is a subgroup, and that $\mathfrak{h} \subset \mathrm{Lie}(M)$ is a Lie subalgebra. Then J is G-cr if and only if J is M-cr and $\mathfrak{h}$ is G-cr if and only if $\mathfrak{h}$ is M-cr.*

Proof. The assertion for J follows from [Ser 05, Proposition 3.2]. The proof for $\mathfrak{h}$ is similar; let us give a sketch. Write X for the building of G. The Lie subalgebra $\mathfrak{h}$ defines a subcomplex Y of X: the simplices of Y are those simplices in X which correspond to parabolic subgroups P with $\mathfrak{h} \subset \mathrm{Lie}(P)$.

Recall [Bo 91, Corollary 14.13] that the intersection $P \cap P'$ of two parabolic subgroups $P, P' \subset G$ contains a maximal torus of G. This implies that $\mathrm{Lie}(P \cap P') = \mathrm{Lie}(P) \cap \mathrm{Lie}(P')$; see e.g. the argument in the first paragraph of [Ja 04, §10.3].

If now $\mathfrak{h} \subset \mathrm{Lie}(P) \cap \mathrm{Lie}(P')$, it follows that $\mathfrak{h} \subset \mathrm{Lie}(P \cap P')$. This shows that the subcomplex Y is convex; see [Ser 05, Prop. 3.1]. Evidently $\mathfrak{h}$ is G-cr if and only if Y is X-cr in the sense of [Ser 05, §2.2].

Choose a parabolic subgroup Q for which M is a Levi factor. Then we may identify the building of M with the residual building of X determined by the parabolic Q; cf. [Ser 05, 2.1.8 and 3.1.7]. Now the claim follows from [Ser 05, Proposition 2.5]. □

2.3. **Cocharacters and parabolic subgroups.** If V is a variety and $f : \mathbf{G}_m \to V$ is a morphism, we write $v = \lim_{t \to 0} f(t)$, and we say that the limit exists, if f extends to a morphism $\tilde{f} : \mathbf{A}^1 \to V$ with $\tilde{f}(0) = v$. If $\mathbf{G}_m$ acts on V, a closed point $w \in V$ determines a morphism $f : \mathbf{G}_m \to V$ via the rule $t \mapsto t \cdot w$; one writes $\lim_{t \to 0} t \cdot w$ as shorthand for $\lim_{t \to 0} f(t)$.

A *cocharacter* of an algebraic group A is a K-homomorphism $\gamma : \mathbf{G}_m \to A$. A linear K-representation (ρ, V) of A yields a linear K-representation $(\rho \circ \gamma, V)$ of $\mathbf{G}_m$. Then V is the direct sum of the weight spaces

$$V(\gamma; i) = \{v \in V \mid (\rho \circ \gamma)(t)v = t^i v, \ \forall\, t \in \mathbf{G}_m\} \tag{2.3.1}$$

for $i \in \mathbf{Z}$. We write $X_*(A)$ for the set of cocharacters of A.

Consider now the reductive group G. If $\gamma \in X_*(G)$, then

$$P_G(\gamma) = P(\gamma) = \{x \in G \mid \lim_{t \to 0} \gamma(t) x \gamma(t^{-1}) \text{ exists}\}$$

is a parabolic subgroup of G whose Lie algebra is $\mathfrak{p}(\gamma) = \sum_{i \geq 0} \mathfrak{g}(\gamma; i)$. Moreover, each parabolic subgroup of G has the form $P(\gamma)$ for some cocharacter γ; for all this cf. [Spr 98, 3.2.15 and 8.4.5].

We note that γ "exhibits" a Levi decomposition of $P = P(\gamma)$. Indeed, $P(\gamma)$ is the semidirect product $Z(\gamma) \cdot U(\gamma)$, where $U(\gamma) = \{x \in P \mid \lim_{t \to 0} \gamma(t) x \gamma(t^{-1}) = 1\}$ is the unipotent radical of $P(\gamma)$, and the reductive subgroup $Z(\gamma) = C_G(\gamma(\mathbf{G}_m))$ is a Levi factor in $P(\gamma)$; cf. [Spr 98, 13.4.2].

Lemma 5. *Let P be a parabolic subgroup of G, let L be a Levi factor of P, let $\gamma \in X_*(L)$ and assume that $P = P(\gamma)$. Then $L = Z(\gamma)$ and the image of γ lies in the connected center of L.*

Proof. Let R be the radical of P. Then the Levi factors of P are precisely the centralizers of the maximal tori of R; cf. [Bo 91, Cor. 14.19]. Since the connected center of a Levi factor of P evidently lies in R, we see that the connected center of each Levi factor is a maximal torus of R.

Now, the centralizer $L_1 = Z(\gamma)$ is a Levi factor of P, so that γ is a cocharacter of the connected center of L_1; in particular, the image of γ lies in R. Moreover, since $L_1 = Z(\gamma)$, the centralizer of the image of γ in R is a maximal torus S of R. It follows that S is the unique maximal torus of R containing the image of γ.

Since the image of γ lies in L and in R, and since L intersects R in a maximal torus of R, it follows that $S = L \cap R$ so that $L = L_1$ as required. □

2.4. **Instability in invariant theory.** Let (ρ, V) be a linear representation [always assumed finite dimensional] of G, and fix a closed G-invariant subvariety $S \subset V$. We are going to describe a precise form – due to Kempf and Rousseau – of the Hilbert-Mumford criteria for the instability of a vector $v \in V$ under the action of G.

Let us first briefly describe our goal: given a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}(G)$, fix a basis $\mathbf{X} = (X_1, \dots, X_d)$ of $\mathfrak{h}$. If the G-orbit of $\mathbf{X}$ in $\bigoplus^d \mathfrak{g}$ is not closed – so that $\mathbf{X}$ is an unstable vector – the results of Kempf and Rousseau permit us to associate to $\mathbf{X}$ a unique parabolic subgroup $P_{\mathbf{X}}$; see Corollary 9 below. If $g \in G$ satisfies $\mathrm{Ad}(g)\mathfrak{h} = \mathfrak{h}$, one of our main objectives is to show that $g \in P_{\mathbf{X}}$. Using g, we get a new basis $\mathrm{Ad}(g)\mathbf{X} = (\mathrm{Ad}(g)X_1, \dots, \mathrm{Ad}(g)X_d)$ of $\mathfrak{h}$, and generalities show that $P_{\mathrm{Ad}(g)\mathbf{X}} = gP_{\mathbf{X}}g^{-1}$. So we want to prove the equality $P_{\mathbf{X}} = P_{\mathrm{Ad}(g)\mathbf{X}}$; it will then follow that $g \in P_{\mathbf{X}}$, as desired.

Return now to our general setting: V is any linear representation of G. For $v \in V$, put

$$|V, v| = \{\lambda \in X_*(G) \mid \lim_{t \to 0} \rho(\lambda(t))v \quad \text{exists}\}.$$

Write $V = \bigoplus_{i \in \mathbf{Z}} V(\lambda; i)$ as in (2.3.1), and write $v = \sum_i v_i$ with $v_i \in V(\lambda; i)$. Then evidently

$$\lambda \in |V, v| \iff v_i = 0 \quad \forall i < 0; \tag{2.4.1}$$

if $\lambda \in |V, v|$ then of course $\lim_{t \to 0} \rho(\lambda(t))v = v_0$.

Now let $S \subset V$ be a G-invariant closed subvariety and suppose that $v \notin S$. Given $\lambda \in |V, v|$, write $v_0 = \lim_{t \to 0} \rho(\lambda(t))v$. If $v_0 \in S$, write $\alpha_{S,v}(\lambda)$ for the order of vanishing of the regular function $(t \mapsto \rho(\lambda(t))v - v_0) : \mathbf{A}^1 \to V$, otherwise write $\alpha_{S,v}(\lambda) = 0$; see [Ke 78, §3] for more details. Then $\alpha_{S,v}(\lambda)$ is a non-negative integer, and $\alpha_{S,v}(\lambda) > 0$ if and only if $v_0 \in S$. Moreover, if $v = \sum_{i \in \mathbf{Z}} v_i$ with $v_i \in V(\lambda; i)$ as before, then

$$v_0 \in S \implies \alpha_{S,v}(\lambda) = \alpha_{\{v_0\},v}(\lambda) = \min\{j > 0 \mid v_j \neq 0\}. \tag{2.4.2}$$

Suppose that $W \subset V$ is a subspace of dimension $d = \dim W$. Let $w_1, \dots, w_d$ be a basis of W, and consider the point $x = (w_1, \dots, w_d)$ of the linear space $X = \bigoplus^d V$; abusing notation somewhat, we write also ρ for the diagonal action $\bigoplus^d \rho$ of G on X. We observe for $\lambda \in X_*(G)$ that we have

$$\lambda \in |X, x| \iff W \subset \sum_{j \geq 0} V(\lambda; j). \tag{2.4.3}$$

Fix $S \subset X = \bigoplus^d V$ a closed and $\rho(G)$-invariant subvariety, and assume that $x = (w_1, \dots, w_d) \notin S$. In this setting one may compute the function $\alpha_{S,x}$ for the diagonal G-action on X using functions $\alpha_{\{v_0\},v}$ for the G-representation V. More precisely, we have:

Lemma 6. *Let $\lambda \in |X, x|$ and suppose $\alpha_{S,x}(\lambda) > 0$. For $w \in W$, write $w_0 = \lim_{t \to 0} \rho(\lambda(t))w$. Then*

$$(*) \quad \alpha_{S,x}(\lambda) = \min_{w \in W} \alpha_{\{w_0\},w}(\lambda).$$

Proof. For $1 \le i \le d$ write $x = \sum_j x^j$ with $x^j \in X(\lambda; j)$.

By assumption, $\lambda \in |X, x|$; by (2.4.1) we see that $x^j = 0$ if $j < 0$. Moreover, using (2.4.2) we see that

$$\alpha_{S,x}(\lambda) = \alpha_{\{x^0\},x}(\lambda) = \min(j > 0 \mid x^j \neq 0). \tag{2.4.4}$$

If we now write $R = \min_{v \in W} \alpha_{\{v_0\},v}(\lambda)$ for the right hand side of $(*)$, then upon considering the components in V of the vectors $x^j \in X = \bigoplus^d V$, one uses (2.4.4) to see that $\alpha_{S,x}(\lambda) \ge R$.

On the other hand, we may choose $v \in W$ such that $R = \alpha_{\{v_0\},v}(\lambda)$. Writing $v = \sum_{j \ge 0} v^j$ with $v^j \in V(\lambda; j)$, we see that

$$R = \alpha_{\{v^0\},v}(\lambda) = \min(j > 0 \mid v^j \neq 0)$$

by (2.4.1). Now write

$$v = \sum_i \beta_i w_i \quad \text{for scalars } \beta_i \in K.$$

Now, $v^R \neq 0$ implies that $x^R \neq 0$; it follows from (2.4.4) that $R \ge \alpha_{S,x}(\lambda)$, and the Lemma is proved. □

Fix a basis $\{w_i\}$ for W and let $x = (w_1, \dots, w_d) \in X$. Write

$$S = \overline{\rho(G)x} - \rho(G)x;$$

then S is closed in X Notice that S is a closed subset, since $\rho(G)x$ is open in $\overline{\rho(G)x}$, and S is G-invariant. We suppose that $\rho(G)x$ is not closed, or equivalently that S is non-empty.

Corollary 7. *Let $h \in G$ satisfy $\rho(h)W = W$. If $x' = \rho(h)x$, then we have $|X, x| = |X, x'|$. Moreover,*

$$\alpha_{S,x}(\lambda) = \alpha_{S,x'}(\lambda)$$

for each $\lambda \in |X, x|$.

Proof. Since by (2.4.3) the sets $|X, x|$ and $|X, x'|$ both consist of all cocharacters λ for which $W \subset \sum_{j \ge 0} V(\lambda; j)$, we have that $|X, x| = |X, x'|$.

Now write $x_0 = \lim_{t \to 0} \rho(\lambda(t))x$ and $x_0' = \lim_{t \to 0} \rho(\lambda(t))x'$. We first claim that $x_0 \in S$ if and only if $x_0' \in S$.

Well, assume that $x_0 \notin S$. Since x_0 lies in the closure of $\rho(G)x$ but not in S, it actually lies in $\rho(G)x$; thus (†) $x_0 = \rho(g)x$ for some $g \in G$.

Since the components in V of the vector $x \in X = \bigoplus^d V$ form a basis of W, one concludes from (†) that

$$\lim_{t \to 0} \rho(\lambda(t))y = \rho(g)y$$

for each $y \in \bigoplus^d W \subset X$. This shows in particular that $x_0' = \rho(g)x' = \rho(gh)x$, so that $x_0' \notin S$. Since the argument just given is symmetric in x and x', it follows that $x_0 \in S$ if and only if $x_0' \in S$.

Recall that $\alpha_{S,x}(\lambda) > 0$ if and only if $x_0 \in S$ and that $\alpha_{S,x'}(\lambda) > 0$ if and only if $x_0' \in S$. Thus to prove the final equality asserted by the corollary, we may suppose that $x_0, x_0' \in S$. Now, according to $(*)$ of Lemma 6 we have

$$\alpha_{S,x}(\lambda) = \min_{w \in W} \alpha_{\{w_0\},w}(\lambda) = \alpha_{S,x'}(\lambda)$$

as required. □

Fix a real-valued G-invariant length function $\lambda \mapsto \|\lambda\|$ on the set $X_*(G)$ of cocharacters of G.

Theorem 8 (Kempf [Ke 78, Theorem 3.4], Rousseau)**.** *Let $z \in X - S$ and assume that $\overline{\rho(G)z} \cap S$ is non-empty. Then the function $\alpha_{S,z}(\lambda)/\|\lambda\|$ assumes a maximal value $B > 0$ on the non-trivial elements of $|X, z|$. Let*

$$\Delta_{S,z} = \{\lambda \in |X, z| \mid \alpha_{S,z}(\lambda) = B \cdot \|\lambda\| \quad \text{and } \lambda \text{ is indivisible}\}.$$

Then

1. *$\Delta_{S,z}$ is non-empty,*
2. *there is a parabolic subgroup $P_{S,z}$ of G such that $P_{S,z} = P(\lambda)$ for each $\lambda \in \Delta_{S,z}$,*
3. *$\Delta_{S,z}$ is a principal homogeneous space under $R_u P_{S,z}$, and*
4. *any maximal torus of $P_{S,z}$ contains a unique cocharacter which lies in $\Delta_{S,z}$.*

Let $H \subset G$ be a subgroup and suppose that W is $\rho(H)$ invariant. Let $x = (w_1, \dots, w_d) \in X$ for a basis $\{w_i\}$ of W.

Corollary 9. *Assume that $\rho(G)x$ is not closed in X, and let*

$$S = \overline{\rho(G)x} - \rho(G)x.$$

Then

1. *$P_{S,x}$ is a proper parabolic subgroup of G,*
2. *$H \subset P_{S,x}$, and*
3. *if $L \subset P_{S,x}$ is a Levi factor, there is a cocharacter λ of the connected center Z of L which lies in $\Delta(S, x)$.*

Proof. Since the image of any $\lambda \in |X, x|$ with $\alpha_{S,x}(\lambda) > 0$ is not central in G, (1) is immediate.

Since the parabolic subgroup $P = P_{S,x}$ is self-normalizing, (2) will follow if we show that $hPh^{-1} = P$ for each $h \in H(k)$. But $hP_{S,x}h^{-1} = P_{S,\rho(h)x}$; see e.g. [Ke 78, Cor. 3.5]. Since $\rho(h)W = W$, Corollary 7 shows that $|X, x| = |X, \rho(h)x|$ and that $\alpha_{S,x}(\lambda) = \alpha_{S,\rho(h)x}(\lambda)$ for all $\lambda \in |X, x| = |X, \rho(h)x|$; thus $\Delta_{S,x} = \Delta_{S,\rho(h)x}$ so that $P_{S,x} = P_{S,\rho(h)x}$ by Theorem 8. Thus indeed $H \subset P_{S,x}$.

Finally, for (3) let S be a maximal torus of L and hence of $P_{S,x}$. By (3) of Theorem 8, S has a cocharacter λ which lies in $\Delta_{S,x}$. Since $P_{S,x} = P(\lambda)$, it follows from Lemma 5 that the image of λ lies in the connected center of L, as required. □

Finally, we record:

Lemma 10. *Assume that $\rho(G)x$ is closed in X and that $\lambda \in |X, x|$. Then the subset*

$$\lim_{t\to 0} \rho(\lambda(t))W = \left\{\lim_{t\to 0} \rho(\lambda(t))w \mid w \in W\right\}$$

satisfies

$$\lim_{t\to 0} \rho(\lambda(t))W = \rho(g)W$$

for some $g \in G$.

Proof. Since $\lambda \in |X, x|$, the limit $x_\lambda = \lim_{t\to 0} \rho(\lambda(t))x$ exists. Since the orbit $\rho(G)x$ is closed, we have $\rho(g)x = x_\lambda$ for some $g \in G$. Since $w_1, \dots, w_d$ is a basis of W, it follows that $\mathrm{Ad}(g)w = \lim_{t\to 0} \rho(\lambda(t))w$ for each $w \in W$, whence the Lemma. □

3. Proof of the main theorem

Recall that G is a reductive group with Lie algebra $\mathfrak{g}$, and that $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra. Fix a basis $X_1, \dots, X_d \in \mathfrak{h}$, and let $\mathbf{X} = (X_1, \dots, X_d) \in \bigoplus^d \mathfrak{g} = Y$. We write (Ad, Y) for the representation $(\bigoplus^d \mathrm{Ad}, \bigoplus^d \mathfrak{g})$ of G.

Proof of part (1) of Theorem 1. Recall that we must show: the Lie algebra $\mathfrak{h}$ is G-cr if and only if the G-orbit of $\mathbf{X}$ is closed in $Y = \bigoplus^d \mathfrak{g}$.

We first suppose that $\mathrm{Ad}(G)\mathbf{X}$ is closed, and we show that $\mathfrak{h}$ is G-cr. Let S be a maximal torus of the centralizer $C_G(\mathfrak{h})$. Then $\mathfrak{h} \subset \mathrm{Lie}(L)$ where $L = C_G(S)$; moreover, L is a Levi factor of a parabolic subgroup of G. It follows from Lemma 4 that $\mathfrak{h}$ is G-cr if and only if $\mathfrak{h}$ is L-cr.

Moreover, it follows from Proposition 2 that $\mathrm{Ad}(L)\mathbf{X}$ is closed in Y. Thus we may replace G by L and so suppose that any torus in G which centralizes $\mathfrak{h}$ is central in G. [Equivalently: $\mathfrak{h}$ is not contained in the Lie algebra of any Levi factor of a proper parabolic subgroup of G.] To show that $\mathfrak{h}$ is G-cr we will show that $\mathfrak{h}$ is not contained in $\mathrm{Lie}(P)$ for any proper parabolic subgroup P of G.

Suppose that $\mathfrak{h} \subset \mathrm{Lie}(P)$ for a parabolic subgroup $P \subset G$; we will show that $P = G$. Write $P = P(\phi)$ for some cocharacter ϕ of G, and write $L = L(\phi)$ for the centralizer in G of the image of ϕ; then L is a Levi factor of P.

Since the G-orbit of $\mathbf{X}$ is closed, Lemma 10 shows that

$$\lim_{t \to 0} \mathrm{Ad}(\phi(t))\mathfrak{h} = \mathrm{Ad}(g)\mathfrak{h}$$

for some $g \in G$. Since $\lim_{t\to 0} \mathrm{Ad}(\phi(t))H \in \mathrm{Lie}(L)$ for each $H \in \mathfrak{h}$, we conclude that $\mathfrak{h} \subset \mathrm{Ad}(g^{-1})\,\mathrm{Lie}(L)$. But then the image of the cocharacter $\mathrm{Int}(g^{-1}) \circ \phi$ is a torus centralizing $\mathfrak{h}$; hence the image of ϕ is central in G so that $P = G$. This proves that $\mathfrak{h}$ is indeed G-cr.

To complete the proof of (i), it remains to show: if the orbit $\mathrm{Ad}(G)\mathbf{X}$ is not closed, then $\mathfrak{h}$ is not G-cr. As in Corollary 9 let $S = \overline{\mathrm{Ad}(G)\mathbf{X}} - \mathrm{Ad}(G)\mathbf{X}$; our assumption means that S is non-empty so that $\alpha_{S,\mathbf{X}}(\lambda) > 0$ for each $\lambda \in \Delta_{S,\mathbf{X}}$. Moreover, $P = P_{S,\mathbf{X}}$ is a proper parabolic subgroup of G.

We have $\mathfrak{h} \subset \mathrm{Lie}(P)$ by (2.4.3). To complete the proof, we suppose $\mathfrak{h}$ is G-cr, and find a contradiction.

Since $\mathfrak{h}$ is G-cr, there is a Levi factor L of P with $\mathfrak{h} \subset \mathrm{Lie}(L)$. By Corollary 9, there is a cocharacter λ of the connected center of L which lies in $\Delta_{S,\mathbf{X}}$. Since $\mathfrak{h} \subset \mathrm{Lie}(L)$, we have $\mathfrak{h} \subset \mathfrak{g}(\lambda; 0)$; thus $\mathbf{X} \in X(\lambda; 0)$. But then $\alpha_{S,\mathbf{X}}(\lambda) = 0$, which is impossible since $\lambda \in \Delta_{S,\mathbf{X}}$. □

Proof of part (2) of Theorem 1. Recall that if $H \subset G$ is a subgroup which is G-cr, we must prove that $\mathfrak{h} = \mathrm{Lie}(H)$ is G-cr.

Let $S \subset C_G(H)$ be a maximal torus. Then $H \subset L = C_G(S)$ and $\mathfrak{h} \subset \mathrm{Lie}(L)$. Applying Lemma 4, it is enough to show that $\mathfrak{h}$ is L-cr; thus we replace G by L and so suppose that H is not contained in a Levi factor of any proper parabolic subgroup of G. Since H is G-cr, we conclude that H is contained in no proper parabolic subgroup of G.

To show that $\mathfrak{h}$ is G-cr, we use part (1) of Theorem 1; it is enough to show that $\mathrm{Ad}(G)\mathbf{X}$ is closed in Y. In fact, we are going to suppose that $\mathrm{Ad}(G)\mathbf{X}$ is not closed and obtain a contradiction. Let $S = \overline{\mathrm{Ad}(G)\mathbf{X}} - \mathrm{Ad}(G)\mathbf{X}$ and let $P = P_{S,\mathbf{X}}$. Since S is assumed non-empty, Corollary 9 shows that P is a proper parabolic subgroup. Moreover, since $\mathrm{Ad}(H)$ leaves $\mathfrak{h}$ invariant, that same corollary shows that $H \subset P$. This contradiction completes the proof. □

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