LEVI DECOMPOSITIONS OF LINEAR ALGEBRAIC GROUPS AND NON-ABELIAN COHOMOLOGY

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To the memory of Gary Seitz (1943-2023)

Contents

Abstract. Let k be a field, and let G be a linear algebraic group over k for which the unipotent radical U of G is defined and split over k. Consider a finite, separable field extension ℓ of k and suppose that the group G_ℓ obtained by base-change has a *Levi decomposition* (over ℓ). We continue here our study of the question previously investigated in (McNinch 2013): does G have a *Levi decomposition* (over k)?

Using non-abelian cohomology we give some condition under which this question has an affirmative answer. On the other hand, we provide an(other) example of a group G as above which has no Levi decomposition over k.

1. Introduction

Let k be a field, and let G be a linear algebraic group over k. Thus G is a group scheme which is smooth and affine over k.

If k_{alg} denotes an algebraic closure of k, the *unipotent radical* of $G_{k_{\mathrm{alg}}}$ is the maximal connected, unipotent, normal subgroup. The unipotent radical of G is defined over k if G has a k-subgroup U such that $U_{k_{\mathrm{alg}}}$ is the unipotent radical of $G_{k_{\mathrm{alg}}}$.

Definition 1.1. We say that G *satisfies condition* ***(R)*** if the unipotent radical U of G is *defined and split* over k. (See Definition 2.1 for the notion of split unipotent group). Write $\pi : G \to G/U$ for the quotient morphism; we say that G/U is the *reductive quotient* of G. Thus G/U is a (not necessarily connected) reductive algebraic group.

Remark 1.2. If k is perfect then **(R)** holds for any linear algebraic group G over k. Indeed, the unipotent radical U is defined over k by Galois descent. Moreover, every (smooth) connected unipotent group over a perfect field k is k-split; see Remark 2.2.

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Definition 1.3. Suppose that G satisfies condition **(R)**. The group G has a *Levi decomposition* (over k) if there is a closed k-subgroup scheme M of G such that the restriction of the quotient mapping determines an isomorphism

$$\pi_{|M} : M \xrightarrow{\sim} G/R.$$

The subgroup M is then a *Levi factor* of G.

If G satisfies condition **(R)** and if M is a Levi factor of G then Proposition 2.7 below shows that G may be identified with the semidirect product $U \rtimes M$ as algebraic groups

Remark 1.4. When k has characteristic 0, G work of Mostow show that G always has a Levi decomposition; see e.g. (McNinch 2010) §3.1. For any field k of characteristic $p > 0$, there are linear algebraic groups G over k with no Levi factor; see e.g. (Conrad, Gabber, and Prasad 2015) A.6 for a construction.

We now fix a linear algebraic k-group G satisfying **(R)**. Suppose that ℓ is a finite, separable field extension of k, and suppose that G_ℓ has a Levi decomposition. We pose the question:

($\diamondsuit$) If G_ℓ has a a Levi decomposition (over ℓ), does G have a Levi decomposition (over k)?

This question about descent of Levi factors was already considered in the paper (McNinch 2013) whose main result gave the following partial answer:

Theorem 1.5. *Assume that ℓ is a finite, Galois field extension of k with Galois group $\Gamma = \operatorname{Gal}(\ell/k)$, and assume that G_ℓ has a Levi decomposition. If $|\Gamma|$ is invertible in k then G has a Levi decomposition.*

In the present paper, we introduce the non-abelian cohomology set $H^1_{\mathrm{coc}}(M, U)$ in Section 3, and in Section 4 we prove the following result providing a different partial answer to ($\diamondsuit$):

Theorem 1.6. *If ℓ is a finite separable extension of k, suppose the following:*

(a) G_ℓ has a Levi decomposition,
(b) the group scheme ${U_\ell}^{M_\ell}$ is trivial, and
(c) $H^1_{\mathrm{coc}}(M_\ell, U_\ell) = 1$.

Then G has a Levi decomposition.

We also prove Corollary 4.5 which gives a reformulation of Theorem 1.6 using a filtration of U. After some preliminaries in Section 5 and Section 6, we prove the following related result in Section 7:

Theorem 1.7. *Suppose the following:*

(a) G_ℓ has a Levi decomposition,
(b) $\operatorname{Inn}(U_\ell)^{M_\ell}$ is trivial,
(c) the center Z of U is a vector group on which G acts linearly, and
(d) $H^1_{\mathrm{coc}}(M_\ell, \operatorname{Inn}(U_\ell)) = 1$.

Then G has a Levi decomposition.

The reader should compare these results with (McNinch 2010) Theorem 5.2. This older result shows that a certain condition involving the vanishing of second cohomology H^2 unconditionally guarantees the existence of a Levi factor. These newer results – Theorem 1.6, Corollary 4.5 and Theorem 1.7 – instead give conditions using vanishing of (some form of) first cohomology to descend Levi factors over finite separable field extensions.

We note that *some* additional hypotheses are required to answer the question ($\diamondsuit$). Indeeed, Section 8 provides an example of an algebraic group G satisfying condition **(R)** for which G_ℓ has a Levi factor for some cyclic Galois extension ℓ of degree p over k, but G has no Levi factor over k.

Every example currently known to the author of a group G satisfying **(R)** for which ($\diamondsuit$) has a negative answer is *not connected*. This suggests the following natural problem for which a solution would be desirable:

Problem 1.8. Let ℓ be a finite, separable field extension of k and G a connected linear algebraic group over k satisfying **(R)**. Either find a proof of the assertion G_ℓ has a Levi factor implies that G has a Levi factor or find an example of a group for which this condition fails.

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2. Preliminaries

We fix an arbitrary field k. Throughout the paper, G will denote a linear algebraic group over k. Thus G is a group scheme which is smooth, affine, and of finite type over k.

If V is a linear representation of G, then for $i \geq 0$, $H^i(G, V)$ denotes the ith *(Hochschild) cohomology* group of V; see e.g. (Jantzen 2003) I.4.

Automorphism group functors. By a k-group functor, we mean a functor from the category of commutative k-algebras to the category of groups. Of course, any group scheme – and in particular, any linear algebraic group – over k is *a fortiori* a k-group functor, but we will consider a few group functors which are in general not representable (i.e. which fail to be group schemes).

For a linear algebraic group G over k, we write $\operatorname{Aut}(G)$ for the k-group functor which assigns to a commutative k-algebra Λ the group $\operatorname{Aut}(G)(\Lambda) = \operatorname{Aut}(G(\Lambda))$.

If Z denotes the (scheme-theoretic) center of G, there is a natural homomorphism of k-group functors $\operatorname{Inn} : G/Z \to \operatorname{Aut}(G)$ whose image determines a normal k-sub-group functor $\operatorname{Inn}(G)$ of $\operatorname{Aut}(G)$; see (Demazure and Grothendieck 2011) XXIV §1.1.

Now, the k-group functor $\operatorname{Out}(G)$ is defined for each Λ by the rule

$$\operatorname{Out}(G)(\Lambda) = \operatorname{Aut}(G)(\Lambda)/\operatorname{Inn}(G)(\Lambda)).$$

The quotient mappings $\operatorname{Aut}(G(\Lambda)) \to \operatorname{Aut}(G(\Lambda))/\operatorname{Inn}(G(\Lambda))$ determine a homomorphism of k-group functors

$$\Psi : \operatorname{Aut}(G) \to \operatorname{Out}(G). \tag{2.1}$$

Unipotent groups. Recall from (Borel 1991) §15.1 the following:

Definition 2.1. A connected, unipotent linear algebraic group U over k is said to be *k-split* provided that there is a sequence

$$1 = U_0 \subset U_2 \subset \cdots \subset U_{m-1} \subset U_m = U$$

of closed, connected, normal k-subgroups of U such that $U_{i+1}/U_i \simeq \mathbf{G}_{a/k}$ for $i = 0, \cdots, m-1$, where $\mathbf{G}_a = \mathbf{G}_{a/k}$ is the additive group.

Remark 2.2. When k is not a perfect field, there are connected unipotent k-groups which are not k-split; for an example, see e.g. (Serre 2002) III.§2.1 Exercise 3. On the other hand, if k is perfect, every connected unipotent k-group is k-split. (Borel 1991) Cor. 15.5(ii).

Proposition 2.3. *Let U be a k-split unipotent group. If V is a normal k-subgroup of U, then U/V is again a k-split unipotent group.*

Proof. The assertion follows from (Borel 1991) Theorem 15.4(i). $\square$

A substantial reason for our focus on split unipotent groups is the following result of Rosenlicht:

Proposition 2.4. *Suppose that U is a connected, k-split unipotent subgroup of G and write $\pi : G \to G/U$ for the quotient morphism. Then there is a morphism of k-varieties*

$$\sigma : G/U \to G$$

which is a section to π – i.e. $\pi \circ \sigma$ is the identity. In particular, the mapping $\pi : G(k) \to (G/U)(k)$ on k-points is surjective.

Proof. See (Springer 1998) Theorem 14.2.6. □

Extensions, group actions and semi-direct products. Let A and M be linear algebraic k-groups.

Definition 2.5. An *extension* of M by A is a linear algebraic k-group E together with a sequence

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} M \to 1. \tag{2.2}$$

where i and π are morphisms of algebraic groups over k, i determines an isomorphism of A onto $\ker \pi$, and the homomorphism π is faithfully flat.

Definition 2.6. If A and M are linear algebraic groups, we say that A is *an M-group* provided that there is a morphism of k-group functors $M \to \mathrm{Aut}(A)$.

If A is a M-group via the homomorphism of k-group functors

$$\alpha : M \to \mathrm{Aut}(A)$$

then we can form the semi-direct product $A \rtimes_\alpha M$; it is an extension of M by A. (We omit the subscript α from $\rtimes_\alpha$ when it is clear from context).

If E is an extension (2.2), observe that the conjugation action of E determines a morphism of group functors $\mathrm{Inn} : E \to \mathrm{Aut}(A)$.

We record the following two results; their proofs are straightforward and left to the reader:

Proposition 2.7. *Let A and M be linear algebraic k-group and consider an extension* (2.2)

$$1 \to A \to G \xrightarrow{\pi} M \to 1.$$

If $s : M \to G$ is a group homomorphism that is a section to π then the multiplication mapping $(x, m) \mapsto xm$ induces an isomorphism

$$A \rtimes_\phi M \xrightarrow{\sim} G$$

of algebraic k-groups, where $\phi : M \to \mathrm{Aut}(A)$ is the composite $\mathrm{Inn} \circ s$.

Proposition 2.8. *Let A and M be linear algebraic k-group and consider an extension* (2.2)

$$1 \to A \to G \xrightarrow{\pi} M \to 1.$$

There is a unique homomorphism of k-group functors $\phi : M \to \mathrm{Out}(A)$ such that for any for any section $s_0 : M \to G$ to π as in Proposition 2.4, for any commutative k-algebra Λ, and for any $m \in M(\Lambda)$, $\phi(m)$ is the class of the inner automorphism $\mathrm{Inn}(s_0(m))$ *in* $\mathrm{Out}(A)$ *.*

Remark 2.9. A unipotent k-group U is *wound* if every mapping $\mathbf{A}^1 \to U$ of k-schemes is constant. A connected, wound unipotent group of positive dimension is not k-split. If M is a connected and reductive k-group and if U is a wound unipotent k-group, then:

($*$) any homomorphism of k-group functors $M \to \mathrm{Aut}(U)$ is trivial.

Indeed, if M is a torus then $(*)$ follows from (Conrad, Gabber, and Prasad 2015) Corollary B.44. Now $(*)$ follows in general since the connected reductive group M is generated by its maximal k-tori – see (Springer 1998) Theorem 13.3.6.

Observation $(*)$ provides some partial justification for our focus on groups satisfying **(R)**.

Linear actions. Let G and U be linear algebraic groups, suppose that U is connected and unipotent, and suppose that U is a G-group.

Definition 2.10. If U is a vector group, the action of G on U is said to be *linear* if there is a G-equivariant isomorphism of algebraic groups $U \simeq \operatorname{Lie}(U)$.

Definition 2.11. A filtration

$$1 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_{m-1} \subset U_m = U$$

by G-invariant closed k-subgroups U_i with U_i normal in U_{i+1} for each i is a *linear filtration* for the action of G if U_{i+1}/U_i is a vector group on which G acts linearly for each $i = 0, \cdots, m-1$.

A linear filtration is a *central linear filtration* if U_{i+1}/U_i is central in U/U_i for each $i \geq 0$.

The following result was proved already in (Stewart 2013) under the assumption that k is algebraically closed.

Theorem 2.12. *Assume that the unipotent radical U of G is defined and split over k.*

(a) If G is connected, there is a linear filtration of U for the action of G.
(b) If U has a linear filtration for the action of $U \rtimes G$ then it has a central linear filtration.

Proof. (a) is the main result of (McNinch 2014).

To see (b), suppose that the subgroups U_i form a linear filtration of U for the action of $U \rtimes G$. We may clearly refine this filtration to arrange that $\operatorname{Lie}(U_i)/\operatorname{Lie}(U_{i+1})$ is an irreducible representation of $U \rtimes G$ for each i. [1]. We claim that this refined filtration is central. We proceed by induction on the length m of the linear filtration. If $m = 1$ then U is abelian and the result is immediate.

Suppose now that $m > 1$ and that one knows that any linear filtration of U for the action of $U \rtimes G$ of length $< m$ for which the factors of consecutive terms form irreducible $U \rtimes G$-representations is central.

Now, the conjugation action of U on U_1 is a *linear* action; thus, the fixed points for the conjugation action of U on U_1 form a G-invariant subgroup scheme which is smooth over k. Since $U_1 \simeq \operatorname{Lie}(U_1)$ is an irreducible G-representation, it follows that U acts trivially on U_1; thus U_1 is central in U. Now, it is clear that

$$1 \subset U_2/U_1 \subset \cdots \subset U_m/U_1 = U/U_1 \tag{2.3}$$

forms a linear filtration of U/U_1 for the action of G for which the factors of consecutive terms form irreducible $U \rtimes G$-representations. Thus by induction (2.3) is a central linear filtration; this completes the proof. ☐

Remark 2.13. In the proof of Theorem 2.12, we constructed a central linear filtration by arranging that the action of $U \rtimes G$ on each quotient U_{i+1}/U_i is irreducible. This condition is sufficient, but not necessary – in general, there are central linear filtrations for which $\operatorname{Lie}(U_{i+1})/\operatorname{Lie}(U_i)$ is a reducible G-representation for some i.

[1] Since U is unipotent, an irreducible representation of $U \rtimes G$ amounts to an irreducible representation of G.

Galois cohomology. Write $\Gamma = \mathrm{Gal}(k_{\mathrm{sep}}/k)$ for the absolute Galois group of k where k_{sep} is a separable closure of k.

Let G be a k-group functor satisfying the conditions spelled out in (Serre 2002) II.1.1. Then Γ acts continuously on the group $G(k_{\mathrm{sep}})$ and we may consider the Galois cohomology set $H^1(k, G) := H^1(\Gamma, G(k_{\mathrm{sep}}))$ (Serre 2002), §5.1.

Proposition 2.14. *Let U be a connected, split unipotent algebraic group over k. Then the Galois cohomology set satisfies $H^1(k, U) = 1$.*

Proof. The necessary tools are recalled in (McNinch 2004) Prop. 30. □

3. Non-abelian cohomology

Let A and M be linear algebraic k-groups and suppose that A is an M-group. Following (Demarche 2015) §2.1, we introduce the cohomology set $H^1_{\mathrm{coc}}(M, A)$ as follows. Let $Z^1_{\mathrm{coc}}(M, A)$ denote the set of regular maps $f : M \to A$ such that for each commutative k-algebra Λ and each $x, y \in M(\Lambda)$, the 1-cocycle condition

$$f(xy) = f(x) \cdot {}^x f(y) \tag{3.1}$$

holds. Two cocycles $f, f' \in Z^1_{\mathrm{coc}}(M, A)$ are *cohomologous* provided there is $u \in U(k)$ such that for each Λ and each $x \in M(\Lambda)$ we have

$$f(x) = u^{-1} \cdot f'(x) \cdot {}^x u.$$

This defines an equivalence relation on $Z^1_{\mathrm{coc}}(M, A)$ and we write $H^1_{\mathrm{coc}}(M, A)$ for the quotient set.

We view $H^1_{\mathrm{coc}}(M, A)$ as a *pointed set*; the marked point $1 \in H^1_{\mathrm{coc}}(M, A)$ is the class of the cocycle in $Z^1_{\mathrm{coc}}(M, A)$ which takes the constant value 1. The pointed set $H^1_{\mathrm{coc}}(M, A)$ is trivial if $H^1_{\mathrm{coc}}(M, A) = \{1\}$; we often indicate this condition by the shorthand $H^1_{\mathrm{coc}}(M, A) = 1$.

One interpretation or application of this cohomology set arises from examination of a semidirect product $G = A \rtimes M$. Consider a linear algebraic group G with normal subgroup A and a quotient mapping $\pi : G \to M = G/A$. We suppose that there is a group homomorphism $s_0 : M \to G$ which is a section to π. According to Proposition 2.7, s_0 determines an isomorphism $G \simeq A \rtimes M$.

Definition 3.1. Consider the set of all homomorphisms of k-groups $M \to G$ which are sections to π; two such homomorphisms s, s' will be considered *equivalent* if there is $a \in A(k)$ such that $s = as'a^{-1}$. Then $\mathrm{Sect}(G \xrightarrow{\pi} M)$ denotes the quotient of the set of all such homomorphisms by this equivalence relation.

Proposition 3.2. *Write $\mu : G \times G \to G$ for the multiplication mapping. For a given homomorphism $s_0 : M \to G$ which is a section to π, the assignment*

$$f \mapsto \mu \circ (f, s_0)$$

– where $(f, s_0) : M \to M \times G$ is the mapping $m \mapsto (f(m), s_0(m))$ – determines a bijection

$$A_{s_0} : H^1_{\mathrm{coc}}(M, A) \to \mathrm{Sect}(G \xrightarrow{\pi} M).$$

Proof. As already observed above, the choice of s_0 determines an isomorphism of linear algebraic groups $G \simeq A \rtimes M$; see Proposition 2.7. Now the result follows from (Demarche 2015) Prop. 2.2.2. □

Remark 3.3. $H^1_{\mathrm{coc}}(M,A)$ is a pointed set – i.e. a set with a distinguished element. That distinguished element is the class of the trivial mapping $(x \mapsto 1) : G \to A$. In the bijection of Proposition 3.2 the section corresponding to the trivial class is s_0.

Remark 3.4. When Z is a vector group with a linear action of M, $H^1_{\mathrm{coc}}(M,Z)$ coincides with the usual Hochschild cohomology group $H^1(M,Z) \simeq H^1(M, \mathrm{Lie}(Z))$. In particular, in that case $H^1_{\mathrm{coc}}(M,Z)$ is a k-vector space.

Suppose now that $A = U$ is a split unipotent M-group and that $Z \subset U$ is a central k-subgroup that is M-invariant. Then U/Z is a split unipotent M-group, and there is a mapping

$$\Delta : H^1_{\mathrm{coc}}(M, U/Z) \to H^2(M,Z) \tag{3.2}$$

where $H^2(M,Z)$ denotes the second Hochschild cohomology; it is defined as follows. First, use Rosenlichts result Proposition 2.4 to choose a regular mapping $s : U/Z \to U$ which is a section to the quotient homomorphism $U \to U/Z$. Let $\alpha = [f] \in H^1_{\mathrm{coc}}(M, A/Z$ with $f \in Z^1_{\mathrm{coc}}(M, A/Z)$

As in (Demazure and Gabriel 1970) II, Subsect. 3.2.3 – see also (McNinch 2010), §4.4 – the rule $(g,h) \mapsto s(f(g))s(f(h))s(f(gh))^{-1}$ determines a Hochschild 2-cocycle whose class in $H^2(G,Z)$ we denote $\Delta(\alpha)$.

Proposition 3.5. *Let U be a split unipotent M-group, and let Z be a central, closed and smooth k-subgroup of U that is M-invariant. Write $i : Z \to U$ and $\pi : U \to U/Z$ for the inclusion and quotient mappings, respectively.*

(a) the sequence of pointed sets

$$H^1(M,Z) \xrightarrow{i_*} H^1_{\mathrm{coc}}(M,U) \xrightarrow{\pi_*} H^1_{\mathrm{coc}}(M,U/Z) \xrightarrow{\Delta} H^2(M,Z)$$

is exact.

(b) If $(U/Z)^M = 1$ then i_* is injective.

Sketch. (a) The proof of the corresponding statement for cohomology of pro-finite groups given in (Serre 2002) I. §5.7 may be applied here *mutatis mutandum.* The main required adaptation is the definition (given above) of the mapping Δ (which required the existence of a regular section $U/Z \to U$).

For (b), suppose that $f_1, f_2 : M \to Z$ are 1-cocycles and that $i_*([f_1]) = i_*([f_2])$. Thus f_1, f_2 are cohomologous in $Z^1_{\mathrm{coc}}(M,U)$, so there is $u \in U(k)$ such that

$$f_1(x) = u^{-1} \cdot f_2(x) \cdot xu$$

for every commutative k-algebra Λ and every $x \in M(\Lambda)$. Passing to the quotient U/Z we see that $1 = u^{-1}xu$ so that the class of u lies in $(U/Z)^M(\Lambda)$. □

Remark 3.6. Assume that ℓ is a finite, Galois extension of k with Galois group $\Gamma = \mathrm{Gal}(\ell/k)$. Then Γ acts on the Galois cohomology $H^1(M_\ell, A_\ell)$ through its action on regular mappings $M_\ell \to A_\ell$.

If A is a vector group on which M acts linearly, then $H^1(M_\ell, A_\ell)$ may be identified with $H^1(M,A) \otimes_k \ell$. In particular, in that case $H^1(M,A)$ may be identified with $H^1(M_\ell, A_\ell)^\Gamma$.

This observation prompts several questions. Suppose U is a split unipotent M-group and that U has a central linear filtration for the action of M.

(a) Under what conditions is it true that $H^1_{\mathrm{coc}}(M,U) = H^1_{\mathrm{coc}}(M_\ell, U_\ell)^\Gamma$?
(b) Under what conditions is it true that the condition $H^1_{\mathrm{coc}}(M,U) = 1$ is equivalent to the condition $H^1_{\mathrm{coc}}(M_\ell, U_\ell) = 1$?

4. Descent of Levi factors

We begin this section by giving the proof of Theorem 1.6 from the introduction.

Proof. Recall that G is a linear algebraic group satisfying condition **(R)**, U is the unipotent radical and $M = G/U$ is the reductive quotient. Moreover, ℓ is a finite, separable field extension of k. We must show that under assumptions (a), (b), and (c), the group G has a Levi decomposition.

First, note that the assumptions are unaffected if we pass to a finite separable extension of ℓ. Thus, we may and will suppose that ℓ is Galois over k; write $\Gamma = \mathrm{Gal}(\ell/k)$ for the Galois group.

According to (a), G_ℓ has a Levi decomposition. Thus we may choose a homomorphism $s : M_\ell \to G_\ell$ which is a section to π. According to (c), we have $H^1_{\mathrm{coc}}(M_\ell, U_\ell) = 1$. Together with Proposition 3.2, this shows that the set $\mathrm{Sect}(G_\ell \xrightarrow{\pi} M_\ell)$ contains a single element. In particular, every homomorphism $u : M_\ell \to G_\ell$ which is a section to π differs from s by conjugation with an element of $U(\ell)$.

There is a natural action of Γ on homomorphisms $M_\ell \to G_\ell$ which determines in turn an action of Γ on $\mathrm{Sect}(G_\ell \xrightarrow{\pi} M_\ell)$. For each $\gamma \in \Gamma$, we thus find an element $u_\gamma \in U(\ell)$ such that ${}^\gamma s = u_\gamma^{-1} \cdot s \cdot u_\gamma$.

We now contend that (♣): u_γ is a 1-cocycle on Γ with values in $U(\ell)$. Well, for $\gamma, \tau \in \Gamma$ we see that

$$ {}^{\gamma\tau} s = u_{\gamma\tau}^{-1} \cdot s \cdot u_{\gamma\tau} \tag{4.1} $$

while on the other hand

$$ \begin{aligned} {}^{\gamma\tau} s &= {}^\gamma(u_\tau^{-1} \cdot s \cdot u_\tau) \\ &= {}^\gamma u_\tau^{-1} \cdot {}^\gamma s \cdot {}^\gamma u_\tau \\ &= {}^\gamma u_\tau^{-1} \cdot u_\gamma^{-1} \cdot s \cdot u_\gamma \cdot {}^\gamma u_\tau \end{aligned} \tag{4.2} $$

Now, assumption (b) guarantees that $U_\ell^{M_\ell}$ is trivial, and it follows that the stabilizer in U_ℓ of the section s is trivial. Thus together (4.1) and (4.2) imply that

$$ u_{\gamma\tau} = u_\gamma \cdot {}^\gamma u_\tau. $$

This confirms (♣). Since U is a split unipotent k-group, $H^1(k, U) = 1$; see Proposition 2.14. Thus there is $u \in U(\ell)$ such that

$$ u_\gamma = u^{-1} \cdot {}^\gamma u \tag{4.3} $$

for each $\gamma \in \Gamma$; i.e. ${}^\gamma u = u u_\gamma$.

Now set $s_0 = u \cdot s \cdot u^{-1} \in \mathrm{Sect}(G_\ell \xrightarrow{\pi} M_\ell)$. We claim that s_0 is a k-homomorphism. It is enough to argue that s is fixed by the Galois group Γ. For $\gamma \in \Gamma$ we note that

$$ \begin{aligned} {}^\gamma s_0 &= {}^\gamma u \cdot s \cdot u^{-1} \\ &= {}^\gamma u \cdot {}^\gamma s \cdot {}^\gamma u^{-1} \\ &= u \cdot u_\gamma \cdot u_\gamma^{-1} \cdot s \cdot u_\gamma \cdot u_\gamma^{-1} \cdot u \\ &= u s u^{-1} = s_0. \end{aligned} $$

Thus $s_0 : M \to G$ is a k-morphism which is a section to π; this shows that G has a Levi factor as required. ∎

In the remainder of this section, we are going to formulate a variant of Theorem 1.6 using a filtration of U. We are going to *assume that U has a central linear filtration*

$$1 = Z_0 \subset Z_1 \subset \cdots Z_m = U$$

for the action of G; see Definition 2.11. Note that such a filtration always exists in case G is connected; see Theorem 2.12.

Proposition 4.1. *For each $n \geq 0$ the homomorphism of k-group functors*

$$\phi_0 : M \to \mathrm{Out}(U)$$

of Proposition 2.8 determines an action of M on the quotient Z_{n+1}/Z_n.

Proof. Since Z_{n+1}/Z_n is abelian, $\mathrm{Out}(Z_{n+1}/Z_n) = \mathrm{Aut}(Z_{n+1}/Z_n)$. For each natural number n, ϕ_0 determines by restriction and passage to the quotient a homomorphism of k-group functors

$$\phi_{0|Z_{n+1}} : M \to \mathrm{Out}(Z_{n+1}/Z_n) = \mathrm{Aut}(Z_{n+1}/Z_n),$$

i.e. an action of M on Z_{n+1}/Z_n. □

Lemma 4.2. *Suppose that $H^1(M, Z_{i+1}/Z_i) = 0$ for each $i = 0, \cdots, m-1$. Then*

$$H^1_{\mathrm{coc}}(M, U) = 1 \quad \textit{and} \quad H^1_{\mathrm{coc}}(M_\ell, U_\ell) = 1.$$

Proof. First observe that for a linear representation V of G, $H^1(G, V) = 0$ if and only if $H^1(G_\ell, V_\ell) = 0$. Now the result follows from Proposition 3.5. □

Remark 4.3. Viewing a finite dimensional linear representation V of M as an algebraic group, the scheme-theoretic fixed-point subgroup V^M coincides with the vector group given by the M-fixed points on the linear representation V. In particular, if V is an irreducible representation of M, the group scheme V^M is equal to $\{0\}$.

Lemma 4.4. *Suppose that $(Z_{i+1}/Z_i)^M = \{1\}$ for each $i = 0, \cdots, m-1$. Then $U^M = \{1\}$ is the trivial group scheme.*

Proof. We proceed by induction on m, the length of the central linear filtration of U. If $m = 0$, $U = 1$ and the result is immediate.

Now suppose that $m > 0$ and that the result is known for connected and split unipotent M-groups having a central linear filtration of length $< m$. Thus by induction we know $(U/Z_1)^M = \{1\}$. Thus U^M is contained in the kernel of the quotient mapping $U \to U/Z_1$, i.e. U^M is contained in Z_1. Since $(Z_1)^M$ is the trivial group scheme, the proof is complete. □

We now obtain a corollary to Theorem 1.6, as follows:

Corollary 4.5. *Assume that U has a central linear filtration for the action of G and a suppose the following:*

(a) G_ℓ has a Levi decomposition (over ℓ),
(bb) the group scheme $(Z_{i+1}/Z_i)^M$ is trivial for $i = 0, \cdots, m-1$, and
(cc) $H^1(M, Z_{i+1}/Z_i) = 0$ for $i = 0, \cdots, m-1$.

Then G has a Levi decomposition.

Proof. Note that according to Lemma 4.4, condition (bb) implies hypothesis (b) of Theorem 1.6. Similarly, according to Lemma 4.2 (cc) implies hypothesis (c) of Theorem 1.6. Thus the result follows from Theorem 1.6. □

5. Automorphisms of extensions

Let A and M be linear algebraic groups over k, and let E and E be extensions of M by A as in Definition 2.5.

Definition 5.1. A *morphism of extensions* $\phi : E \to E'$ is a morphism of algebraic groups for which the diagram

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "\pi"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & A \arrow[r, "i'"] & E' \arrow[r, "\pi"] & M \arrow[r] & 1 \end{tikzcd}$$

is commutative.

Remark 5.2. If $\phi : E \to E$ is a morphism of extensions, then ϕ is necessarily an isomorphism of algebraic groups $E \xrightarrow{\sim} E'$.

Write $\mathrm{Autext}(E)$ for the group of automorphisms of E. Let Z be the (schematic) center of A. Since Z is characteristic in A, E acts on Z by conjugation. Since A acts trivially on Z, the action of E on Z factors through $M \simeq E/A$.

Write $Z^1_{\mathrm{coc}}(M, Z)$ for the Hochschild 1-cocycles as in Section 3. Since Z is commutative, $Z^1_{\mathrm{coc}}(M, Z)$ is a group. The following result is a consequence of (Florence and Arteche 2020), Prop. 2.3.

Proposition 5.3. *There is a canonical isomorphism of groups* $Z^1_{\mathrm{coc}}(M, Z) \xrightarrow{\sim} \mathrm{Autext}(E)$.

Now suppose that ℓ is a finite, separable field extension of k.

Theorem 5.4. *Assume that the center Z of A is a vector group and that the action of M on Z is linear. If the extensions E_ℓ and E'_ℓ of M_ℓ by A_ℓ are isomorphic, then E and E' are isomorphic extensions of M by A.*

Proof. Write k_{sep} for a separable closure of k containing ℓ and write $\mathscr{E}$ for the set of isomorphism classes of extensions of M by A over k which after scalar extension to k_{sep} become isomorphic to the extension $E_{k_{\mathrm{sep}}}$ of $M_{k_{\mathrm{sep}}}$ by $A_{k_{\mathrm{sep}}}$.

As in (Serre 2002), III.§1, one knows that there is a bijection

$$\mathscr{E} \xrightarrow{\sim} H^1(k, \mathrm{Autext}(E)) := H^1(\mathrm{Gal}(k_{\mathrm{sep}}/k), \mathrm{Autext}(E_{k_{\mathrm{sep}}})). \tag{5.1}$$

Thus, the Theorem will follow if we argue that the Galois cohomology set $H^1(k, \mathrm{Autext}(E))$ is trivial – i.e. contains a unique element.

By assumption, Z is a vector group with linear action of M, so that $Z^1(M, Z)$ is a k-vector space (possibly of infinite dimension). Now Proposition 5.3 shows that $\mathrm{Autext}(E) = Z^1(M, Z)$ is a k-vector space, and it follows from "additive Hilbert 90" that

$$H^1(k, \mathrm{Aut}(E)) \simeq H^1(k, Z^1(A, Z))$$

is trivial; see for example (McNinch 2013) (4.1.2). ∎

6. Automorphisms and cohomology

Let A and M be a linear algebraic k-groups, and suppose that A is a M-group via the mapping

$$\phi : M \to \mathrm{Aut}(A).$$

Let Z denote the center of A as a group scheme. Then $\mathrm{Inn}(A) \simeq A/Z$ is also an M-group via ϕ; for $h \in \mathrm{Inn}(A)(\Lambda)$ and $g \in M(\Lambda)$, we have ${}^g h = \phi(g) h \phi(g)^{-1}$.

Denote by $\phi_0 = \Psi \circ \phi$ the homomorphism of group functors

$$M \xrightarrow{\phi} \mathrm{Aut}(A) \xrightarrow{\Psi} \mathrm{Out}(A)$$

where $\Psi : \mathrm{Aut}(A) \to \mathrm{Out}(A)$ is the natural map of (2.1).

Consider those homomorphisms of k-group functors $\theta : M \to \mathrm{Aut}(A)$ satisfying

$$(*) \quad \Psi \circ \theta_1 = \phi_0.$$

We say that two such homomorphisms θ_1 and θ_2 are equivalent if they are conjugate by $\mathrm{Inn}(A)(k)$; i.e. if there is $h \in \mathrm{Inn}(A)(k)$ for which

$$\theta_1(g) = h^{-1}\theta_2(g)h$$

for each commutative k-algebra Λ and each $g \in M(\Lambda)$. We write $\mathrm{Lift}(\phi_0)$ for the quotient of the set of all homomorphisms $M \to \mathrm{Aut}(A)$ satisfying $(*)$ by the equivalence relation just described.

Proposition 6.1. *Write $\mu : \mathrm{Aut}(A) \times \mathrm{Aut}(A) \to \mathrm{Aut}(A)$ for the group operation. For $f \in Z^1_{\mathrm{coc}}(M, A)$, define $\Phi_f : M \to \mathrm{Aut}(A)$ by the rule*

$$\Phi_f = \mu \circ (f, \phi) : M \to \mathrm{Aut}(A) \times \mathrm{Aut}(A) \to \mathrm{Aut}(A).$$

Then the assignment $f \mapsto \Phi_f$ determines a bijection

$$\Phi : H^1_{\mathrm{coc}}(G, \mathrm{Inn}(A)) \to \mathrm{Lift}(\phi_0)$$

Proof. For any 1-cocycle $f \in Z^1_{\mathrm{coc}}(G, M)$, one checks that the mapping $\Phi_f : G \to \mathrm{Aut}(A)$ is homomorphism of k-group functors contained in $\mathrm{Lift}(f\phi)$.

We now claim for $f_1, f_2 \in Z^1_{\mathrm{coc}}(M, A)$ that f_1 and f_2 are cohomologous if and only if Φ_{f_1} and Φ_{f_2} are equivalent.

$(\Rightarrow)$: By assumption there is $h \in \mathrm{Inn}(U)(k)$ such that for each commutative k-algebra Λ and each $g \in M(\Lambda)$ that

$$f_1(g) = h^{-1} f_2(g)\, {}^g h.$$

Now observe that

$$\begin{aligned}\Phi_{f_1}(g) &= f_1(g)\phi(g) = h^{-1} f_2(g)\, {}^g h \cdot \phi(g)\\ &= h^{-1} f_2(g)\phi(g)h\phi(g)^{-1}\phi(g) = h^{-1} f_2(g)\phi(g)h\\ &= h^{-1}\Phi_{f_2}(g)h\end{aligned}$$

so that indeed Φ_{f_1} and Φ_{f_2} are equivalent.

$(\Leftarrow)$: By assumption there is $h \in \mathrm{Inn}(A)(k)$ for which

$$\Phi_{f_1} = h^{-1}\Phi_{f_2} h.$$

Then for each commutative k-algebra Λ and each $g \in M(\Lambda)$ we have

$$\begin{aligned}f_1(g) &= \Phi_{f_1}(g) \cdot \phi(g)^{-1} = h^{-1}\Phi_{f_2}(g)h \cdot \phi(g)^{-1}\\ &= h^{-1}\Phi_{f_2}(g)\phi(g)^{-1}\, {}^g h = h^{-1} f_2(g)\, {}^g h\end{aligned}$$

so that f_1 and f_2 are cohomologous.

It now follows that $f \mapsto \Phi_f$ determines a well-defined injective mapping

$$\Phi : H^1_{\mathrm{coc}}(M, \mathrm{Inn}(A)) \to \mathrm{Lift}(\phi_0).$$

To see that Φ is surjective, suppose $\theta : M \to \mathrm{Aut}(A)$ represents a class in $\mathrm{Lift}(\phi_0)$. For each commutative k-algebra Λ and each $g \in M(\Lambda)$, we have $\theta(g)\phi(g)^{-1} \in \mathrm{Inn}(A)(\Lambda)$. Thus we have a morphism of k-functors $f : M \to \mathrm{Inn}(A)$ given by the rule

$$f(g) = \theta(g)\phi(g)^{-1}.$$

By the Yoneda Lemma, the assignment f is a morphism of varieties, and a calculation confirms that f is a 1-cocycle for the action of M on $\mathrm{Inn}(A)$ determined by ϕ. Then $[\theta] = [\Phi_f] = \Phi([f])$ which proves that Φ is surjective. □

7. Descent of Levi factors, part 2

In this section, we are going to prove Theorem 1.7. We first prove the following:

Lemma 7.1. *Let M, A be linear algebraic groups, and suppose that A is an M-group via the homomorphism $\phi : M \to \mathrm{Aut}(A)$ of k-group functors. Let $x \in A(k)$ and consider the mapping $\phi_1 : M \to \mathrm{Aut}(A)$ given for each commutative k-algebra Λ and each $g \in M(\Lambda)$ by the rule $\phi_1(g) = \mathrm{Inn}(x)\phi(g)\,\mathrm{Inn}(x)^{-1}$. Then there is a k-isomorphism of extensions of M by A:*

$$A \rtimes_\phi M \simeq A \rtimes_{\phi_1} M.$$

Proof. Write $G = A \rtimes_\phi M$ for the semidirect product constructed using the action defined by ϕ. Now, the mapping $\phi : M \to \mathrm{Aut}(A)$ may be identified with the composite

$$M \xrightarrow{m \mapsto (1,m)} G = A \rtimes_\phi M \xrightarrow{\mathrm{Inn}} \mathrm{Aut}(A)$$

and $\phi_1 : M \to \mathrm{Aut}(A)$ identifies with the composite

$$M \xrightarrow{m \mapsto (x,1)(1,m)(x,1)^{-1}} A \rtimes_\phi M \xrightarrow{\mathrm{Inn}} \mathrm{Aut}(A).$$

Write $s_1 : M \to G = A \rtimes_\phi M$ for the section given by the rule

$$s_1(m) = (x,1)(1,m)(x,1)^{-1}$$

It now follows from Proposition 2.7 that the product mapping

$$((a,m) \mapsto a \cdot s_1(m)) : A \times M \to G$$

determines an isomorphism $A \rtimes_{\phi_1} M \xrightarrow{\sim} G = A \rtimes_\phi M$ of extensions, as required. □

We now prove Theorem 1.7 from Section 1:

Proof. By assumption (a), G_ℓ has a Levi factor M_ℓ; this choice determines a homomorphism

$$\phi : M_\ell \to \mathrm{Aut}(U_\ell)$$

such that $\phi_{0,\ell} = \Psi \circ \phi$ where $\phi_0 : M \to \mathrm{Out}(U)$ is the mapping determined by Proposition 2.8 and $\Psi : \mathrm{Aut}(U) \to \mathrm{Out}(U)$ is the natural mapping of (2.1).

There is a natural action of the Galois group Γ on $\mathrm{Aut}(U_\ell)$ and on $\mathrm{Out}(U_\ell)$ for which Ψ is equivariant. For any $\gamma \in \Gamma$ it follows that

$$\Psi \circ {}^\gamma\phi = \phi_0$$

i.e. in the notation of Proposition 6.1, ${}^\gamma\phi$ determines a class in $\mathrm{Lift}(\phi_{0,\ell})$.

According to Proposition 6.1 there is a bijection $H^1_{\mathrm{coc}}(M_\ell, \mathrm{Inn}(U_\ell)) \xrightarrow{\sim} \mathrm{Lift}(\phi_0)$. Since $H^1_{\mathrm{coc}}(M_\ell, \mathrm{Inn}(U_\ell)) = 1$ it follows that classes of the automorphisms ${}^\gamma\phi$ in $\mathrm{Lift}(\phi_0)$ all coincide; i.e. all ${}^\gamma\phi$ are equivalent.

By the definition of the equivalence relation defining $\mathrm{Lift}(\phi_0)$, we find for each $\gamma \in \Gamma$ an element $h_\gamma \in \mathrm{Inn}(U)(\ell)$ such that

$$^\gamma\phi = h_\gamma^{-1} \cdot \phi \cdot h_\gamma.$$

If $\gamma, \tau \in \Gamma$ we see that

$$^{\gamma\tau}\phi = h_{\gamma\tau}^{-1} \cdot \phi \cdot h_{\gamma\tau}, \tag{7.1}$$

while on the other hand

$$\begin{aligned} ^\gamma({}^\tau\phi) &= {}^\gamma(h_\tau^{-1} \cdot \phi \cdot h_\tau) \\ &= {}^\gamma h_\tau^{-1} \cdot {}^\gamma\phi \cdot {}^\gamma h_\tau \\ &= {}^\gamma h_\tau^{-1} \cdot h_\gamma^{-1}\phi \cdot h_\gamma \cdot {}^\gamma h_\tau. \end{aligned} \tag{7.2}$$

By assumption (b) we know that the stabilizer in $\mathrm{Inn}(U)$ of the automorphism ϕ is trivial. Thus taken together (7.1) and (7.2) imply that

$$h_{\gamma\tau} = h_\gamma {}^\gamma h_\tau;$$

i.e. h_γ is a 1-cocycle on Γ with values in $\mathrm{Inn}(U)(\ell)$. Since U is connected and split unipotent, so is $\mathrm{Inn}(U)$; see Proposition 2.3. Thus $H^1_{\mathrm{coc}}(M_\ell, \mathrm{Inn}(U_\ell)) = 1$ by Proposition 2.14.

It follows that the cocycle h_γ is trivial. Thus there is $h \in \mathrm{Inn}(U)(\ell)$ such that for each $\gamma \in \Gamma$ we have

$$h_\gamma = h^{-1} \cdot {}^\gamma h$$

We now claim that the mapping $\phi_1 : M_\ell \to \mathrm{Aut}(U_\ell)$ defined by

$$\phi_1 = h \cdot \phi \cdot h^{-1}$$

is Γ-stable. For $\gamma \in \Gamma$ we have

$$^\gamma\phi_1 = {}^\gamma(h \cdot \phi \cdot h^{-1}) = {}^\gamma h \cdot {}^\gamma\phi \cdot {}^\gamma h^{-1} = hh_\gamma \cdot h_\gamma^{-1}\phi h_\gamma \cdot h_\gamma^{-1}h^{-1} = \phi_1.$$

Thus ϕ_1 is Γ-stable and hence defines a morphism $\phi_1 : M \to \mathrm{Aut}(U)$ of k-group functors which we may use to define a semidirect product $G_1 = U \rtimes_{\phi_1} M$ over k.

Now, the center Z of U is a connected and split unipotent group; thus $H^1(\ell, Z) = 1$. It follows that the mapping $U(\ell) \to \mathrm{Inn}(U)(\ell)$ is surjective, so we may choose an element $u \in U(\ell)$ for which $\mathrm{Inn}(u) = h \in \mathrm{Inn}(U)(\ell)$.

Thus we have

$$\phi_1 = \mathrm{Inn}(u) \cdot \phi \cdot \mathrm{Inn}(u)^{-1}.$$

It now follows from Lemma 7.1 that there is an isomorphism of extensions

$$G_\ell = U_\ell \rtimes_\phi M_\ell \simeq G_{1,\ell} = U_\ell \rtimes_{\phi_1} M_\ell$$

of M_ℓ by U_ℓ.

According to Theorem 5.4, assumption (c) implies that the extension G_ℓ has a unique k-form. Since G and G_1 are both k-forms of this extension, it follows that $G \simeq G_1$ are k-isomorphic extensions and in particular are k-isomorphic algebraic groups; since G_1 has a Levi factor over k, we conclude that G has a Levi factor over k as well. □

8. An example

In (McNinch 2013) §5 we gave an example of an extension

$$1 \to W \to E \to \mathbf{Z}/p\mathbf{Z} \to 1$$

with E commutative and W a connected, commutative unipotent group of exponent p^2. The group E was constructed by *twisting*, and it provided a negative answer to the question ($\diamondsuit$) from Section 1. Namely, for a suitable finite galios extension ℓ of k the group E_ℓ has a Levi factor, but E had no Levi factor.

We conclude the present paper with another example of a linear algebraic group over k which provides a negative answer to the question ($\diamondsuit$).

The example below gives a non-commutative extension of a finite abelian p-group by a connected, non-commutative unipotent group; in this case, the construction of the extension is perhaps slightly more straightforward.

Suppose that the characteristic of k is $p > 2$. Consider the additive polynomial $X^p - X \in k[X]$ defining the *Artin-Schreier* mapping $\mathscr{P}$: for any commutative k-algebra Λ, this mapping $\mathscr{P} : \Lambda \to \Lambda$ is given by the rule $x \mapsto x^p - x$.

Recall that if $s \in k$ is not in the image of $\mathscr{P} : k \to k$ then the polynomial $F(X) = X^p - X - s \in k[X]$ is irreducible. If α is a root of $F(X)$ in an extension field of k then $\ell = k(\alpha)$ is a Galois extension of k with $\mathrm{Gal}(\ell/k) \simeq \mathbf{Z}/p\mathbf{Z}$.

Let V be a vector space of dimension 2 over k with a basis e, f, and write $\beta : V \times V \to k$ for the unique non-degenerate symplectic form satisfying $\beta(e, f) = 1 = -\beta(f, e)$. Viewing $\mathscr{P} \circ \beta$ as a *factor system*, we define a unipotent group H as an extension of V by $\mathbf{G}_a$; see (Serre 1988) VII.1.4. Explicitly, for a commutative k-algebra Λ we have

$$H(\Lambda) = \Lambda \times V \otimes_k \Lambda$$

with operation

$$(t, v) \cdot (s, w) = (t + s + \mathscr{P}(\beta(v, w)), v + w) = (t + s + \beta(v, w)^p - \beta(v, w), v + w)$$

for $v, w \in V \otimes \Lambda$ and $s, t \in \Lambda$.

Thus H is the non-abelian central extension

$$0 \to \mathbf{G}_a \xrightarrow[t \mapsto (0,t)]{i} H \xrightarrow[(v,t) \mapsto v]{} V \to 0. \tag{8.1}$$

Write Z for the center of H; then $Z \simeq \mathbf{G}_a$ is the image of the mapping i of (8.1).

Fix $t \in k$ and let $V_{0,t} = \langle te, f \rangle \subset V$, so that $V_{0,t} \simeq (\mathbf{Z}/pZ)^2$. Let μ_t be the central extension of $V_{0,t}$ by $Z \simeq \mathbf{G}_a$ defined by β (*not* by $\mathscr{P} \circ \beta$)). Thus there is an exact sequence

$$0 \to \mathbf{G}_a \to \mu_t \to V_{0,t} = (\mathbf{Z}/p\mathbf{Z})^2 \to 0$$

and the group operation is given by

$$(a, v) \cdot (b, w) = (a + b + \beta(v, w), v + w)$$

for $v, w \in V_{0,t} \otimes \Lambda = V_{0,t}$ and $a, b \in \Lambda$.

Write E for the fiber product $E = H \times_{\mathbf{G}_a} \mu_t$; thus E is an extension of $V_{0,t} \simeq (\mathbf{Z}/p\mathbf{Z})^2$ by H. By the definition of the fiber product, there is a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{G}_a \arrow[r] \arrow[d] & \mu_t \arrow[r] \arrow[d] & (\mathbf{Z}/p\mathbf{Z})^2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H \arrow[r] & H \times_{\mathbf{G}_a} \mu_t \arrow[r, "\pi"] & (\mathbf{Z}/p\mathbf{Z})^2 \arrow[r] & 0
\end{tikzcd}$$

Proposition 8.1. *If $X^p - X + t$ has no root in k, then the group $E = H \times_{\mathbf{G}_a} \mu_t$ has no Levi factor over k. If α is a root $X^p - X + t$ and $\ell = k(\alpha)$ then E_ℓ has a Levi factor.*

Sketch. We may represent elements of $E(k)$ as tuples (a, v, w) where $v \in V_{0,t}$, $w \in V$ and $a \in k$. We have

$$(a, v, w) \cdot (a', v', w') = (a + a' + \beta(v, v') + \mathscr{P}\beta(w, w)', v + v', w + w')$$

Now, any elements $\widetilde{e}, \widetilde{f}$ of $E(k)$ mapping to $te, f \in V_{0,t}$ via π must have the form $\widetilde{e} = (a, te, v)$ for some $v \in V$ and $a \in k$ and $\widetilde{f} = (b, f, w)$ for some $w \in V$ and $b \in k$.

We see that

$$\widetilde{e} \cdot \widetilde{f} = (a, te, v) \cdot (b, f, w) = (a + b + t + \mathscr{P}\beta(v, w), te + f, v + w)$$

while

$$\widetilde{f} \cdot \widetilde{e} = (b, f, w) \cdot (a, te, v) = (a + b + -t - \mathscr{P}\beta(v, w), te + f, v + w)$$

Since the characteristic of k is not 2, $\widetilde{e} \cdot \widetilde{f} = \widetilde{f} \cdot \widetilde{e}$ if and only if

$$0 = \mathscr{P}\beta(v, w) + t = \beta(v, w)^p - \beta(v, w) + t.$$

If $X^p - X + t$ has no root in k, it follows that the group $\langle \widetilde{e}, \widetilde{f} \rangle$ is non-abelian for any choice of $\widetilde{e}, \widetilde{f}$. This shows that E has no Levi factor.

On the other hand, E_ℓ always has a Levi factor since we may take $\widetilde{e} = (0, te, \alpha e)$ and $\widetilde{f} = (0, f, f)$; then $\langle \widetilde{e}, \widetilde{f} \rangle \simeq (\mathbf{Z}/p\mathbf{Z})^2$ so that $\langle \widetilde{e}, \widetilde{f} \rangle$ provides a Levi factor. □

Remark 8.2. The group E of Proposition 8.1 fails to satisfy hypotheses (b) and (c) of Theorem 1.6. Indeed, let $M = E/H \simeq (\mathbf{Z}/p\mathbf{Z})^2$ be the reductive quotient of E. Then:

- M_ℓ acts trivially on H_ℓ. Thus, $H_\ell^{M_\ell} = H_\ell \neq \{1\}$, so that condition (b) fails to hold.
- The cohomology group $H^1_{\mathrm{coc}}(\mathbf{Z}/p\mathbf{Z}, \mathbf{G}_a)$ is non-trivial. Using a Künneth formula, we see that $H^1_{\mathrm{coc}}(M, \mathbf{G}_a) \neq 1$. Now use Proposition 3.5 to conclude that $H^1_{\mathrm{coc}}(M, H) \neq 1$ and $H^1_{\mathrm{coc}}(M_\ell, H_\ell) \neq 1$. Thus condition (c) fails to hold.

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