ON THE NILPOTENT ORBITS OF A REDUCTIVE GROUP OVER A LOCAL FIELD

GEORGE MCNINCH

Abstract. Let $\mathcal{K}$ be the field of fractions of a complete discrete valuation ring $\mathcal{A}$ with perfect residue field k of characteristic p, and let G be a connected and reductive algebraic group over $\mathcal{K}$ which splits over an unramified extension of $\mathcal{K}$.

Suppose that $\mathcal{P}$ is a parahoric group scheme over $\mathcal{A}$ with generic fiber $\mathcal{P}_{\mathcal{K}} = G$. A nilpotent section $\mathcal{X} \in \mathrm{Lie}(\mathcal{P})$ is *balanced* if the fibers $C_{\mathcal{K}}$ and C_k are smooth group schemes of the same dimension, where $C = C_{\mathcal{P}}(\mathcal{X})$ is the scheme theoretic centralizer of $\mathcal{X}$ for the adjoint action of $\mathcal{P}$. The identity component of the centralizer $C_{\mathcal{P}}(\mathcal{X})$ of a balanced nilpotent section is *smooth* over $\mathcal{A}$. If X_0 is a nilpotent element in the Lie algebra of the reductive quotient of the special fiber $\mathcal{P}_k$, we give conditions for the existence and conjugacy of balanced nilpotent sections $\mathcal{X}$ of $\mathrm{Lie}(\mathcal{P})$ with $X_0 = \mathcal{X}_k$.

The construction of balanced sections given here provides useful qualitative information about the parametrization of $G(\mathcal{K})$-orbits on the nilpotent elements of $\mathrm{Lie}(G)$ described in [DeB02].

Contents

1. Introduction

Throughout this paper, the symbol $\mathcal{A}$ will denote a complete discrete valuation ring with maximal ideal $\mathfrak{m}$, with field of fractions $\mathcal{K}$ (a "local field"), and with residue field $k = \mathcal{A}/\mathfrak{m}$.

1.1. **Rational nilpotent orbits and an overview.** Let G be connected and reductive over the local field $\mathcal{K}$, and let $\mathcal{P}$ be a parahoric group scheme with generic fiber $\mathcal{P}_{\mathcal{K}} = G$. Recall that $\mathcal{P}$ is in particular a smooth and affine $\mathcal{A}$-group scheme of finite type over $\mathcal{A}$; see section 2.5. In general, $\mathcal{P}$ is not a reductive group scheme over $\mathcal{A}$ since its special fiber $\mathcal{P}_k$ need not be reductive.

The paper seeks to provide information about the $G(\mathcal{K})$-orbits on nilpotent elements of $\mathrm{Lie}(G) = \mathrm{Lie}(G)(\mathcal{K})$. Following the lead of Debacker [DeB02], we relate these orbits to the nilpotent orbits for the reductive quotients of the special fibers $\mathcal{P}_k$ for the various parahoric group schemes attached to G.

This relationship is achieved by using certain *nilpotent sections* of $\mathrm{Lie}(\mathcal{P})$– i.e. elements of $\mathrm{Lie}(\mathcal{P})(\mathcal{A})$. We say that a section $\mathcal{X} \in \mathrm{Lie}(\mathcal{P})$ is *nilpotent* provided that its image $\mathcal{X}_{\mathcal{K}}$ in $\mathrm{Lie}(\mathcal{P}_{\mathcal{K}})$ is nilpotent. We observe – see Lemma 2.1.1 – that if $\mathcal{X} \in \mathrm{Lie}(\mathcal{P})$ is nilpotent, then also $\mathcal{X}_k \in \mathrm{Lie}(\mathcal{P}_k)$ is nilpotent.

Date: October 17, 2018.

Write $C = C_{\mathcal{P}}(\mathcal{X})$ for the scheme theoretic centralizer of the section $\mathcal{X}$. Then the generic fiber $C_{\mathcal{K}}$ of C is the centralizer $C_{\mathcal{P}_{\mathcal{K}}}(\mathcal{X}_{\mathcal{K}})$ – a $\mathcal{K}$-group scheme, and the special fiber C_k is the centralizer $C_{\mathcal{P}_k}(\mathcal{X}_k)$ – a k-group scheme.

Definition 1.1.1. The section $\mathcal{X}$ is *balanced* for the adjoint action of $\mathcal{P}$ if C_k is a smooth group scheme over k, if $C_{\mathcal{K}}$ is a smooth group scheme over $\mathcal{K}$, and if $\dim C_k = \dim C_{\mathcal{K}}$.

Let $\mathcal{X}$ be a nilpotent section and suppose that $C_{\mathcal{K}}$ and C_k are known to be smooth; if $\mathcal{P} = \mathcal{G}$ is reductive and if the fibers $\mathcal{G}_{\mathcal{K}}$ and $\mathcal{G}_k$ are *standard*, these centralizers are indeed known to be smooth. Under these assumptions, we relate in section 2.2 the condition that $\mathcal{X}$ is balanced to the condition that the identity component C^0 of the group scheme C is smooth over $\mathcal{A}$.

One of the main goals of the present paper is to give existence and conjugacy results for balanced nilpotent sections of $\mathrm{Lie}(\mathcal{P})$.

1.2. **Distinguished nilpotent elements.** Before formulating our main results, we are going to pause to remind the reader of (portions of) the Bala-Carter Theorem [Car93] [Pre03] [Jan04]; recall that this result gives a precise description of the *geometric* nilpotent orbits of a standard reductive group.

Thus for the moment we let $\mathcal{F}$ be an algebraically closed field, and we write H for a standard reductive algebraic group over $\mathcal{F}$ [1]. A nilpotent element X in $\mathrm{Lie}(H)$ is said to be *distinguished* if a maximal torus of the centralizer in H of X is central in H.

Consider pairs $(L, \mathcal{O}_L$ where L is a Levi factor of a parabolic subgroup of H and $\mathcal{O}_L \subset \mathrm{Lie}(L)$ is a distinguished nilpotent L-orbit. To such a pair, one associates the H-orbit $\mathrm{Ad}(H)\mathcal{O}$.

Now, a weak form of the Bala-Carter Theorem – see [Jan04, §4] – says that this association determines a bijection between the H-conjugacy classes of such pairs $(L, \mathcal{O}_L)$ and the nilpotent H-orbits.

Now let us return to the reductive group G over the local field $\mathcal{K}$. Recall that a $\mathcal{K}$-torus T is said to be *unramified* provided that $T_{\mathcal{L}}$ is split for some unramified extension $\mathcal{L} \supset \mathcal{K}$. Equivalently, the inertia subgroup $I \subset \Gamma_{\mathcal{K}} = \mathrm{Gal}(\mathcal{K})$ is contained in the kernel of the action of $\Gamma_{\mathcal{K}}$ on the group $X^*(T_{\mathcal{K}_{\mathrm{sep}}})$ of chararacters of $T_{\mathcal{K}_{\mathrm{sep}}}$.

Let $X \in \mathrm{Lie}(G) = \mathrm{Lie}(G)(\mathcal{K})$ be a ($\mathcal{K}$-rational) nilpotent element, and let $C = C_G(X)$ be the centralizer of X. We say that X is *nr-distinguished* if any unramified torus in $C_G(X)$ is central in G.

1.3. **The main results for reductive group schemes.** We first state results which hold under the assumption that there is a *reductive* group scheme $\mathcal{G}$ with generic fiber $\mathcal{G}_{\mathcal{K}} = G$. For example, this condition holds when G is $\mathcal{K}$-split.

These main results are proved under an assumption on $\mathcal{G}$ specified in conditions (SG1) and (SG2) of section 4.1. Briefly, these conditions stipulate that the fibers $\mathcal{G}_k$ and $\mathcal{G}_{\mathcal{K}}$ are *geometrically standard reductive groups*; see section 3.1 for the notion of (geometrically) standard reductive group over a field.

Suppose for the remainder of section 1.3 that conditions (SG1) and (SG2) of section 4.1 hold for $\mathcal{G}$.

We now give the statements of the main results of this paper; see the subsequent section section 1.5 for a slightly more detailed overview of the arguments of the paper.

Theorem 1.3.1. *Let $X_0 \in \mathrm{Lie}(\mathcal{G}_k)$ be a nilpotent element.*

(a) There is a balanced, nilpotent section $\mathcal{X} \in \mathrm{Lie}(\mathcal{G})$ such that the image $\mathcal{X}_k$ in $\mathrm{Lie}(\mathcal{G}_k)$ coincides with X_0.
(b) There is an $\mathcal{A}$-homomorphism $\phi : \mathbf{G}_m \to \mathcal{G}$ such that $\mathcal{X} \in \mathrm{Lie}(\mathcal{G})(\phi; 2)$, ϕ_k is a cocharacter associated with $\mathcal{X}_k$ and $\phi_{\mathcal{K}}$ is a cocharacter associated with $\mathcal{X}_{\mathcal{K}}$.
(c) Let $\mathcal{X}, \mathcal{X}' \in \mathrm{Lie}(\mathcal{G})$ be balanced nilpotent sections with $\mathcal{X}_k = \mathcal{X}'_k = X_0$. Then there is an element $g \in \mathcal{G}(\mathcal{A})$ such that $\mathcal{X}' = \mathrm{Ad}(g)\mathcal{X}$.
(d) If X_0 is distinguished *in $\mathrm{Lie}(\mathcal{G}_k)$ if and only if $\mathcal{X}_{\mathcal{K}}$ is* distinguished *in $\mathrm{Lie}(G)$.*

Assertions (a) and (b) of Theorem 1.3.1 follow from Theorem 4.5.2, and assertion (c) follows from Corollary 7.3.2. See section 3.2 for the notion of a cocharacter associated with a nilpotent element.

[1] the condition "standard" amounts to the requirement that the characterisitic of $\mathcal{F}$ is not "too small" with respect to the root datum of H; see section 3.1 for the precise definition

1.4. **The main result for parahoric group schemes.** Now let $\mathcal{P}$ be any parahoric $\mathcal{A}$-group scheme with generic fiber $\mathcal{P}_{\mathcal{K}} = G$, and let $X_0 \in \mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k)$ be a nilpotent element.

Since $G = \mathcal{G}_{\mathcal{K}}$ is assumed to split over an unramified extension of $\mathcal{K}$, we have:

Theorem 1.4.1 ([McN18]). *There is a reductive subgroup scheme $\mathcal{M} \subset \mathcal{P}$ for which $\mathcal{M}_k$ is a Levi factor of $\mathcal{P}_k$, and $\mathcal{M}_{\mathcal{K}}$ is - in particular - a reductive subgroup containing a maximal torus T of G which is maximally split.*

We observe in Proposition 6.2.2 that $\mathcal{M}$ also satisfies conditions (SG1) and (SG2) of section 4.1.

Since $\mathcal{M}_k$ is a Levi factor of $\mathcal{P}_k$, we may and will identify X_0 with an element of $\mathrm{Lie}(\mathcal{M}_k)$. Now apply Theorem 1.3.1 to find a nilpotent section $\mathcal{X} \in \mathrm{Lie}(\mathcal{M})$ balanced for the action of $\mathcal{M}$ for which $\mathcal{X}_k = X_0$, together with an $\mathcal{A}$-homomorphism $\phi : \mathbf{G}_m \to \mathcal{M}$ with the properties stated in (b) of that Theorem.

Let h be the maximum of the Coxeter numbers of the irreducible components of the Dynkin diagram of the reductive group $G_{\overline{\mathcal{K}}}$ where $\overline{\mathcal{K}}$ is an algebraic closure of $\mathcal{K}$, and let p denote the characteristic of the residue field k. Of course, the characteristic of $\mathcal{K}$ is then either 0 or p.

Theorem 1.4.2. *Suppose that $p > 2h - 2$.*

(a) *$\mathcal{X} \in \mathrm{Lie}(\mathcal{P})$ is balanced for the action of $\mathcal{P}$.*
(b) *If $\mathcal{X}' \in \mathrm{Lie}(\mathcal{P})$ is any balanced nilpotent section for which $\mathcal{X} + \mathrm{Lie}(R_u\mathcal{P}_k) = \mathcal{X}' + \mathrm{Lie}(R_u\mathcal{P}_k)$ in $\mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k)$, then there is an element $g \in \mathcal{P}(\mathcal{A})$ such that $\mathcal{X}' = \mathrm{Ad}(g)\mathcal{X}$.*
(c) *If X_0 is* distinguished *in $\mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k$ then $\mathcal{X}_{\mathcal{K}}$ is* u-distinguished *in $\mathrm{Lie}(\mathcal{M}_{\mathcal{K}}$.*

Assetion (a) of Theorem 1.4.2 follows from Theorem 6.4.3 together with Proposition 3.4.3, and assertion (b) follows from Corollary 7.4.2.

Finally, one wants to "realize" nilpotent elements $X_1 \in \mathrm{Lie}(G) = \mathrm{Lie}(\mathcal{G}_{\mathcal{K}})$ as the value $\mathcal{X}_{\mathcal{K}}$ at the generic point of a balanced nilpotent section $\mathcal{X} \in \mathrm{Lie}(\mathcal{P})$ for a suitable parahoric group scheme $\mathcal{P}$. Here our proof basically follows the argument of Prasad that is given in the proof of [DeB02, Lemma 4.5.3], but we obtain additional precision on the required hypotheses on G.

Theorem 1.4.3. *Suppose that $p > 2h - 2$ and let $X_1 \in \mathrm{Lie}(G)$ be nilpotent. Then there is a parahoric group scheme $\mathcal{P}$ with generic fiber $\mathcal{P}_{\mathcal{K}} = G$ and a nilpotent section $\mathcal{X} \in \mathrm{Lie}(\mathcal{P})$ balanced for the action of $\mathcal{P}$ with $\mathcal{X}_{\mathcal{K}} = X_1$.*

This result is proved in Theorem 8.3.1

1.5. **Overview.** We now give a somewhat more detailed overview of the content of the paper and in particular of the arguments used in obtaining the results of the preceding section.

After some recollections on group schemes in section 2, we recall in section 3 some important aspects of the theory of nilpotent orbits for what we call a *geometrically standard* reductive group.

For a reductive group scheme $\mathcal{G}$ over $\mathcal{A}$ and for a nilpotent element $X_0 \in \mathrm{Lie}(\mathcal{G}_k)$, the proof of the *existence* of a balanced nilpotent section $\mathcal{X}$ lifting X_0 is carried out in section 4; more precisely, Theorem 1.3.1(a) is a consequence of Theorem 4.5.2. The proof Theorem 4.5.2 – and the arguments given in the first 4 sections of section 4 – constitute an improved version of results used in the proof of [McN08, Theorem A]. In the present language, results proved in [McN08] showed that there is a balanced nilpotent section $X \in \mathrm{Lie}(\mathcal{G})$ whose image in $\mathrm{Lie}(\mathcal{G}_k)$ is $\mathcal{G}(\overline{k})$-conjugate to X_0. See section 9 for further discussion of the results of [McN08].

In section 5 we consider a balanced nilpotent section $\mathcal{X} \in \mathrm{Lie}(\mathcal{G})$ together with an $\mathcal{A}$ homomorphism $\phi : \mathbf{G}_{m,\mathcal{A}} \to \mathcal{G}$ for which ϕ_k is a cocharacter associated to $\mathcal{X}_k$ and $\phi_{\mathcal{K}}$ is a cocharacter associated to $\mathcal{X}_{\mathcal{K}}$. We argue that if $(\mathcal{X}_k)^{[p]} = 0$, there is an $\mathcal{A}$-homomorphism $\Phi : \mathrm{SL}_{2,\mathcal{A}} \to \mathcal{G}$ for which $d\Phi$ maps the upper triangular nilpotent section $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of $\mathrm{Lie}(\mathrm{SL}_{2,\mathcal{A}})$ to X and for which the restriction of Φ to the diagonal maximal torus in $\mathrm{SL}_{2,\mathcal{A}}$ coincides with ϕ.

Now let $\mathcal{P}$ be any parahoric group scheme over $\mathcal{A}$ with generic fiber $G = \mathcal{P}_{\mathcal{K}}$. Give $X_0 \in \mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k)$ we use a Levi decomposition of the special fiber $\mathcal{P}_k$ to identify X_0 with an element of $\mathrm{Lie}(\mathcal{P}_k)$. The existence of a Levi decomposition for the special fiber $\mathcal{P}_k$ was established by the author in [McN10]; see also [McN14] and [McN13]. But the most useful result was obtained more recently. Namely, we showed in [McN18] that there is a reductive subgroup scheme $\mathcal{M} \subset \mathcal{P}$ such that $\mathcal{M}_k$ is a Levi factor of $\mathcal{P}_k$. We argue in Proposition 6.2.2 that $\mathcal{M}$ satisifies the conditions enumerated in section 4.1.

Given $X_0 \in \mathrm{Lie}(\mathcal{M}_k) \subset \mathrm{Lie}(\mathcal{P}_k)$, we use the existence result for balanced nilpotent sections for reductive group schemes – Theorem 4.5.2 – to find a section $\mathcal{X} \in \mathrm{Lie}(\mathcal{M})$ with $\mathcal{X}_k = X_0$ which is balanced for the action of $\mathcal{M}$ together with an $\mathcal{A}$-homomorphism $\phi : \mathbf{G}_m \to \mathcal{M}$ such that ϕ_k is a cocharacter associated to $\mathcal{X}_k$ and $\phi_{\mathcal{K}}$ is a cocharacter associated to $\mathcal{X}_{\mathcal{K}}$.

We propose to show that $\mathcal{X}$ is balanced for the action of $\mathcal{P}$. In fact, we don't know how to argue that without some further assumptions. If we impose the condition that $p > 2h-2$, then we find as in Section 5 an $\mathcal{A}$-homomorphism $\mathrm{SL}_{2,\mathcal{A}} \to \mathcal{M}$. Moreover, the condition on p implies that weights of a split maximal torus of $\mathrm{SL}_{2,\mathcal{A}}$ on $\mathrm{Lie}(\mathcal{P})$ are all $< p$. Under this condition, the fibers of the $\mathrm{SL}_{2,\mathcal{A}}$-module $\mathrm{Lie}(\mathcal{P})$ are restricted semisimple modules – see Proposition 5.3.2; using this condition, we are then able to argue that $X \in \mathrm{Lie}(\mathcal{P})$ is indeed balanced for the action of $\mathcal{P}$; see Theorem 6.4.3.

Having settled the *existence* of balanced nilpotent sections, we take up the question of *conjugacy* in section 7. Here, we adapt the argument of [DeB02, §5.2]. We give in Section 7.2 a condition for conjugacy of balanced nilpotent sections which depends on the Moy-Prasad filtration of $\mathcal{P}(\mathcal{A})$ and $\mathrm{Lie}(\mathcal{P})$ – see Section 7.1 – as well as the "mock exponential mapping" of [Adl98]. When $\mathcal{P} = \mathcal{G}$ is reductive, we confirm that this condition holds when $\mathcal{G}$ satisfies (SG1) and (SG2) of Section 4.1; see Theorem 7.3.1. For general $\mathcal{P}$, we impose the condition that $p > 2h-2$. As before, this permits us to exploit the existence of an $\mathcal{A}$-homomorphism $\Phi : \mathrm{SL}_{2,\mathcal{A}} \to \mathcal{P}$ for which the fibers of $\mathrm{Lie}(\mathcal{P})$ are restricted semisimple modules for $\mathrm{SL}_{2,k}$ and $\mathrm{SL}_{2,\mathcal{K}}$. In turn, the $\mathcal{A}$-homomorphism Φ permits us to deduce the required condition for conjugacy in this case; see Theorem 7.4.1.

In section 8 we explain the relevance of balanced nilpotent sections to the parametrization given in [DeB02] of orbits of $G(\mathcal{K})$ on the Lie algebra $\mathrm{Lie}(G)$. Here we prove Proposition 8.2.3 relating balanced nilpotent sections with the "lifting" described by DeBacker. We also prove Theorem 8.3.1 showing that every rational nilpotent $X_1 \in \mathrm{Lie}(G)$ can be "realized" as the value at the generic fiber of some balanced nilpotent section for some parahoric group scheme $\mathcal{P}$ with $\mathcal{P}_{\mathcal{K}} = G$.

In section 9 we consider the scheme theoretic centralizer of a balanced nilpotent section. Here we give a corrected and streamlined proof of [McN08, Theorem A] that the root datum of the reductive quotient of the centralizer of a nilpotent element Y depends only on the Bala-Carter label of Y, and is independent of "very good" characteristic; see Theorem 9.1.1. As a consquence, we observe in section 9.2 that there are in general nilpotent elements $X_1 \in \mathrm{Lie}(G)$ which are not the value $\mathcal{X}_{\mathcal{K}}$ of any balanced nilpotent section $\mathcal{X} \in \mathrm{Lie}(\mathcal{G})$; thus one really must use general parahorics to "detect" all rational nilpotent classes.

Finally, Appendix A contains a generalization of a result found in [MS03]. This result concerns the cocharacters of a reductive group G associated to a nilpotent element $X \in \mathrm{Lie}(G)$; see section 3.2 for the notion of an associated cocharacter. If $M \subset G$ is a reductive subgroup of type $C(\mu)$ – see section 6.1 – and if $X \in \mathrm{Lie}(M)$ we show that a cocharacter of M associated to X is also associated to X in G. The corresponding result was obtained for a "pseudo-Levi subgroup" $M \subset G$ in [MS03].

1.6. **Notation and terminology.**

1.6.1. Throughout the paper, we write G for a connected and reductive algebraic group over a field $\mathcal{F}$. For the most part, $\mathcal{F} = \mathcal{K}$. We always write h for the maximum value of the Coxeter number of an irreducible component of the root system of $G_{\overline{\mathcal{F}}}$.

1.6.2. Recall [Jan03, (I.2.7)] that if Λ is a commutative ring and if $\mathcal{H}$ is an affine Λ-group scheme, then a module - or representation - for $\mathcal{H}$ is a Λ-module M together with an action of $\mathcal{H}$ on the Λ-functor determined by M. In particular, for any commutative Λ-algebra A, the group $\mathcal{H}(A)$ acts linearly on $M(A) = M \otimes_\Lambda A$. Alternatively, one may view M as a *comodule* for the Hopf algebra $\Lambda[\mathcal{H}]$.

1.6.3. We now formulate a somewhat general notion of "balanced section". Suppose that $\mathcal{H}$ is a group scheme which is smooth, affine and of finite type over $\mathcal{A}$. Consider an $\mathcal{H}$-module $\mathcal{L}$ which is a finitely generated free $\mathcal{A}$-module.

Let $x \in \mathcal{L}$, write $x_{\mathcal{K}}$ for the image of x in $\mathcal{L}_{\mathcal{K}} = \mathcal{L} \otimes_{\mathcal{A}} K$, and write x_k for the image of $x \in \mathcal{L}_k = \mathcal{L} \otimes_{\mathcal{A}} k = \mathcal{L}/\mathfrak{m}\mathcal{L}$.

Definition 1.6.1. The element $x \in \mathcal{L}$ is said to be *balanced* for the action of $\mathcal{H}$ if the scheme-theoretic stabilizer $C = \mathrm{Stab}_{\mathcal{H}}(x)$ has the following properties: (i) $C_{\mathcal{K}}$ is a smooth group scheme over $\mathcal{K}$, (ii) C_k is a smooth group scheme over k, and (iii) $\dim C_{\mathcal{K}} = \dim C_k$.

1.6.4. If Λ is a commutative ring, and if D is a diagonalizable group scheme over Λ with character group X, recall that any D-module M can be written as a direct sum $M = \bigoplus_{\lambda \in X} M_\lambda$ where D acts on the *weight space* M_λ according to the character λ. If $\mathcal{H}$ is a Λ-group scheme, if $\phi : \mathbf{G}_m \to \mathcal{H}$ is a Λ-homomorphism, and if M is an $\mathcal{H}$-module, we identify the character group of $\mathbf{G}_m$ with $\mathbf{Z}$, and for $i \in \mathbf{Z}$ we often write $M(\phi; i)$ for the i-weight space for the action of $\mathbf{G}_m$ on M determined by ϕ.

2. Generalities on group schemes and balanced sections

2.1. **Nilpotent sections of certain group schemes.** Recall from the introduction that this paper is concerned in part with elements - or sections - of the Lie algebra of parahoric group schemes.

More generally, consider a connected and reductive algebraic group G over $\mathcal{K}$, and let $\mathcal{H}$ be a smooth, affine group scheme of finite type over $\mathcal{A}$ with an identification $\mathcal{H}_{\mathcal{K}} = G$.

Lemma 2.1.1. *Let $\mathcal{X} \in \mathrm{Lie}(\mathcal{H})$ be a section. If $\mathcal{X}_{\mathcal{K}} \in \mathrm{Lie}(\mathcal{H}_{\mathcal{K}})$ is nilpotent, then also $\mathcal{X}_k \in \mathrm{Lie}(\mathcal{H}_k)$ is nilpotent.*

Proof. Recall that if H is a linear algebraic group over the field $\mathcal{F}$, then an element $Y \in \mathrm{Lie}(H)$ is nilpotent if and only if $0 = Y^N \in \mathrm{End}_{\mathcal{F}}(V)$ for some sufficiently large $N > 0$ and some faithful linear representation V of H.

Since $\mathcal{H}$ is affine and smooth - in particular, flat - over $\mathcal{A}$, it follows from [BT84, (1.4.5)] that $\mathcal{H}$ has a faithful linear representation on a free $\mathcal{A}$-module M of finite rank. Then $M_k = M \otimes_{\mathcal{A}} k$ affords a faithful linear representation of $\mathcal{H}_k$, and $M_{\mathcal{K}} = M \otimes_{\mathcal{A}} \mathcal{K}$ affords a faithful linear representation of $\mathcal{H}_{\mathcal{K}}$.

Identifying $\mathcal{H}$ as a closed subgroup scheme of $\mathrm{GL}(M)$, the nilpotence of $\mathcal{X}_{\mathcal{K}}$ implies that $0 = \mathcal{X}_{\mathcal{K}}^N \in \mathrm{End}_{\mathcal{K}}(M_{\mathcal{K}})$ for some $N > 0$. But then $0 = \mathcal{X}^N \in \mathrm{End}_{\mathcal{A}}(M)$, and hence $0 = \mathcal{X}_k^N \in \mathrm{End}_k(M_k)$ so that indeed $\mathcal{X}_k$ is nilpotent as well. □

Remark 2.1.2. The Lemma applies when $\mathcal{H} = \mathcal{G}$ is reductive, or when $\mathcal{H} = \mathcal{P}$ is a parahoric group scheme with generic fiber G; see section 2.5 below for details on parahoric group schemes.

Definition 2.1.3. A section $\mathcal{X} \in \mathrm{Lie}(\mathcal{H})$ is said to be *nilpotent* just in case the generic fiber $\mathcal{X}_{\mathcal{K}} \in \mathrm{Lie}(\mathcal{H}_{\mathcal{K}})$ is nilpotent.

Thus the Lemma shows that the fiber $\mathcal{X}_k$ of a nilpotent section $\mathcal{X}$ is nilpotent.

2.2. **Stabilizers and smoothness.** Let $\mathcal{A}$ be an integral domain with field of fractions $\mathcal{K}$ and let $\mathcal{G}$ be a group scheme over $\mathcal{A}$ which is smooth at all points over its unit section. Recall [BT84, §1.2.12] that *identity component* $\mathcal{G}^0$ of $\mathcal{G}$ is the union of the identity components $\mathcal{G}_p^0$ of the fibers $\mathcal{G}_p$ for $p \in \mathrm{Spec}(\mathcal{A})$; it is an open subgroup scheme of $\mathcal{G}$ which is smooth over $\mathcal{A}$ and has connected fibers. In particular, the generic fiber $(\mathcal{G}^0)_{\mathcal{K}}$ is precisely the identity component of $\mathcal{G}_{\mathcal{K}}$.

Now suppose that $\mathcal{A}$ is a discrete valuation ring with residue field k and field of fractions $\mathcal{K}$.

Proposition 2.2.1. *Let X be a Noetherian $\mathcal{A}$—scheme. Suppose that*

(i) $X_{\mathcal{K}}$ *is an irreducible and reduced $\mathcal{K}$-variety, and X_k is an irreducible and reduced k-variety,*
(ii) $\dim X_{\mathcal{K}} = \dim X_k$, *and*
(iii) X *has a section $s \in X(\mathcal{A})$.*

Then X is flat over $\mathcal{A}$.

Proof. This follows by application of [Prop. 6.1 GY03]. □

Proposition 2.2.2. *Let $\mathcal{G}$ be a flat group scheme of finite type over $\mathcal{A}$ such that the generic fiber $\mathcal{G}_{\mathcal{K}}$ is affine. Then $\mathcal{G}$ is affine if and only if it is separated.*

Proof. This result – due to Raynaud – is proved in [PY06, Prop. 3.1]. □

Corollary 2.2.3. *Let $\mathcal{H}$ be a scheme which is affine over $\mathcal{A}$, such that the identity components $\mathcal{G}^0_{\mathcal{K}}$ and $\mathcal{G}^0_k$ are smooth. Then the identity component $\mathcal{G}^0$ is affine over $\mathcal{A}$.*

Proof. Indeed, first observe that $\mathcal{G}^0$ is smooth and in particular flat over $\mathcal{A}$. Since $\mathcal{G}$ is affine over $\mathcal{A}$, it is separated over $\mathcal{A}$. Since $\mathcal{G}^0$ is an open subgroup scheme of $\mathcal{G}$, $\mathcal{G}^0$ is itself separated over $\mathcal{A}$. Now $(\mathcal{G}^0)_{\mathcal{K}} = \mathcal{G}^0_{\mathcal{K}}$ is closed in $\mathcal{G}_{\mathcal{K}}$ and is thus affine. So the statement of the Corollary follows from the result of Raynaud Proposition 2.2.2. □

The statement and proof of the following result were communicated to me by Brian Conrad.

Theorem 2.2.4. *Let $\mathcal{G}$ be a group scheme which is separated and of finite type over $\mathcal{A}$ for which the fibers $\mathcal{G}_{\mathcal{K}}$ and $\mathcal{G}_k$ are each smooth of the same dimension. Then there is a locally closed subgroup scheme $\mathcal{M} \subset \mathcal{G}$ such that:*

(a) $\mathcal{M}$ is smooth and of finite type over $\mathcal{A}$,
(b) $\mathcal{M}_{\mathcal{K}} = (\mathcal{G}_{\mathcal{K}})^0$ and $\mathcal{M}_k = (\mathcal{G}_k)^0$.

If $\mathcal{G}$ is affine over $\mathcal{A}$, then also $\mathcal{M}$ is affine over $\mathcal{A}$.

Proof. Let $\mathcal{H}$ be the schematic closure of $(\mathcal{G}_{\mathcal{K}})^0$ in $\mathcal{G}$. Since $\mathcal{A}$ is a DVR, $\mathcal{H}$ is an $\mathcal{A}$-flat closed subgroup scheme of $\mathcal{G}$ with $\mathcal{H}_{\mathcal{K}} = (\mathcal{G}_{\mathcal{K}})^0$; see [BT84, §I.2.7]. Now, the $\mathcal{A}$-flatness of $\mathcal{H}$ guarantees that the fibers $\mathcal{H}_{\mathcal{K}}$ and $\mathcal{H}_k$ have the same dimension; see [EGAIV$_{III}$, (14.3.10)] or [Mat89, Thm 15.1, 15.5].

Hence, the closed k-subgroup $(\mathcal{H}_k)^0$ of $(\mathcal{G}_k)^0$ has the same dimension as $\mathcal{H}_{\mathcal{K}} = (\mathcal{G}_{\mathcal{K}})^0$, whose dimension is the same as that of $G_{\mathcal{K}}$, which in turn by hypothesis has the same dimension as $\mathcal{G}_k$.

Since $(\mathcal{H}_k)^0$ and $(\mathcal{G}_k)^0$ are connected k-group schemes of the same dimension, we deduce that the smooth $(\mathcal{G}_k)^0$ is the underlying reduced scheme of $(\mathcal{H}_k)^0$. But $\mathcal{H}_k \subset \mathcal{G}_k$ implies $(\mathcal{H}_k)^0 \subset (\mathcal{G}_k)^0$ as closed subscheme, so equality is forced; thus $(\mathcal{H}_k)^0 = (\mathcal{G}_k)^0$ is smooth. Since $\mathcal{H}_k$ is an algebraic group scheme over k, [Knu+98, Prop. 21.10 (4)] shows that in fact $\mathcal{H}_k$ is already smooth over k.

Since $\mathcal{H}$ is flat over $\mathcal{A}$, it follows from [SGA3$_I$, VI$_B$ Thm. 3.10] that the union $\mathcal{M} = \mathcal{H}^0_{\mathcal{K}} \cup \mathcal{H}^0_k$ is a smooth open $\mathcal{A}$-subgroup scheme of $\mathcal{H}$. In particular, $\mathcal{M}$ is locally closed in $\mathcal{G}$.

Finally, to see that $\mathcal{M}$ is affine over $\mathcal{A}$ when $\mathcal{G}$ is, apply Proposition 2.2.2. □

Theorem 2.2.5. *Let $x \in \mathcal{L}$ be an element which is balanced for the action of $\mathcal{G}$ and let $\mathrm{Stab}_{\mathcal{G}}(x)$ be the scheme theoretic stabilizer of x in $\mathcal{G}$ Then there is a locally closed subgroup scheme $\mathcal{H}$ which is smooth, affine, and of finite type over $\mathcal{A}$ for which $\mathcal{H}_{\mathcal{K}} = \mathrm{Stab}^0_{\mathcal{G}_{\mathcal{K}}}(x_{\mathcal{K}})$ and $\mathcal{H}_k = \mathrm{Stab}^0_{\mathcal{G}_k}(x_k)$.*

Proof. This follows by applying of Theorem 2.2.4 to the scheme-theoretic stabilizer $\mathrm{Stab}_{\mathcal{G}}(x)$. □

2.3. **Group schemes of multiplicative type.** Let $\mathcal{G}$ and $\mathcal{M}$ be groups schemes of finite type over $\mathcal{A}$. Suppose moreover that $\mathcal{G}$ is smooth and affine over $\mathcal{A}$, and that $\mathcal{M}$ is of multiplicative type over $\mathcal{A}$. Consider the functors

$$F : \mathbf{Sch}_{\mathcal{A}} \to \mathbf{Sets} \quad \text{and} \quad H = \underline{\mathrm{Hom}}_{\mathcal{A}\text{-gr}}(\mathcal{M}, \mathcal{G}) : \mathbf{Sch}_{\mathcal{A}} \to \mathbf{Sets}$$

where $\mathbf{Sch}_{\mathcal{A}}$ is the category of schemes over $\mathcal{A}$, **Sets** is the category of sets, and for an $\mathcal{A}$-scheme T, $F(T)$ is the set of subgroup schemes of $\mathcal{G}$ of multiplicative type over T and $H(T) = \mathrm{Hom}_{T\text{-gr}}(\mathcal{M}_T, \mathcal{G}_T)$ is the set of homomorphisms of group schemes $\mathcal{M}_T \to \mathcal{G}_T$ over T.

We record the following result:

Theorem 2.3.1. *(a) The functors F and $H = \underline{\mathrm{Hom}}_{\mathcal{A}\text{-gr}}(\mathcal{M}, \mathcal{G})$ are representable by $\mathcal{A}$-schemes which are smooth and separated over $\mathcal{A}$.*
(b) If $S_0 \subset \mathcal{G}_k$ is a k-torus, there is an $\mathcal{A}$-torus $S \subset \mathcal{G}$ for which $S_k = S_0$.

(c) If $\lambda_0 : \mathbf{G}_m \to \mathcal{G}_k$ *is a* k*-homomorphism, there is an* $\mathcal{A}$*-homomorphism* $\lambda : \mathbf{G}_m \to \mathcal{G}$ *for which* $\lambda_k = \lambda_0$*.*

Proof. The assertions in (a) are proved in [SGA3_{II}, Exp. XI Thm 4.1 and Cor. 4.2]. Now (b) and (c) are consequences of (a) together with the scheme-theoretic version of Hensel's Lemma – see [SGA3_{II}, Exp. XI Prop 1.10 and Cor. 1.11]. □

For a group scheme $\mathcal{G}$ of finite type over $\mathcal{A}$, a closed subgroup scheme $\mathscr{T} \subset \mathcal{G}$ is called a *maximal torus* if: (i) $\mathscr{T}$ is an $\mathcal{A}$-torus, (ii) $\mathscr{T}_k$ is a maximal torus of $\mathcal{G}_k$ and (iii) $\mathscr{T}_{\mathcal{K}}$ is a maximal torus of $\mathcal{G}_{\mathcal{K}}$.

Corollary 2.3.2. *Let* $\mathcal{G}$ *be reductive over* $\mathcal{A}$*. Then* $\mathcal{G}$ *possesses a maximal torus* $\mathscr{T}$*.*

Proof. Choose a maximal torus S_0 of $\mathcal{G}_k$ and use Theorem 2.3.1 to find a torus $\mathscr{S} \subset \mathcal{G}$ with $\mathscr{S}_k = S_0$. We now argue that $\mathscr{S}$ is a maximal torus – i.e. that $\mathscr{S}_{\mathcal{K}}$ is a maximal torus of $\mathcal{G}_{\mathcal{K}}$. Since $\mathcal{G}$ is reductive, it follows from [SGA3_{II}, Exp XII, Théorèm 1.7] that $\mathcal{G}$ has a maximal torus locally in the étale topology of $\text{Spec}(\mathcal{A})$. In particular, the dimension of a maximal torus of $\mathcal{G}_{\mathcal{K}}$ coincides with that of a maximal torus of $\mathcal{G}_k$. Since $\dim \mathscr{S}_{\mathcal{K}} = \dim \mathscr{S}_k$, it follows that $\mathscr{S}_{\mathcal{K}}$ is a maximal torus of $\mathcal{G}_{\mathcal{K}}$, as required. □

Proposition 2.3.3. *Suppose that* $\lambda_0 : \mathbf{G}_m \to \text{der}(\mathcal{G}_k)$ *is a* k*-homomorphism with values in the* derived group *of* $\mathcal{G}_k$*. Then there is an* $\mathcal{A}$*-homomorphism* $\lambda : \mathbf{G}_m \to \mathcal{G}$ *such that* $\lambda_k = \lambda_0$ *and such that* $\lambda_{\mathcal{K}}$ *takes values in the derived group* $\text{der}(\mathcal{G}_{\mathcal{K}})$*.*

Proof. According to [SGA3_{III}, Théorème 6.2.1], there is a closed subgroup scheme $\text{der}(\mathcal{G})$ which is semisimple over $\mathcal{A}$ for which $\text{der}(\mathcal{G})_k$ is the derived group of $\mathcal{G}_k$ and $\text{der}(\mathcal{G})_{\mathcal{K}}$ is the derived group of $\mathcal{G}_{\mathcal{K}}$. Now the result follows from Theorem 2.3.1. □

2.4. **Parabolic subgroup schemes of a reductive group scheme.** Let $\mathcal{G}$ be a reductive group scheme over $\mathcal{A}$ with connected fibers. We first recall - see [SGA3_{III}, Exp XXII Def 5.11.1] - that a subgroup scheme $\mathcal{H} \subset \mathcal{G}$ is said to be *of type (R)* provided that $\mathcal{H}$ is smooth over $\mathcal{A}$ with connected fibers, $\mathcal{H}_k$ contains a maximal torus of $\mathcal{G}_k$, and $\mathcal{H}_{\mathcal{K}}$ contains a maximal torus of $\mathcal{G}_{\mathcal{K}}$.

Now, according to [SGA3_{III}, Exp XXVI §1] a subgroup scheme $P \subset \mathcal{G}$ is a *parabolic subgroup scheme* if P is smooth over $\mathcal{A}$, if $P_k \subset \mathcal{G}_k$ is a parabolic subgroup, and if $P_{\mathcal{K}} \subset \mathbf{G}_{\mathcal{K}}$ is a parabolic subgroup. In particular, a parabolic subgroup scheme of $\mathcal{G}$ is of type (R).

If P is a parabolic subgroup scheme of $\mathcal{G}$, then according to [SGA3_{III}, Exp. XXVI Prop. 1.6] there is a closed subgroup scheme $R_uP \subset P$ such that R_uP is smooth over $\mathcal{A}$ with connected fibers, $(R_uP)_k$ is the unipotent radical of P_k, and $(R_uP)_{\mathcal{K}}$ is the unipotent radical of $P_{\mathcal{K}}$.

Proposition 2.4.1. *Let* $\mathscr{S} \subset \mathcal{G}$ *be an* $\mathcal{A}$*-torus. Then the centralizer* $\mathcal{M} = C_{\mathcal{G}}(\mathscr{S})$ *is a reductive* $\mathcal{A}$*-subgroup scheme of* $\mathcal{G}$ *with connected fibers. Moreover,* $\mathcal{M}$ *is a subgroup scheme of* $\mathcal{G}$ *of type (R).*

Proof. Indeed, according to [SGA3_{II}, Exp XI, Cor 5.3], $\mathcal{M}$ is a closed subgroup scheme of $\mathcal{G}$ which is smooth over $\mathcal{A}$. Now [SGA3_{III}, Exp XIX, §1.3] shows that $\mathcal{M}_k$ and $\mathcal{M}_{\mathcal{K}}$ are connected and reductive subgroups of $\mathcal{G}_k$ and $\mathcal{G}_{\mathcal{K}}$ respectively, and the Proposition follows. □

Lemma 2.4.2. *Let* $\mathcal{L}$ *be a free* $\mathcal{A}$*-module of finite rank, suppose that* $\psi : \mathbf{G}_m \to \text{GL}(\mathcal{L})$ *is an* $\mathcal{A}$*-homomorphism, and let* $\mathcal{M} \subset \mathcal{L}$ *be an* $\mathcal{A}$*-submodule such that*

$$\mathcal{M}_{\mathcal{K}} = \sum_{i \geqslant m} \mathcal{L}_{\mathcal{K}}(\psi_{\mathcal{K}}; i) \quad \textit{and} \quad \mathcal{M}_k = \sum_{i \geqslant m} \mathcal{L}_k(\psi_k; i).$$

for some integer m*. Then* $\mathcal{M} = \sum_{i \geqslant m} \mathcal{L}(\psi; i)$*.*

Proof. Since $\mathcal{L} = \bigoplus_i \mathcal{L}(\psi; i)$, we have

$$(\flat) \quad \mathcal{M} \subset \mathcal{L} \cap \left(\sum_{i \geqslant m} \mathcal{L}_{\mathcal{K}}(\psi_{\mathcal{K}}; i) \right) = \sum_{i \geqslant m} \mathcal{L}(\psi; i).$$

Since $\mathcal{M}_k = \sum_{i \geqslant m} \mathcal{L}_k(\psi_k; i)$ it follows from the Nakayama Lemma that equality holds in $(\flat)$. □

Proposition 2.4.3. *Let $\psi : \mathbf{G}_m \to \mathcal{G}$ be an $\mathcal{A}$-homomorphism. There is a unique subgroup scheme $P_{\mathcal{G}}(\psi) \subset \mathcal{G}$ of type (R) with the property that*

$$(\sharp) \quad \operatorname{Lie} P_{\mathcal{G}}(\psi) = \sum_{i \geqslant 0} \operatorname{Lie}(\mathcal{G})(\psi; i).$$

Moreover,

(a) $P = P_{\mathcal{G}}(\psi)$ *is a parabolic subgroup scheme of* $\mathcal{G}$*,*
(b) $C_{\mathcal{G}}(\psi)$ *is a Levi factor of* P*, and*
(c) $\operatorname{Lie}(R_u P) = \sum_{i>0} \operatorname{Lie}(G)(\psi; i)$.

Proof. First recall [Spr98, Prop. 8.4.5 and Theorem 13.4.2] that if G is a connected and reductive group over a field $\mathcal{F}$ and if $\phi : \mathbf{G}_m \to G$ is an $\mathcal{F}$-homomorphism, then there is a parabolic subgroup $P_G(\phi)$ for which

$$P_G(\phi)(\mathcal{F}_{\mathrm{sep}}) = \{g \in G(\mathcal{F}_{\mathrm{sep}}) \mid \lim_{t \to 0} \operatorname{Int}(\phi(t))g \text{ exists}\}.$$

Moreover, $\operatorname{Lie}(P_G(\phi)) = \sum_{i \geqslant 0} \operatorname{Lie}(G)(\phi; i)$, $\operatorname{Lie}(R_u P_G(\phi)) = \sum_{i>0} \operatorname{Lie}(G)(\phi; i)$, and $C_G(\phi)$ is a Levi factor of $P_G(\phi)$.

Let us write $P_{\mathcal{K}}$ for the parabolic subgroup $P_{\mathcal{G}_{\mathcal{K}}}(\psi_{\mathcal{K}})$ of $\mathcal{G}_{\mathcal{K}}$ determined by $\psi_{\mathcal{K}}$. Consider the functor $\underline{\mathrm{Par}}$ defined for an $\mathcal{A}$-scheme S by the rule

$$\underline{\mathrm{Par}}(S) = \text{set of all } S\text{-parabolic subgroup schemes of } \mathcal{H}_S$$

It follows from [SGA3$_{III}$, Exp. XXVI, Cor. 3.5] that $\underline{\mathrm{Par}}$ is represented by a scheme which is smooth, projective and of finite type over $\mathcal{A}$.

Since $\underline{\mathrm{Par}}$ is projective and since $\mathcal{A}$ is a discrete valuation ring, the $\mathcal{K}$-points of this $\mathcal{A}$-scheme coincide with its $\mathcal{A}$-points; see e.g. [Liu02, Thm 3.3.25]. Thus, the $\mathcal{K}$-point $P_{\mathcal{K}} \in \underline{\mathrm{Par}}(\mathcal{K})$ determines a unique $\mathcal{A}$-point $P \in \underline{\mathrm{Par}}(\mathcal{A})$. Since $\operatorname{Lie}(P)$ is an $\mathcal{A}$-lattice in $\operatorname{Lie}(P_{\mathcal{K}})$ and since $\operatorname{Lie}(P)$ is contained in $\operatorname{Lie}(\mathcal{G})$, it follows immediately from Lemma 2.4.2 that $(\sharp)$ holds.

Since P is smooth, the discussion in [BT84, (I.2.6)] shows that it is equal to the schematic closure in $\mathcal{G}$ of its generic fiber $P_{\mathcal{K}}$. Similarly, $\mathcal{M} = C_{\mathcal{G}}(\psi)$ is the schematic closure of $M_{\mathcal{K}} = C_{\mathcal{G}_{\mathcal{K}}}(\psi_{\mathcal{K}})$. Since $M_{\mathcal{K}} \subset P_{\mathcal{K}}$, it follows that $\mathcal{M}$ is a closed subgroup scheme of P.

Now, $\mathcal{M}$ is a reductive subgroup scheme of P, and according to the discussion in the first paragraph of this proof, $\mathcal{M}_k$ is a Levi factor of P_k and $\mathcal{M}_{\mathcal{K}}$ is a Levi factor of $P_{\mathcal{K}}$. Thus indeed $\mathcal{M}$ is a Levi factor of P and (b) holds.

Finally, recall the smooth subgroup scheme $R_u P$ has the property that

$$\operatorname{Lie}((R_u P)_{\mathcal{K}}) = \sum_{i>0} \operatorname{Lie}(\mathcal{G}_{\mathcal{K}})(\psi_{\mathcal{K}}; i) \quad \text{and} \quad \operatorname{Lie}((R_u P)_k) = \sum_{i>0} \operatorname{Lie}(\mathcal{G}_k)(\psi_k; i).$$

It now follows from Lemma 2.4.2 that (c) holds. □

2.5. **Parahoric group schemes.** Let G be a connected and reductive algebraic group over $\mathcal{K}$. We are going to consider the *parahoric group schemes* $\mathcal{P}$ attached to G. In particular, $\mathcal{P}$ is a smooth, affine $\mathcal{A}$-group scheme, and the generic fiber $\mathcal{P}_{\mathcal{K}}$ coincides with the given group G.

We first suppose that G is *split* over $\mathcal{K}$. Fix a maximal split torus T of G, and write $(X, Y, \Phi, \Phi^\vee)$ for the root datum of G with respect to T. In particular, $\Phi \subset X^*(T)$ is the set of *roots* of T in $\operatorname{Lie}(G)$. For $\alpha \in \Phi$, there is a corresponding $\mathcal{K}$-subgroup U_α - the root subgroup - normalized by T.

For a root α, a *pinning* of α will mean a central $\mathcal{K}$-isogeny $\psi_\alpha : SL_2 \to G_\alpha$ satisfy certain requirements. First write S for the diagonal maximal torus of SL_2 identified with $\mathbf{G}_m$, and write U^+ for the unipotent upper-triangular subgroup of SL_2. Then the requirements are as follows:

(P1) ψ_α maps S to T and the restriction of ψ_α to S identifies with the co-root $\alpha^\vee \in X_*(T)$ [2], and
(P2) ψ_α maps U^+ isomorphically to U_α.

[2] for the fixed identification of S and $\mathbf{G}_m$

We fix an identification $U^+ \simeq \mathbf{G}_a$. Then upon restriction to the root subgroups of SL_2, ψ_α determines $\mathcal{K}$-isomorphisms $\psi_{\alpha,\pm} : \mathbf{G}_a \to U_{\pm\alpha}$.

Recall [BT84, (3.2.2)] that a Chevalley system is a collection of pinnings (ψ_α) for each root $\alpha \in \Phi^+$ satisfying an additional "compatibility" property spelled out e.g. in [BT84, (3.2.2) condition (Ch 2)]. See also [McN18, §4.1]. Let us fix such a Chevalley system.

Let U be a 1 dimensional commutative unipotent group scheme over $\mathcal{K}$, and let $\mathbf{G}_a \xrightarrow{\psi} U$ be a fixed isomorphism. For $n \in \mathbf{Z}$, consider the fractional ideal $I = \pi^n\mathcal{A} \subset \mathcal{K}$. Then I determines a smooth $\mathcal{A}$-group scheme I_{add} with generic fiber $\mathbf{G}_{a\mathcal{K}}$. By transport of structure along ψ, one finds a smooth $\mathcal{A}$-group scheme $\mathscr{U}_n$ with an identification $\mathscr{U}_{n,\mathcal{K}} = U$. Moreover, the isomorphism ψ determines an isomorphism $I = \pi^n\mathcal{A} \xrightarrow{\psi} \mathscr{U}_n(\mathcal{A})$. Compare [BT84, (4.3.2)] and [McN18, §4.3].

More generally, for $r \in \mathbf{Q}$, write $\lceil r \rceil = \min\{n \in \mathbf{Z} \mid n \geqslant r\}$ for the "ceiling function", and write $\mathscr{U}_r = \mathscr{U}_{\lceil r \rceil}$. Thus $\mathscr{U}_r(\mathcal{A}) = \{\psi(a) \mid \nu(a) \geqslant r\}$ where $\nu : \mathcal{K}^\times \to \mathbf{Z}$ is the (normalized) valuation of $\mathcal{K}$, and $\mathscr{U}_r = (\pi^{\lceil r \rceil}\mathcal{A})_{\text{add}}$. [3]

Let $V = X_*(T) \otimes \mathbf{Q}$. For $x \in V$ and $\alpha \in \Phi$, consider the group scheme $\mathscr{U}_{\alpha,x} = (\mathscr{U}_\alpha)_{\langle \alpha,x \rangle}$. Thus $\mathscr{U}_{\alpha,x}$ is obtained via $\psi_{\alpha,+}$ from I_{add} where $I = \pi^{\lceil \langle \alpha,x \rangle \rceil}\mathcal{A}$.

Let $\mathscr{T}$ denote the split $\mathcal{A}$-torus with generic fiber T

Theorem 2.5.1. *Let G be split over $\mathcal{K}$, and fix $x \in V$. There is an group scheme $\mathcal{P}_x$ which is affine, smooth and of finite type over $\mathcal{A}$ such that the following hold:*

- *(a) the generic fiber $\mathcal{P}_{x,\mathcal{K}}$ identifies with G;*
- *(b) the inclusion $T \to G$ (resp. $U_a \to G$ for $a \in \Phi$) prolongs to an isomorphism from $\mathscr{T}$ (resp. $\mathscr{U}_{a,x}$) to a closed $\mathcal{A}$-subgroup scheme of $\mathcal{P}_x$;*
- *(c) for each system of positive roots Φ^+ of Φ and for each order of Φ^+ (resp. of $\Phi^- = -\Phi^+$), the product mapping is an isomorphism of schemes from $\prod_{a \in \Phi^+} \mathscr{U}_{a,x}$ to a closed subgroup scheme $\mathscr{U}^+$ (resp. $\mathscr{U}^-$) of $\mathcal{P}_x$;*
- *(d) the product mapping determines an isomorphism of schemes from $\mathscr{U}^- \times \mathscr{T} \times \mathscr{U}^+$ to an open sub-scheme U of $\mathcal{P}_x$;*
- *(e) the fibers of $\mathcal{P}_x$ are connected.*

Moreover, $\mathcal{P}_x$ is – up to isomorphism – the unique $\mathcal{A}$-group scheme satisfying (a),(b),(c) and (d). Finally, if $x \in X_*(T) \subset V$, $\mathcal{P}_x$ is a split reductive group scheme over $\mathcal{A}$.

Proof. Recall the notion of a *schematic root datum* $\mathscr{D} = (\mathscr{U}_\alpha)_{\alpha \in \Phi}$ for G – see [BT84, §3.1], and see also [McN18, §4.2]. In the present language, we confirmed in [McN18, Theorem 4.3.4] that the collection $\mathscr{D}_x = (\mathscr{U}_{\alpha,x})_{\alpha \in \Phi}$ is a schematic root datum for G with respect to T.

Now [BT84, Théorème 3.8.1] provides the required smooth, affine group scheme $\mathcal{P}_x$ with the properties (a),(b),(c) and (d).

Write U for the open sub-scheme described in (d). Since $\mathscr{T}$ and each $\mathscr{U}_{\alpha,x}$ have connected fibers, also the fibers of U are connected. In view of [BT84, Prop. 4.6.5], the fibers of $\mathcal{P}_x$ are connected as well. The uniqueness of $\mathcal{P}_x$ follows from [BT84, Theorem 3.8.3].

When $x = 0 \in V$, $\mathscr{D}_0$ is the schematic root datum determined by the choice of Chevalley system; see e.g. [McN18, Theorem 4.3.1]. Since $\mathcal{P}_0$ has connected fibers by (e), the uniqueness of $\mathcal{P}_0$ implies that $\mathcal{P}_0$ identifies with the split reductive group scheme $\mathcal{G}$.

When $x = \phi \in X_*(T) \subset V$, it follows from [McN18, Theorem 4.3.4(c)] $\mathscr{D}_x$ is obtained from $\mathscr{D}_0$ by applying the inner automorphism $\mathrm{Int}(\phi(\pi))$, and hence that $\mathcal{P}_x \simeq \mathcal{P}_0$ is split reductive over $\mathcal{A}$, as required. □

Remark 2.5.2. Recall that the root system Φ defines a system of *facets* in $V = X_*(T) \otimes \mathbf{Q}$; see e.g. [McN18, §2.4]. If $x, x' \in V$ lie in the same facet, the corresponding schematic root data $\mathscr{D}_x$ and $\mathscr{D}_{x'}$ – and hence also the parahoric group schemes $\mathcal{P}_x$ and $\mathcal{P}_{x'}$ – coincide.

[3] Of course, the notation $\mathscr{U}_r$ can be made meaningful for $r \in \mathbf{R}$, but we only require this notion for rational r.

Now suppose that G splits over an unramified extension of $\mathcal{K}$. According to [BT84, §5.1.12] there is a finite, unramified, Galois extension $\mathcal{L}$ of $\mathcal{K}$, a maximal $\mathcal{K}$-split torus S of G and a maximal $\mathcal{K}$-torus T of G containing S for which $T_{\mathcal{L}}$ is $\mathcal{L}$-split.

Write $\Gamma = \mathrm{Gal}(\mathcal{L}/\mathcal{K})$, write $\mathcal{B}$ for the integral closure of $\mathcal{A}$ in $\mathcal{L}$, and let $V_{\mathcal{L}} = X_*(T_{\mathcal{L}}) \otimes \mathbf{Q}$. Let F be a Γ-invariant facet in $V_{\mathcal{L}}$ – see e.g. [McN18, §2.4] for the notion of facet for $V_{\mathcal{L}}$– and let $\mathcal{Q} = \mathcal{Q}_F = \mathcal{Q}_x$ be the corresponding parahoric group scheme for $G_{\mathcal{L}}$ determined by any point $x \in F$.

Proposition 2.5.3 (Étale descent)**.** *There is a smooth, affine $\mathcal{A}$-group scheme $\mathcal{P}$ with connected fibers such that*

(i) the generic fiber $\mathcal{P}_{\mathcal{K}}$ may be identified with G*, and*
(ii) $\mathcal{Q} \simeq \mathcal{P} \otimes_{\mathcal{A}} \mathcal{B}$.

Proof. Since $\mathcal{K} \subset L$ is unramified, this follows from [BT84, §4.6.30 and §5.1.8]. □

Definition 2.5.4. Assume that G splits over an unramified extension $\mathcal{L}$ of $\mathcal{K}$. If $\mathcal{L} = \mathcal{K}$ – so that G is $\mathcal{K}$-split – the parahoric group schemes associated to G and T are precisely the $\mathcal{A}$-group scheme $\mathcal{P}_x$ of Theorem 2.5.1 for $x \in V$. For general unramified extension $\mathcal{L}$ of $\mathcal{K}$, the parahoric group schemes associated to G are the $\mathcal{A}$-group schemes obtained via étale descent Proposition 2.5.3 from parahoric $\mathcal{B}$-groups $\mathcal{Q}_x$ where x lies in a Γ-stable facet of $V_{\mathcal{L}}$.

3. Nilpotent elements for a reductive group over a field

In this section, we are going to recall results concerning the nilpotent orbits of a geometrically standard reductive group over a ground field $\mathcal{F}$ of characteristic $p \geqslant 0$.

3.1. **Standard reductive groups.** We now describe the notion of "standard" reductive groups; the terminology follows the "standard hypotheses" considered by J. C. Jantzen.

The class $\mathcal{C} = \mathcal{C}_{\mathcal{F}}$ of *standard reductive groups* over $\mathcal{F}$ consists of all connected and reductive linear algebraic groups over $\mathcal{F}$ satisfying the following properties:

(S1) $\mathcal{C}$ contains all simple $\mathcal{F}$-groups in *very good* characteristic; see Remark 3.1.1 below.
(S2) If G_1 and G_2 are in $\mathcal{C}$ then $G_1 \times G_2$ is in $\mathcal{C}$.
(S3) If G is in $\mathcal{C}$ and H is a reductive F-group, and if there is a separable isogeny between G and H, then H is in $\mathcal{C}$.
(S4) If G is in $\mathcal{C}$ and $D \subset G$ is a diagonalizable subgroup scheme, then $C_G(D)^o$ is in $\mathcal{C}$.
(S5) If $G \simeq H \times T$ for a reductive $\mathcal{F}$-group H and a $\mathcal{F}$-torus T, then G is in $\mathcal{C}$ if and only if H is in $\mathcal{C}$.

We say that G is *geometrically standard* if $G_{\mathcal{E}}$ is standard for some finite and separable field extension $\mathcal{E}$ of $\mathcal{F}$.

Remark 3.1.1. Recall if H is a split simple group over F, the characteristic p of F is *good* for H provided that p does not divide the index of the root lattice in the weight lattice; see [SS70].

If H is any split semisimple group, one may apply [Knu+98, Theorems 26.7 and 26.8] to see that there is a (possibly inseparable) central isogeny

$$\prod_{i=1}^{m} H_i \to H$$

where each H_i is a simple group over F. The characteristic is *good* - respectively *very good* – for H just in case it is good - respectively very good - for each H_i.

Remark 3.1.2. See [MT16, §4] for some discussion reconciling this version of the definition of "standard" with that found in older papers of the author, and for comparison with the notion of "pretty good primes" introduced by S. Herpel [Her13] and with the "standard hypotheses" of J. C. Jantzen.

Remark 3.1.3. For any $n \geqslant 1$, the group GL_n is standard. The group SL_n is D-standard if and only if p does not divide n. See [McN05, Remark 3].

Proposition 3.1.4. *Let* G *be geometrically standard over* $\mathcal{F}$*.*

(a) The (scheme theoretic) center $Z = Z(G)$ *of* G *is smooth over* $\mathcal{F}$.
(b) There is a G*-invariant non-degenerate bilinear form* β *on* $\mathrm{Lie}(G)$*, and* $\mathrm{Lie}(G)$ *is a completely reducible* G*-module.*
(c) Let G *be a standard reductive group over* $\mathcal{F}$*, let* $x \in G(\mathcal{F})$ *and let* $X \in \mathfrak{g}$*. Then* $C_G(x)$ *and* $C_G(X)$ *are smooth over* $\mathcal{F}$*. In other words,*

$$\dim C_G(x) = \dim \mathfrak{c}_{\mathfrak{g}}(x) \quad \textit{and} \quad \dim C_G(X) = \dim \mathfrak{c}_{\mathfrak{g}}(X).$$

Proof. For standard reductive groups, (a) follows from [MT09, (3.4.2)], and (b) and (c) follow from [MT07, Prop. 12].

Since smoothness is a geometric property, (a) and (c) follow at once for geometrically standard groups. For (b), also completely reducibility is a geometric property. As to the remaining assertion, recall that a non-degenerate bilinear form on $\mathrm{Lie}(G)$ amounts to a G-module isomorphism $\mathrm{Lie}(G) \to \mathrm{Lie}(G)^\vee$. Such an isomorphism exists over $\mathcal{F}$ if G is standard. If $\mathcal{F}$ is infinite, the $\mathcal{F}$-homomorphisms $\mathrm{Hom}_G(\mathrm{Lie}(G), \mathrm{Lie}(G)^\vee)$ form a dense subset of the variety of all G-homomorphisms, and it follows that the non-empty, open subvariety $\mathrm{Isom}_G(\mathrm{Lie}(G), \mathrm{Lie}(G)^\vee)$ must have an $\mathcal{F}$-rational point. If $\mathcal{F}$ is finite, note that $\mathrm{Isom}_G(\mathrm{Lie}(G), \mathrm{Lie}(G)^\vee)$ is a torsor over the group $A = \mathrm{Aut}_G(\mathrm{Lie}(G))$. Since $\mathrm{Lie}(G)$ is a semisimple G-module, A is a connected (and reductive) $\mathcal{F}$-group. Hence the Lang-Steinberg Theorem implies that $\mathrm{Isom}_G(\mathrm{Lie}(G), \mathrm{Lie}(G)^\vee)$ has an $\mathcal{F}$-rational point, as required; see e.g. [Ser02, III §2]. □

3.2. **Nilpotent centralizers and orbits.** Throughout section 3.2, G will denote a *geometrically standard* reductive algebraic group over the ground field $\mathcal{F}$. Write $\mathfrak{g} = \mathrm{Lie}(G)$ for the Lie algebra of G. See section 3.1 above for details on geometrically standard reductive groups.

Let $X \in \mathfrak{g}$ be a nilpotent element. Write $C = C_G(X)$ for the centralizer of X.

Definition 3.2.1. A cocharacter $\phi : \mathbf{G}_m \to G$ is *associated with* X provided that
(a) $X \in \mathfrak{g}(\phi; 2)$, and
(b) there is a maximal torus $S \subset C$ such that the image of ϕ lies in the derived group $\mathrm{der}(M)$ where $M = C_G(S)$.

Write $N = N(X)$ for the stabilizer in G of the line $[X]$ in the projective space $\mathbf{P}(\mathfrak{g})$.

Proposition 3.2.2. *(a) The subgroup* N *is smooth over* $\mathcal{F}$*.*
(b) For each maximal torus S *of* N*, there is a unique cocharacter* $\phi \in X_*(S)$ *associated to* X*. In particular, there is a cocharacter associated with* X*.*
(c) If ϕ *is a cocharacter associated with* X*, the image of* ϕ *centralizes some maximal* $\mathcal{F}$*-torus of the connected centralizer* $H = C_G(X)$*, and the centralizer* $M = C_H(\phi)$ *of the image of* ϕ *in* H *is a Levi factor of* H*.*
(d) If S *is a maximal* $\mathcal{F}$*-torus of* $C_G(X)$*, there is a cocharacter associated with* X *whose image centralizes* S*.*
(e) The unipotent radical $U = R_u(C)$ *is defined and split over* $\mathcal{F}$*, and*
(f) Any two associated cocharacters for X *are conjugate by an element of* $U(\mathcal{F})$*.*

Proof. First suppose that G is a standard reductive group. In that case, (a) follows from [McN04, Lemma 23]. Geometrically, (b) and (c) follow from the work of Premet [Pre03] giving a modern proof of the Bala-Carter Theorem; for the proof over a ground field, see [McN04, Thm. 26, Cor. 20, and Cor. 29]. (d) and (e) follow from [McN05, Prop/Defn 21, parts (3) and (4)].

Now, the unipotent radical U of $C = C_G(X)$ is defined and split unipotent over a finite galois extension $\mathcal{E}$ of $\mathcal{F}$. Then U is already defined over $\mathcal{F}$ – see [Spr98, p. 14.4.5] – and in fact is $\mathcal{F}$-split – see [Spr98, Theorem 14.3.8].

The preceding results show that the collection of cocharacters associated with X is a torsor for the unipotent radical U of the centralizer $C = C_G(X)$. Since U is split unipotent over $\mathcal{F}$, one knows that $H^1(\mathcal{F}, U)$ is trivial, and it follows at once that there is a cocharacter ϕ of G associated with X that is defined over $\mathcal{F}$. The remaining assertions in the geometrically standard case are now a consequence of the existence of ϕ. □

An cocharacter associated to X determines the dimension of the orbit of X, as follows:

Proposition 3.2.3. *Let ϕ be a cocharacter associated to X, let $P(\phi)$ be the parabolic subgroup determined by ϕ, and write $C_G(X)$ for the centralizer of X in G. Then:*

(a) $C_G(X) \subset P(\phi)$, *and*
(b) we have
$$\dim C_G(X) = \dim_{\mathcal{F}}(\mathfrak{g}(\phi;0) + \mathfrak{g}(\phi;1))$$
.

Proof. Indeed [Jan04, Prop. 5.9] shows that (a) holds, that the P-orbit $\mathcal{O}$ of X is smooth, and that $\mathcal{O}$ is dense in
$$R = \sum_{i \geqslant 2} \mathfrak{g}(\phi;i).$$
The result now follows since $\mathrm{Lie}(P(\phi)) = \sum_{i\geqslant 0} \mathfrak{g}(\phi;i)$. □

3.3. **Optimal SL_2-homomorphisms over a field.** Consider the semisimple $\mathcal{F}$-group $\mathrm{SL}_{2,\mathcal{F}}$. Write E_1 for the nilpotent section $E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{Lie}(\mathrm{SL}_{2/\mathcal{F}})$, and write $\mathscr{S}_1 \subset \mathrm{SL}_{2,\mathcal{F}}$ for the diagonal torus; thus $\mathscr{S} \simeq \mathbf{G}_{m/\mathcal{F}}$.

If $p > 0$ recall that $\mathrm{Lie}(G)$ is a p-Lie algebra. Thus, there is a p-operation $Y \mapsto Y^{[p]}$ on $\mathrm{Lie}(G)$, and an element Y is nilpotent if and only if $Y^{[p]^j} = 0$ for some $j \geqslant 1$. For notational convenience, when the characteristic of $\mathcal{F}$ is zero, we *define* $Y^{[p]}$ to be 0, for *nilpotent* $Y \in \mathrm{Lie}(G)$.

Theorem 3.3.1. *Let G be a geometrically standard reductive group over $\mathcal{F}$, let $X \in \mathrm{Lie}(G)$ be nilpotent, and let ϕ be a cocharacter associated with X. If $X^{[p]} = 0$, then there is a unique $\mathcal{F}$-homomorphism $\Psi : \mathrm{SL}_{2,\mathcal{F}} \to G$ such that $d\Psi(E_1) = X_1$ and $\Psi_{|S_1} = \phi_1$.*

Proof. For a standard reductive group over $\mathcal{F}$, this is the main result of [McN05]. For geometrically standard G the result follows by Galois descent from the result for standard G. □

3.4. **Numerics.** Finally, we are going to record some numerical facts about nilpotent elements. Write h for the maximum value of the Coxeter number of an irreducible component of the geometric root system of G – i.e. of the root system of $G_{\overline{\mathcal{F}}}$ where $\overline{\mathcal{F}}$ is an algebraic closure of $\mathcal{F}$.

Proposition 3.4.1. *Let $X \in \mathrm{Lie}(G)$ be nilpotent, and let ϕ be a cocharacter associated to X. Then $X^{[p]} = 0$ if and only if $\mathrm{Lie}(G)(\phi;i) = 0$ for all $i \geqslant 2p$.*

Proof. This follows from [McN05, Prop. 24]. □

With notations as in section 3.3, we have the following:

Proposition 3.4.2. *Let $X \in \mathrm{Lie}(G)$ be nilpotent. Then*

(a) If $p \geqslant h$, then $X^{[p]} = 0$.
(b) Let ϕ be a cocharacter associated to X, and suppose that $\mathrm{Lie}(G)(\phi;i) \neq 0 \implies |i| \leqslant p$. Then $X^{[p]} = 0$.
(c) If $X^{[p]} = 0$, there is a homomorphism $\Phi : \mathrm{SL}_{2,\mathcal{F}} \to G$ for which $d\Phi(E_1) = X$ and $\Phi_{|S_1} = \phi$.

Proof. For(a), it suffices to prove the result after extending the ground field. Thus we may and will suppose that G is split reductive over $\mathcal{F}$. Now let $X_1 \in \mathrm{Lie}(G)$ be a regular nilpotent element, and let ϕ_1 be a cocharacter associated to X_1. One knows that if $\mathrm{Lie}(G)(\phi_1;j) \neq 0$, then $j \leqslant 2h-2$. By definition we have $X_1 \in \mathrm{Lie}(G)(\phi_1;2)$. Moreover, $(X_1)^{[p]} \in \mathrm{Lie}(G)(\phi_1;2p)$. Since we have $2p > 2h-2$, it follows that $X_1^{[p]} = 0$. Thus, the morphism $Y \mapsto Y^{[p]}$ from the nilpotent variety to itself vanishes on the dense, open subvariety of regular elements and so is identically zero. This completes the proof.

For (b), note that $X \in \mathrm{Lie}(G)(\phi;2)$ so that $X^{[p]} \in \mathrm{Lie}(G)(\phi;2p)$. Since $2p > p$, we deduce $X^{[p]} = 0$ as required.

Finally, (c) follows from Theorem 3.3.1. □

For $n \in \mathbf{Z}_{\geqslant 0}$, write $L(n) = L_{\mathcal{F}}(n)$ for the simple $SL_{2,\mathcal{F}}$-module of highest weight n; see [Jan03, §II.2].

Proposition 3.4.3. *Let* $p > 2h - 2$, *let* $X \in \mathrm{Lie}(G)$ *be nilpotent, and let* ϕ *be a cocharacter associated to* X. *Using (a) and (c) of Proposition 3.4.2, find a homomorphism* $\Phi : SL_{2,\mathcal{F}} \to G$ *with the indicated properties.*

(a) As a module for $SL_{2,\mathcal{F}}$, *the adjoint representation* $\mathrm{Lie}(G)$ *is restricted semisimple. More precisely, there is an isomorphism of* $SL_{2,\mathcal{F}}$*-modules*

$$\mathrm{Lie}(G) \simeq \bigoplus_{i=1}^{d} L(n_i)$$

where $0 \leqslant n_i \leqslant p - 1$ *for each* i.

(b) In particular, if $\mathrm{Lie}(G)(\phi; i) \neq 0$ *then* $-p < i < p$.

Proof. First note that (b) is an immediate consequence of (a). Since $SL_{2,\mathcal{F}}$ is split reductive over $\mathcal{F}$, two semimsimple $SL_{2,\mathcal{F}}$-modules which become isomorphic after scalar extension of $\mathcal{F}$ are already isomorphic over $\mathcal{F}$. Thus it suffices to prove the result after scalar extension, so we may and will suppose that G is split over $\mathcal{F}$.

If $-2p + 2 \leqslant i \leqslant 2p - 2$, [McN05, Prop. 30] shows that $\mathrm{Lie}(G)(\phi; i) \neq 0$; see also [Sei00, Prop. 4.1 and 4.2]. Arguing as in [McN05, Prop. 34 and 35], we see that as a module for $SL_{2,\mathcal{F}}$, the adjoint representation $\mathrm{Lie}(G)$ is a *tilting module* and moreover that it is a direct sum of indecomposable tilting modules $T(d)$ having highest weights $d \leqslant 2p - 2$; see [Sei00, §2] and [Don93].

If $X_1 \in \mathrm{Lie}(G)$ is regular nilpotent, then $\mathrm{ad}(X_1)^{2h-2} = 0$; now a density argument implies that $\mathrm{ad}(Y)^{2h-2} = 0$ for every nilpotent element $Y \in \mathrm{Lie}(G)$. In particular it follows that $\mathrm{ad}(X)^{p-1} = 0$.

For any $p \leqslant d \leqslant 2p - 2$, $\dim T(d) = 2p$ and E_1 acts on $T(d)$ with two Jordan blocks each of size p; see [McN03a, Prop. 5] or the description in [Sei00, Lemma 2.3]. Thus the only tilting gmodules that can appear in the decomposition of $\mathrm{Lie}(G)$ as $SL_{2,\mathcal{F}}$ are those of the form $T(d)$ with $d < p$. But those tilting modules are simple – i.e. $T(d) = L(d)$ for $0 \leqslant d < p$. The result now follows. □

4. Existence of balanced nilpotent sections for a reductive group scheme

4.1. **Assumptions.** For the remainder of the manuscript, we consider a reductive group scheme $\mathcal{G}$ over $\mathcal{A}$ with connected fibers. We write $G = \mathcal{G}_{\mathcal{K}}$ for the generic fiber; thus G is a reductive group over $\mathcal{K}$, and according to **??**, G splits over an unramified extension of $\mathcal{K}$.

We consider the assumptions:

(SG1) The fiber $\mathcal{G}_k$ is a geometrically standard reductive group over k, and
(SG2) The fiber $\mathcal{G}_{\mathcal{K}}$ is a geometrically standard reductive group over $\mathcal{K}$.

4.2. **Morphisms of $\mathcal{A}$-schemes which are dominant on the fibers.** Suppose that $\mathcal{F}$ is a field, that V and W are $\mathcal{F}$-varieties, and that $f : V \to W$ is a morphism over $\mathcal{F}$. We say that f is *dominant* if the image $f(V)$ is dense in W.

Proposition 4.2.1. *[Liu02, Exer. 2.4.11] Suppose that* V *and* W *are affine over* $\mathcal{F}$, *with respective coordinate rings* $\mathcal{F}[V]$ *and* $\mathcal{F}[W]$. *The morphism* $f : V \to W$ *is dominant if and only if the comorphism* $f^* : \mathcal{F}[W] \to \mathcal{F}[V]$ *is* injective.

Proposition 4.2.2. *Let* R, S *be finitely generated commutative* $\mathcal{A}$*-algebras which are* flat *- or equivalently (since* $\mathcal{A}$ *is a discrete valuation ring),* torsion free *- as* $\mathcal{A}$*-modules, and let* $f : R \to S$ *be an* $\mathcal{A}$*-algebra homomorphism. Assume that* $f_k : R_k = R \otimes_{\mathcal{A}} k \to S_k = S \otimes_{\mathcal{A}} k$ *is* injective. *Then* f *is injective, and in particular* $f_{\mathcal{K}} : R_{\mathcal{K}} \to S_{\mathcal{K}}$ *is injective.*

Proof. Since S is torsion free as an $\mathcal{A}$-module, also the image $B = \mathrm{im}(f)$ of f is a torsion free and hence flat $\mathcal{A}$-module.

Write $I = \ker f$, and write . So there is a short exact sequence of $\mathcal{A}$-modules

$$0 \to I \to R \xrightarrow{f} B \to 0.$$

Tensoring this sequence over $\mathcal{A}$ with k, we find an exact sequence

$$\mathrm{Tor}_1^{\mathcal{A}}(B,k) \to I \otimes_{\mathcal{A}} k \to R \otimes_{\mathcal{A}} k \xrightarrow{f_k} B \otimes_{\mathcal{A}} k \to 0.$$

Since B is a flat $\mathcal{A}$-module, $\mathrm{Tor}_1^{\mathcal{A}}(B,k) = 0$. Since f_k is injective, we deduce that $I/\pi I = I \otimes_{\mathcal{A}} k = 0$, where $\pi\mathcal{A}$ is the maximal ideal of $\mathcal{A}$; put another way, we know that $I = \pi I$.

Now, R is a finitely generated $\mathcal{A}$-algebra, hence R is Noetherian. In particular, I is a finitely generated R-module. In order to show that $I = 0$, it is enough to show that the localization $I_{\mathfrak{m}}$ is 0 for each maximal ideal $\mathfrak{m}$ of R.

Since $\pi \in \mathfrak{m}$ for each maximal ideal $\mathfrak{m} \subset R$, the condition $I = \pi I$ implies that $I_{\mathfrak{m}} = \mathfrak{m} I_{\mathfrak{m}}$; since $I_{\mathfrak{m}}$ is a finitely generated $R_{\mathfrak{m}}$-module, Nakayama's Lemma implies that $I_{\mathfrak{m}} = 0$.

Conclude now that $I = 0$. This proves that f - and *a fortiori* $f_{\mathcal{K}}$ - is injective, as required. □

Recall that an affine $\mathcal{A}$-scheme $Y = \mathrm{Spec}(\mathcal{A}[Y])$ is smooth over $\mathcal{A}$ if $\mathcal{A}[Y]$ is a flat $\mathcal{A}$-module which is finitely generated as $\mathcal{A}$-algebra, and if its fibers Y_k and Y_K "are" smooth varieties; see e.g. [BT84, (1.2.9)].

Proposition 4.2.3. *Let X, Y be schemes which are affine, smooth and of finite type over $\mathcal{A}$, and let $f : X \to Y$ be an $\mathcal{A}$-morphism. Suppose that the morphism $f_k : X_k \to Y_k$ obtained by base change is* dominant*. Then the morphism $f_{\mathcal{K}} : X_{\mathcal{K}} \to Y_{\mathcal{K}}$ is dominant, as well.*

Proof. Write $\mathcal{A}[X]$ and $\mathcal{A}[Y]$ for the affine coordinate rings of the schemes X and Y, and write $k[X]$ and $k[Y]$ for the coordinate rings of the affine k-varieties X_k and Y_k obtained by base change; thus e.g. $k[X] = \mathcal{A}[X] \otimes_{\mathcal{A}} k$. Similarly, write $\mathcal{K}[X]$ and $\mathcal{K}[Y]$ for the coordinate rings of the affine $\mathcal{K}$-varieties $X_{\mathcal{K}}$ and $Y_{\mathcal{K}}$.

Since the morphism f_k is dominant, Proposition 4.2.1 shows that the comorphism $f_k^* : k[Y] \to k[X]$ is *injective*. Since X and Y are smooth over $\mathcal{A}$, $\mathcal{A}[X]$ and $\mathcal{A}[Y]$ are free $\mathcal{A}$-modules. Proposition 4.2.2 permits us to conclude that the comorphisms $f^* : \mathcal{A}[Y] \to \mathcal{A}[X]$ and $f_{\mathcal{K}}^* : \mathcal{K}[Y] \to \mathcal{K}[X]$ are each injective, and now a second application of Proposition 4.2.1 shows that $f_{\mathcal{K}} : X_{\mathcal{K}} \to Y_{\mathcal{K}}$ is dominant. □

4.3. **Richardson orbits.** Let us begin our discussion of Richardson orbits by considering a field $\mathcal{F}$ and a reductive group G over $\mathcal{F}$.

Proposition 4.3.1. *Let $Q \subset G$ be an $\mathcal{F}$-parabolic subgroup. Then Q has an open dense orbit on* $\mathrm{Lie}(R_Q)$*.*

Proof. When G is $\mathcal{F}$-split, this follows e.g. from [Jan04, §4.9]. Since there is a *unique* open dense Q-orbit $\mathcal{O}$ on $\mathrm{Lie}\, R_u Q$, Galois descent shows that $\mathcal{O}$ is defined over $\mathcal{F}$. □

Remark 4.3.2. (a) The dense Q-orbit $\mathcal{O}$ on $\mathrm{Lie}(R_u Q)$ is called the *Richardson orbit*; an element $X \in \mathcal{O}(F) \subset \mathrm{Lie}(R_u Q)$ is known as a *Richardson element* for Q.

(b) There is always an $\mathcal{F}$-rational Richardson element. Indeed, when $\mathcal{F}$ is infinite an $\mathcal{F}$-open subset $\mathscr{U}$ of affine space $\mathrm{Lie}\, R_u P \simeq \mathbf{A}^N$ has an $\mathcal{F}$-rational point as soon as $\mathscr{U}(\overline{\mathcal{F}})$ is non-empty.

Now suppose that $\mathcal{G}$ is a reductive group scheme over $\mathcal{A}$ with connected fibers. Let us fix a parabolic subgroup scheme $\mathcal{Q}$. Recall from section 2.4 that there is a smooth subgroup scheme $R_u\mathcal{Q}$ whose fibers are the unipotent radicals $R_u\mathcal{Q}_k$ and $R_u\mathcal{Q}_{\mathcal{K}}$.

Let us write $\mathcal{R} = \mathrm{Lie}\, R_u\mathcal{Q}$.

Proposition 4.3.3. *Suppose that $X_0 \in \mathcal{R}_k$ is a Richardson element for $\mathcal{Q}_k$. Let $\mathcal{X} \in \mathcal{R}$ be any element with $\mathcal{X}_k = X_0$ – i.e. for which $\mathcal{X} \equiv X_0 \pmod{\pi\mathcal{R}}$. Then $\mathcal{X}_{\mathcal{K}} \in \mathcal{R}_K$ is a Richardson element for $\mathcal{Q}_K$.*

Proof. By abuse of notation, we also write $\mathcal{R}$ for the $\mathcal{A}$-scheme isomorphic to $\mathbf{A}^N$ whose $\mathcal{A}$-points identify with $\mathcal{R}$, where N is the rank of $\mathcal{R}$ as $\mathcal{A}$-module. Consider the $\mathcal{A}$-morphism

$$\alpha : \mathcal{Q} \to \mathcal{R}$$

given by $\alpha(g) = \mathrm{Ad}(g)\mathcal{X}$.

The morphism $\alpha_k : \mathcal{Q}_k \to \mathcal{R}_k$ obtained by base change from α is then given by the rule $\alpha_k(g) = \mathrm{Ad}(g)X_0$; since X_0 is a representative of the dense $\mathcal{Q}_k$ orbit on $\mathcal{R}_k$, it follows that α_k is a dominant morphism.

Since $\mathcal{Q}$ and $\mathcal{R}$ are schemes which are affine, smooth, and of finite type over $\mathcal{A}$, it now follows from Proposition 4.2.3 that $\alpha_{\mathcal{K}} : \mathcal{Q}_{\mathcal{K}} \to \mathcal{R}_{\mathcal{K}}$ is dominant. In view of Proposition 4.3.1, this proves that $\mathcal{X}_{\mathcal{K}}$ is a Richardson element, as required. □

4.4. **Distinguished parabolics and geometrically distinguished nilpotent elements.** If G is a connected and reductive group over a field $\mathcal{F}$, let us recall that a parabolic subgroup $P \subset G$ is *distinguished* provided that

$$\dim \mathrm{der}(P)/R_u P = \dim R_u P - \dim \mathrm{der}(R_u P).$$

On the other hand, a nilpotent element $X \in \mathrm{Lie}(G)$ is *geometrically distinguished* provided that a maximal torus of $C_G(X)$ is central in G.

Theorem 4.4.1. *Suppose that* G *is a geometrically standard reductive group.*

(a) Let $X \in \mathrm{Lie}(G)$ *be a geometrically distinguished nilpotent element and let* $\phi : \mathbf{G}_m \to G$ *be a cocharacter associated to* X*. Then* $P = P_G(\phi)$ *is a distinguished parabolic subgroup of* G *and* $X \in \mathrm{Lie}\, R_u P$ *is a Richardson element.*

(b) Let Q *be a distinguished parabolic subgroup of* G*. Then any Richardson element in* $\mathrm{Lie}\, R_u Q$ *is a geometrically distinguished nilpotent element.*

Proof. (a) follows from [Pre03, Prop 2.5], and (b) follows from [Car93, Cor. 5.2.4]. □

Remark 4.4.2. The preceding Theorem is a crucial part of the Bala-Carter Theorem parametrizing the geometric nilpotent orbits of G; it was first proved in good characteristic by K. Pommerening [Pom77; Pom80]. A more conceptual proof of the Bala-Carter Theorem was given later by Premet [Pre03].

4.5. **Existence of balanced nilpotent sections of a reductive group scheme.** Let $\mathcal{G}$ be a reductive group scheme over $\mathcal{A}$ with connected fibers satisfying the conditions (SG1) and (SG2) of Section 4.1.

Proposition 4.5.1. *If* $\mathcal{Q} \subset \mathcal{G}$ *is a parabolic subgroup scheme, then* $\mathcal{Q}_k$ *is a distinguished parabolic subgroup of* $\mathcal{G}_k$ *if and only if* $\mathcal{Q}_{\mathcal{K}}$ *is a distinguished parabolic subgroup of* $\mathcal{G}_{\mathcal{K}}$*.*

Proof. Since $\mathcal{G}_k$ and $\mathcal{G}_{\mathcal{K}}$ are assumed to be *geometrically standard*, this follows from the characterization of distinguished parabolics given in [Jan04, §4.10]. □

Consider a triple $(\mathcal{X}, \mathscr{S}, \phi)$ for which $\mathcal{X} \in \mathrm{Lie}(\mathcal{G})$, $\mathscr{S} \subset \mathcal{G}$ is a closed $\mathcal{A}$-subgroup scheme which is an $\mathcal{A}$-torus, and $\phi : \mathbf{G}_m \to \mathcal{L}$ is an $\mathcal{A}$-homomorphism, where $\mathcal{L} = C_{\mathcal{G}}(\mathscr{S})$. We say that $(\mathcal{X}, \mathscr{S}, \phi)$ is a *balanced triple* if the following conditions hold:

(B1) $\mathcal{X}$ is balanced for the adjoint action of $\mathcal{G}$ and $\mathcal{X}_k = X_0$,
(B2) the cocharacter $\phi_{\mathcal{K}}$ is associated with the nilpotent element $\mathcal{X}_{\mathcal{K}} \in \mathrm{Lie}(\mathcal{G}_{\mathcal{K}})$ determined by $\mathcal{X}$,
(B3) the cocharacter ϕ_k is associated with the nilpotent element $\mathcal{X}_k \in \mathrm{Lie}(\mathcal{G}_k)$ determined by $\mathcal{X}$, and
(B4) $\mathcal{X}_k$ is distinguished in $\mathrm{Lie}(\mathcal{L}_k)$ and $\mathcal{X}_{\mathcal{K}}$ is distinguished in $\mathrm{Lie}(\mathcal{L}_{\mathcal{K}})$.

Theorem 4.5.2. *Fix a nilpotent element* $X_0 \in \mathrm{Lie}(\mathcal{G}_k)$ *and a maximal torus* S_0 *of the centralizer* $C_{\mathcal{G}_k}(X_0)$*. There is a* balanced triple $(\mathcal{X}, \mathscr{S}, \phi)$ *such that* $\mathcal{X}_k = X_0$ *and* $\mathscr{S}_k = S_0$*.*

Proof. Write $L_0 = C_{\mathcal{G}_k}(X_0)$. Using Proposition 3.2.2, we choose a cocharacter ϕ_0 of L_0 associated with X_0; thus also ϕ_0 is associated with X_0 in $\mathrm{Lie}(\mathcal{G}_k)$; cf. Theorem A.5 in the Appendix to this manuscript.

Now use Theorem 2.3.1 to choose a torus $\mathscr{S} \subset \mathcal{G}$ for which $S_0 = \mathscr{S}_k$. Set $\mathcal{L} = C_{\mathcal{G}}(\mathscr{S})$; according to Proposition 2.4.1, $\mathcal{L}$ is a reductive group scheme with connected fibers. Moreover, since the fibers of $\mathcal{G}$ are geometrically standard reductive groups, also the fibers of $\mathcal{L}$ are geometrically standard reductive groups; see section 3.1.

By definition ϕ_0 takes values in the derived group $\mathrm{der}(\mathcal{L}_k) = \mathrm{der}(L_0)$. Using Proposition 2.3.3 we may find an $\mathcal{A}$-morphism $\phi : \mathbf{G}_m \to \mathcal{L}$ such that $\phi_0 = \phi_k$ and such that $\phi_{\mathcal{K}}$ takes values in the derived group $\mathrm{der}(\mathcal{L}_{\mathcal{K}})$. Note that (B3) holds by construction.

Now let $Q = P_{\mathcal{L}}(\phi)$ and $P = P_{\mathcal{G}}(\phi)$ be the parabolic subgroup schemes of $\mathcal{L}$ and $\mathcal{G}$ determined by ϕ as in Proposition 2.4.3.

Since a maximal torus of $C_{\mathcal{L}_k}(X_0)$ is central in $\mathcal{L}$, X_0 is distinguished in $\mathrm{Lie}(\mathcal{L}_k)$. It follows from Theorem 4.4.1 that $X_0 \in \mathrm{Lie}(R_u Q_k)$ is a Richardson element and that Q_k is a distinguished parabolic subgroup of $\mathcal{L}_k$. It follows from Proposition 4.5.1 that $Q_{\mathcal{K}}$ is a distinguished parabolic subgroup of $\mathcal{L}_{\mathcal{K}}$. Let $\mathcal{X} \in \mathrm{Lie}(\mathcal{L})(\phi;2) \subset \mathrm{Lie}(R_u Q)$ by any element with $\mathcal{X}_k = X_0$. It follows from Proposition 4.3.3 that $\mathcal{X}_{\mathcal{K}} \in \mathrm{Lie}(R_u Q_{\mathcal{K}})$ is Richardson. Since $Q_{\mathcal{K}}$ is a distinguished parabolic, a second application of Theorem 4.4.1 shows that $\mathcal{X}_{\mathcal{K}}$ is distinguished in $\mathrm{Lie}(\mathcal{L}_{\mathcal{K}})$; this confirms condition (B4). Since $\mathcal{X}_{\mathcal{K}} \in \mathrm{Lie}(\mathcal{L}_{\mathcal{K}})(\phi_{\mathcal{K}};2)$ and since the image of $\phi_{\mathcal{K}}$ lies in $\mathrm{der}(\mathcal{L}_{\mathcal{K}})$, it follows that $\phi_{\mathcal{K}}$ is a cocharacter associated to $\mathcal{X}_{\mathcal{K}}$ in $\mathcal{L}_{\mathcal{K}}$. It now follows from the result in the Appendix Theorem A.5 that $\phi_{\mathcal{K}}$ is a cocharacter associated to $\mathcal{X}_{\mathcal{K}}$ in $\mathcal{G}_{\mathcal{K}}$, as well. Thus condition (B2) holds

Finally, since $\mathcal{G}_k$ and $\mathcal{G}_{\mathcal{K}}$ are geometrically standard reductive groups, the centralizers $C_{\mathcal{G}_k}(\mathcal{X}_k)$ and $C_{\mathcal{G}_{\mathcal{K}}}(\mathcal{X}_{\mathcal{K}})$ are smooth – see Proposition 3.1.4. Since $\phi_{\mathcal{K}}$ is associated with $\mathcal{X}_{\mathcal{K}}$ and ϕ_k is associated with $\mathcal{X}_k$, it follows from Proposition 3.2.3 that the dimension of $C_{\mathcal{G}_k}(\mathcal{X}_k)$ coincides with that of $C_{\mathcal{G}_{\mathcal{K}}}(\mathcal{X}_{\mathcal{K}})$. This shows that $\mathcal{X}$ is indeed balanced; this confirms condition (B1) and completes the proof of the Theorem. □

Remark 4.5.3. The previous theorem implies the validity of Theorem 1.3.1(a) from the introduction.

5. Optimal SL_2-homomorphisms over $\mathcal{A}$

Recall that $\mathcal{G}$ denotes a reductive group scheme over $\mathcal{A}$ with generic fiber $G = \mathcal{G}_{\mathcal{K}}$. Throughout this section, we require that $\mathcal{G}$ satisfy conditions (SG1) and (SG2) from section 4.1.

5.1. **Notation.** We first establish some notations for the semisimple $\mathcal{A}$-group scheme $\mathrm{SL}_{2,\mathcal{A}}$, and we reconcile those notations with those given in section 3.3

Write E for the nilpotent section $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{Lie}(\mathrm{SL}_{2/\mathcal{A}})$, and write $\mathscr{S} \subset \mathrm{SL}_{2,\mathcal{A}}$ for the diagonal torus; thus $\mathscr{S} \simeq \mathbf{G}_{m/\mathcal{A}}$.

For a field $\mathcal{F}$, recall from section 3.3 that we write $E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{Lie}(\mathrm{SL}_{2/\mathcal{F}})$, and we write S_1 for the diagonal torus. Thus for $\mathcal{F} \in \{k, \mathcal{K}\}$ we have $E_1 = E_{\mathcal{F}}$ and $S_1 = \mathscr{S}_{\mathcal{F}}$.

5.2. **Optimal** SL_2**-homomorphisms for a reductive group scheme over** $\mathcal{A}$**.** Let $(\mathcal{X}, \mathscr{S}, \phi)$ be a balanced triple for the reductive group scheme $\mathcal{G}$, as in section 4.5. Recall that the scheme theoretic centralizer $\mathcal{L} = C_{\mathcal{G}}(\mathscr{S})$ is a reductive subgroup scheme of $\mathcal{G}$, that $\mathcal{X} \in \mathrm{Lie}(\mathcal{L})$ and that ϕ factors through the inclusion $\mathcal{L} \subset \mathcal{G}$.

Since the fibers of $\mathcal{G}$ are geometrically standard reductive groups, it follows from definitions that the fibers of $\mathcal{L}$ are geometrically standard reductive groups – see section 3.1. For simplicity of exposition in this section, we assume that $\mathcal{G} = \mathcal{L}$; this amounts to the condition that $\mathcal{X}_k$ and $\mathcal{X}_{\mathcal{K}}$ are geometrically distinguished nilpotent elements.

Recall that ϕ_k is a cocharacter associated with $\mathcal{X}_k$ and thatx $\phi_{\mathcal{K}}$ is a cocharacter associated with $\mathcal{X}_{\mathcal{K}}$. Write p for the characteristic of the residue field k.

Lemma 5.2.1. *If* $(\mathcal{X}_k)^{[p]} = 0$*, then* $(\mathcal{X}_{\mathcal{K}})^{[p]} = 0$*.*

Proof. We apply Proposition 3.4.1. According to that result, the condition $(\mathcal{X}_k)^{[p]} = 0$ shows that the weights of the image of ϕ_k on $\mathrm{Lie}(\mathcal{G}_k)$ are all $< p$. But then it is immediate that the weights of the image of $\phi_{\mathcal{K}}$ on $\mathrm{Lie}(\mathcal{G}_{\mathcal{K}})$ are all $< p$ and the same result shows that $(\mathcal{X}_{\mathcal{K}})^{[p]} = 0$. □

The goal of this section is to prove the following theorem:

Theorem 5.2.2. *If* $\mathcal{X}_k^{[p]} = 0$*, there is a unique* $\mathcal{A}$*-homomorphism*

$$\Phi : \mathrm{SL}_{2/\mathcal{A}} \to \mathcal{G}$$

such that $d\Phi(E) = \mathcal{X}$*, and* $\Phi_{|S} = \phi$*, where* $\mathcal{G}$*.*

Before giving the proof of Theorem 5.2.2, we first establish some preliminary results.

Proposition 5.2.3. *Let* $\mathcal{H}$ *and* $\mathcal{H}'$ *be group schemes which are smooth, affine, and of finite type over* $\mathcal{A}$*. Make the following assumptions:*

(C1) There is a $\mathcal{K}$*-homomorphism* $\psi_1 : \mathcal{H}_\mathcal{K} \to \mathcal{H}'_\mathcal{K}$ *and a* k*-homomorphism* $\psi_0 : \mathcal{H}_k \to \mathcal{H}'_k$*.*

(C2) There is a finite, unramified extension $\mathcal{L}$ *of* $\mathcal{K}$ *with ring of integers* $\mathcal{B}$ *together with a* $\mathcal{B}$*-homomorphism* $\Psi : \mathcal{H}_\mathcal{B} \to \mathcal{H}'_\mathcal{B}$ *such that* $\Psi_\mathcal{L} : \mathcal{H}_\mathcal{L} \to \mathcal{H}'_\mathcal{L}$ *coincides with* $\phi_{1,\mathcal{L}}$*, and such that* $\Psi_\ell : \mathcal{H}_\ell \to \mathcal{H}'_\ell$ *coincides with* $\phi_{0,\ell}$*, where* ℓ *is the residue field of* $\mathcal{B}$*.*

Then there is a unique $\mathcal{A}$*-homomorphism* $\Phi : \mathcal{H} \to \mathcal{H}'$ *such that* $\Psi = \Phi_\mathcal{B}$ *– i.e.* Ψ *arises by base-change from* Φ*; moreover,* $\psi_0 = \Phi_k$ *and* $\psi_1 = \Phi_\mathcal{K}$*.*

Proof. After possibly replacing $\mathcal{L}$ and $\mathcal{B}$ by a larger unramified extension, we may and will suppose that ℓ is a Galois extension of k. Write $\Gamma = \mathrm{Gal}(\ell/k)$; since $\mathcal{L}$ is unramified over $\mathcal{K}$, also $\mathcal{L}$ is a Galois extension of $\mathcal{K}$, and $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$ identifies with Γ.

Since $\mathcal{H}$ is smooth - in particular, flat - over $\mathcal{A}$, $\mathcal{A}[\mathcal{H}]$ identifies with an $\mathcal{A}$-subalgebra of $\mathcal{K}[\mathcal{H}_\mathcal{K}]$. Similarly, $\mathcal{B}[\mathcal{H}_\mathcal{B}]$ identifies with a $\mathcal{B}$-subalgebra of $\mathcal{L}[\mathcal{H}_\mathcal{L}]$. Moreover, $\mathcal{K}[\mathcal{H}_\mathcal{K}] = \mathcal{L}[\mathcal{H}_\mathcal{L}]^\Gamma$, $\mathcal{A}[\mathcal{H}] = \mathcal{B}[\mathcal{H}_\mathcal{B}]^\Gamma$ and $k[\mathcal{H}_k] = \ell[\mathcal{H}_\ell]^\Gamma$. Identical considerations hold for $\mathcal{H}'$.

Now, by our assumptions, $\Psi^* : \mathcal{B}[\mathcal{H}'_\mathcal{B}] \to \mathcal{B}[\mathcal{H}_\mathcal{B}]$ has the property that $\Psi^* \otimes 1_\mathcal{L} = {\psi_{1,\mathcal{L}}}^*$. Since ${\psi_{1,\mathcal{L}}}^*$ is fixed by the action of Γ, it follows that $\Psi^* \otimes 1_\mathcal{L}$ is fixed by the action of Γ, so indeed, $\Psi^* \otimes 1_\mathcal{L}$ determines by restriction an $\mathcal{A}$-algebra mapping $\Phi^* : \mathcal{A}[\mathcal{H}'] \to \mathcal{A}[\mathcal{H}]$, and hence yields an $\mathcal{A}$-homomorphism $\Phi : \mathcal{H} \to \mathcal{H}'$ of group schemes with $\Phi_\mathcal{B} = \Psi$.

It follows by transitivity of base-change that $\Phi_\mathcal{L} = \psi_{1,\mathcal{L}}$ so that $\Phi_\mathcal{K} = \psi_1$ by Galois descent. Similarly, $\Phi_\ell = \psi_{0,\ell}$ so that $\Phi_k = \psi_0$ by Galois descent; this completes the proof. □

Proof of Theorem 5.2.2. Now, according to our assumption together with Lemma 5.2.1 we know that $(\mathcal{X}_k)^{[p]} = 0$ and $(\mathcal{X}_\mathcal{K})^{[p]} = 0$. In view of our assumptions, Theorem 3.3.1 yields a unique $\mathcal{K}$-homomorphism $\psi_1 : \mathrm{SL}_{2,\mathcal{K}} \to \mathcal{G}_\mathcal{K}$ such that $d\psi_1(E_\mathcal{K}) = \mathcal{X}_\mathcal{K}$ and $\psi_{1,S_\mathcal{K}} = \phi_\mathcal{K}$, and a unique k-homomorphism $\psi_0 : \mathrm{SL}_{2,k} \to \mathcal{G}_k$ such that $d\psi_0(E_k) = \mathcal{X}_k$ and $\psi_{0,S_k} = \phi_k$.

We are going to apply Proposition 5.2.3. We choose a finite unramified extension $\mathcal{L}$ of $\mathcal{K}$ which is a splitting field for $\mathcal{G}_\mathcal{K}$; see **??**. Write $\mathcal{B}$ for the ring of integers of $\mathcal{L}$.

In fact, we may choose a maximal $\mathcal{A}$-torus $\mathscr{T}$ of $\mathcal{G}$ containing the image of ϕ – argue as in Corollary 2.3.2. We may and will suppose that $\mathscr{T}_\mathcal{B}$ is a $\mathcal{B}$-split torus. It now follows from [SGA3$_{III}$, Exp. XXII Prop. 2.2] that $\mathcal{G}_\mathcal{B}$ is a split reductive group scheme over $\mathcal{B}$.

If ℓ is the residue field of $\mathcal{B}$, then $\mathcal{X}_\ell$ is distinguished in $\mathrm{Lie}(\mathcal{G}_\ell)$. Thus, Proposition 5.2.4 yields a $\mathcal{B}$-homomorphism $\Psi : \mathrm{SL}_{2,\mathcal{B}} \to \mathcal{G}_\mathcal{B}$ for which $d\Psi(E) = X$ and $\Psi_{|\mathscr{S}} = \phi_\mathcal{B}$. It follows from the uniqueness in Theorem 3.3.1 that $\Psi_\ell = \psi_{0,\ell}$ and $\Psi_\mathcal{L} = \psi_{1,\mathcal{L}}$. Thus Proposition 5.2.3 yields an $\mathcal{A}$-homomorphism $\Phi : \mathrm{SL}_2 \to \mathcal{G}$ with the required properties. □

Proposition 5.2.4. *Let* $\mathcal{G}$ *be a split reductive group scheme over* $\mathcal{A}$ *whose fibers are standard reductive groups, let* $\mathcal{X} \in \mathrm{Lie}(\mathcal{G})$ *be a balanced nilpotent section for which* $\mathcal{X}_k$ *is geometrically distinguished, and let* $\phi : \mathbf{G}_m \to \mathscr{T}$ *be an* $\mathcal{A}$*-homomorphism for some* $\mathcal{A}$*-split maximal torus* $\mathscr{T}$ *of* $\mathcal{G}$ *such that* ϕ_k *is associated with* $\mathcal{X}_k$ *and* $\phi_\mathcal{K}$ *is associated with* $\mathcal{X}_\mathcal{K}$*. If* $(\mathcal{X}_k)^{[p]} = 0$*, there is a homomorphism of group schemes* $\Psi : \mathrm{SL}_{2,\mathcal{A}} \to \mathcal{G}$ *such that* $d\Psi(E) = \mathcal{X}$ *and* $\Psi_{|S} = \phi$*.*

Proof. Write $R = \mathbf{Z}_{(p)}$ and note that there is a unique unital ring homomorphism $R \to \mathcal{A}$. Let $\mathcal{G}_R$ be a split reductive group scheme over R for which $\mathcal{G} = \mathcal{G}_R \times \mathrm{Spec}(\mathcal{A})$. There is a split maximal torus $\mathscr{T}_R$ of $\mathcal{G}_R$ such that $\mathscr{T}_R \times \mathrm{Spec}(\mathcal{A})$ identifies with $\mathscr{T}$. In particular, we may view ϕ also as a cocharacter of $\mathcal{G}_R$. As in Proposition 2.4.3, write $P \subset \mathcal{G}$ for the parabolic $\mathcal{A}$-subgroup scheme of $\mathcal{G}$ determined by ϕ, and let $P_R \subset \mathcal{G}_R$ be the parabolic R-subgroup scheme of $\mathcal{G}_R$ determined by ϕ.

Note that $P = P_R \times \mathrm{Spec}(\mathcal{A})$. Moreover, write $U_R \subset P_R$ and $U \subset P$ for the smooth subgroup schemes of [SGA3$_{III}$, XXVI Prop. 1.21] whose fibers give the unipotent radicals of the fibers of P_R resp. P.

Write $\mathrm{Lie}(\mathcal{G}) = \bigoplus_{n \in \mathbf{Z}} \mathrm{Lie}(\mathcal{G})(\phi; n)$ as a sum of weight spaces for the image of ϕ. According to [McN05, Prop. 24], the condition that $(\mathcal{X}_k)^{[p]} = 0$ implies that $\mathrm{Lie}(\mathcal{G})(\phi; n) = 0$ when $n \geqslant 2p$. Hence also $\mathrm{Lie}(\mathcal{G}_R)(\phi; n) = 0$ when $n \geqslant 2p$. Now a second application of *loc. cit.* shows that the nilpotence class of the fiber $U_{\mathbf{Q}}$ of U_R is $< p$.

Now [Sei00, Prop. 5.1] shows that the exponential isomorphism yields an isomorphism of R-group schemes $\exp : \mathrm{Lie}(U_R) \to U_R$. Hence by base change $R \to \mathcal{A}$, we get an exponential isomorphism $\exp : \mathrm{Lie}(U) \to U$. Applying these considerations to the opposite parabolic subgroup schemes, we have an exponential $\mathcal{A}$-isomorphism $\exp(\mathrm{Lie}(^-)) \to U^-$.

Arguing as in [McN03b, Lemma 10], we find a nilpotent element $\mathcal{Y} \in \mathrm{Lie}(\mathcal{G})(\phi; -2)$ such that $(\mathcal{X}, \mathcal{Y}, H = d\phi(1))$ is an $\mathfrak{sl}_2$-triple over $\mathcal{A}$.

Let $\Psi_1 : \mathrm{SL}_{2,\mathcal{K}} \to \mathcal{G}_{\mathcal{K}}$ be the homomorphism of Theorem 3.3.1 with $d\Psi_1(E_1) = \mathcal{X}_{\mathcal{K}}$ and $\Psi_{1|S_1} = \phi_{\mathcal{K}}$. The "big cell" of SL_2 is the $\mathcal{A}$-subscheme $\Omega = U_1^- S_1 U_1$ where $U_1^{\pm} \simeq \mathbf{G}_{a,\mathcal{A}}$ and $T_1 \simeq \mathbf{G}_{m,\mathcal{A}}$. The restriction of Ψ_1 to $\Omega_{\mathcal{K}}$ is given by $(s, t, u) \mapsto \exp(s\mathcal{Y})\phi_{\mathcal{K}}(t)\exp(u\mathcal{X})$; thus $\Psi_{1|\Omega_{\mathcal{K}}}$ arises by base change from an $\mathcal{A}$-morphism $\Omega \to \mathcal{G}$. Now the argument for [Ser96, Prop. 2] yields the required $\mathcal{A}$-homomorphism $\Psi : \mathrm{SL}_2 \to \mathcal{G}$. (The argument of *loc. cit.* uses that $\mathrm{SL}_{2,\mathbf{Z}}$ is covered by the big cell $\Omega_{\mathbf{Z}}$ together with $w\Omega_{\mathbf{Z}}$ for a suitable $w \in \mathrm{SL}_2(\mathbf{Z})$). □

Remark 5.2.5. In the above proof, we have essentially just adapted the argument given in [McN03b, Theorem 13].

5.3. **Complete reducibility and $\mathrm{SL}_{2,\mathcal{A}}$-homomorphisms.** We collect in this section some results about modules for SL_2. We begin with a "homological" result.

Proposition 5.3.1. *Let $\mathcal{M}$ and $\mathscr{N}$ be $\mathrm{SL}_{2,\mathcal{A}}$-modules that are free and of finite rank as $\mathcal{A}$-modules. Suppose that*

$$\mathrm{Ext}^1_{\mathrm{SL}_{2,k}}(\mathcal{M}_k, \mathscr{N}_k) = 0.$$

If $\phi_0 : \mathcal{M}_k \to \mathscr{N}_k$ is an isomorphism of $\mathrm{SL}_{2,k}$)-modules, there is an isomorphism $\phi : \mathcal{M} \to \mathscr{N}$ of $\mathrm{SL}_{2,\mathcal{A}}$-modules for which $\phi_0 = \phi_k$.

Proof. Under our assumption on Ext^1, it follows from [McN00, Prop. 3.3.1, Prop 3.3.2] that the natural mapping $\mathrm{Hom}_{\mathrm{SL}_{2,\mathcal{A}}}(\mathcal{M}, \mathcal{L}) \to \mathrm{Hom}_{\mathrm{SL}_{2,k}}(\mathcal{M}_k, \mathcal{L}_k)$ is surjective. Now choose an $\mathrm{SL}_{2,\mathcal{A}}$-module homomorphism $\phi : \mathcal{M} \to \mathscr{N}$ such that ϕ_k is the isomorphism ϕ_0. It follows from the Nakayama lemma that ψ is an isomorphism, as required. □

Let $\mathscr{S}$ be the diagonal maximal torus of $\mathrm{SL}_{2,\mathcal{A}}$; thus $\mathscr{S}$ is $\mathcal{A}$-isomorphic to $\mathbf{G}_m$. We write the $\mathrm{SL}_{2,\mathcal{A}}$-module $\mathcal{L}$ as a direct sum of its $\mathscr{S}$-weight spaces: $\mathcal{L} = \bigoplus_{n \in \mathbf{Z}} \mathcal{L}_n$.

We identify the character groups

$$X^*(\mathscr{S}) = X^*(\mathscr{S}_k) = X^*(\mathscr{S}_{\mathcal{K}})$$

with $\mathbf{Z}$. Any $\mathscr{S}$-module M may be written as a direct sum

$$M = \bigoplus_{n \in \mathbf{Z}} M_n$$

of its weight spaces; see [Jan03, p. I.2.11].

Let $\mathcal{L}$ be a module for the $\mathcal{A}$-group scheme $\mathrm{SL}_{2,\mathcal{A}}$ such that $\mathcal{L}$ is free of finite rank as an $\mathcal{A}$-module. We may consider the $\mathrm{SL}_{2,\mathcal{F}}$-modules $\mathcal{L}_{\mathcal{F}} = \mathcal{L} \otimes_{\mathcal{A}} \mathcal{F}$ for $\mathcal{F} \in \{k, \mathcal{K}\}$.

Proposition 5.3.2. *Assume for $n \in \mathbf{Z}$ that $\mathcal{L}_n \neq 0 \implies |n| < p$. Then:*

(a) $\mathcal{L}_{\mathcal{K}}$ is a semisimple representation for $\mathrm{SL}_{2,\mathcal{K}}$, and $\mathcal{L}_k$ is a semisimple representation for $\mathrm{SL}_{2,k}$.

(b) If $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{Lie}(\mathrm{SL}_{2,\mathcal{A}}$, then

$$\dim_k \ker(E_k : \mathcal{L}_k \to \mathcal{L}_k) = \dim_{\mathcal{K}} \ker(E_{\mathcal{K}} : \mathcal{L}_{\mathcal{K}} \to \mathcal{L}_{\mathcal{K}}).$$

(c) Let $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$*. Then* $\mathcal{L} = \ker(F) + E.\mathcal{L}$ *and* $\ker(F) \cap E.\mathcal{L} = \{0\}$*.*

Proof. Recall [Jan03, §II.2] that for $\mathcal{F} \in \{k, \mathcal{K}\}$, the simple modules $L_{\mathcal{F}}(n)$ for $SL_{2,\mathcal{F}}$ are indexed by their highest weight $n \in \mathbf{Z}_{\geqslant 0}$, where we have identified $X^*(\mathscr{S}_{\mathcal{F}}) = \mathbf{Z}$. In view of the assumptions on the $\mathscr{S}$-weights of $\mathcal{L}$, all composition factors of $\mathcal{L}_{\mathcal{F}}$ have the form $L_{\mathcal{F}}(n)$ for $n < p$. It now follows from the *linkage principle* [Jan03, Cor. II.6.7] that $\operatorname{Ext}^1_{SL_{2,\mathcal{F}}}(L, L') = 0$ for any two composition factors L, L' of $\mathcal{L}_{\mathcal{F}}$ We then see immediately that $\mathcal{L}_{\mathcal{F}}$ is semisimple for $\mathcal{F} \in \{k, \mathcal{K}\}$, and (a) is proved.

As to (b), note that the linkage principle argument just given implies for $\mathcal{F} \in \{k, \mathcal{K}\}$ and $0 \leqslant n < p$ that the Weyl module $V_{\mathcal{F}}(n)$ coincides with the simple module $L_{\mathcal{F}}(n)$; see [Jan03, §II.2] for the definition of the Weyl module, and recall that the highest weight of any composition factor of the radical of a Weyl module is $\leqslant n$. Now, there is an $SL_{2,\mathcal{A}}$-module $V_{\mathcal{A}}(n)$ which is free of finite rank over $\mathcal{A}$ such that

$$V_{\mathcal{F}}(n) = V_{\mathcal{A}}(n) \otimes_{\mathcal{A}} \mathcal{F}$$

for $\mathcal{F} \in \{k, \mathcal{K}\}$; see [Jan03, §II.8.3].

In view of the semisimplicity, we may write choose an $SL_{2,k}$-isomorphism $\phi_0 : \bigoplus_{i=1}^t L_k(n_i) \to \mathcal{L}_k$ for certains weights $n_i < p$. Let $\mathcal{M} = \bigoplus_{i=1}^t V_{\mathcal{A}}(n_i)$. Then $\operatorname{Ext}^1_{SL_{2,k}}(\mathcal{M}_k, \mathcal{L}_k) = 0$, so that Proposition 5.3.1 yields an isomorphism $\phi : \mathcal{M} \to \mathcal{L}$ of $SL_{2,\mathcal{A}}$-modules for which $\phi_0 = \phi_k$.

Since $V_{\mathcal{A}}(n_i) \otimes_{\mathcal{A}} \mathcal{K} = V_{\mathcal{K}}(n_i)$ is a simple $SL_{2,\mathcal{K}}$-module for $n_i < 0$ and since $\mathcal{M}_{\mathcal{K}} \simeq \mathcal{L}_{\mathcal{K}}$, it follows that the number t of irreducible summands of $\mathcal{L}_k$ is the same as the number of irreducible summands of $\mathcal{L}_{\mathcal{K}}$.

Now an elementary description of simple $SL_{2,\mathcal{F}}$-modules show first of all that $\dim_k \ker E_k = t = \dim_{\mathcal{K}} \ker E_{\mathcal{K}}$ and moreover that $\mathcal{L} = \ker(F) \oplus E.\mathcal{L}$. Thus (b) and (c) hold. □

6. Existence of balanced nilpotent sections for a parahoric group scheme

Recall that $\mathcal{G}$ denotes a reductive group scheme over $\mathcal{A}$ with generic fiber $G = \mathcal{G}_{\mathcal{K}}$.

6.1. Subgroups of a reductive group of type $C(\mu)$.

In this section, we consider a certain class of reductive subgroups of a reductive group, namely the subgroups of type $C(\mu)$. This class of groups was studied in [McN18] and is slightly larger than the class of pseudo-Levi subgroups considered in [MS03].

For a field $\mathcal{F}$, $\mu_n = \mu_{n,\mathcal{F}}$ denotes the group scheme of n-th roots of unity in $\mathcal{F}$; it is the affine scheme with coordinate algebra $\mathcal{F}[T]/\langle T^n - 1\rangle$. The group scheme μ_n is *diagonalizable* over $\mathcal{F}$, and the group of characters $X^*(\mu_n)$ identifies with $\mathbf{Z}/n\mathbf{Z}$.

Let G be a connected and reductive group over the field $\mathcal{F}$. By a μ-homomorphism with values in G, we mean an equivalence class of homomorphisms of group schemes $\phi : \mu_n \to G$, where

$$\phi_1 : \mu_n \to G \quad \text{and} \quad \phi_2 : \mu_m \to G$$

are equivalent provided that there is N with $m \mid N$, $n \mid N$ and for which the homomorphisms

$$\mu_N \to \mu_n \xrightarrow{\phi_1} G \quad \text{and} \quad \mu_N \to \mu_m \xrightarrow{\phi_2} G$$

coincide, where for any divisor $d \mid N$, $\mu_N \to \mu_d$ denotes the $\mathcal{F}$-homomorphism which on character groups is given by the mapping $\mathbf{Z}/d\mathbf{Z} \to \mathbf{Z}/N\mathbf{Z}$ such that $1 + d\mathbf{Z} \mapsto \dfrac{N}{d} + N\mathbf{Z}$.

Recall:

Proposition 6.1.1. *If T is a split torus over $\mathcal{F}$, then the group of all μ-homomorphisms with values in T identifies with $X_*(T) \otimes \mathbf{Q}/\mathbf{Z}$.*

Proof. [McN18, Prop. 3.4.4]. □

Following [McN18, §3], we say that a connected subgroup M of G is of type $C(\mu)$ if M is the identity component of the centralizer in G of the image of a homomorphism $\phi : \mu_n \to G$ [4]. Any subgroup of type $C(\mu)$ is reductive and contains a maximal torus of G [McN18, Theorem].

When G is $\mathcal{F}$-split, we have the following description of the subgroups of type $C(\mu)$ containing a fixed maximal split torus:

Theorem 6.1.2. *Assume that G is split reductive over $\mathcal{F}$, and write T for a maximal split $\mathcal{F}$-torus. Suppose that ψ be a μ-homomorphism with values in T, and write $M = C_G^o(\psi)$. Then M is a reductive subgroup of G containing the maximal torus T. Moreover, the root system of M is the subsystem Φ_y of Φ, where $y \in X_*(T) \otimes \mathbf{Q}$ represents the element $\psi^* \in X_*(T) \otimes \mathbf{Q}/\mathbf{Z}$, and*

$$\Phi_y = \{\alpha \in \Phi \mid \langle \alpha, y \rangle \in \mathbf{Z}\}.$$

In particular,

$$M = \langle T, U_\alpha \mid \alpha \in \Phi_y \rangle$$

where U_α is the 1-dimensional unipotent subgroup on which T acts via the character α.

Proof. [McN18, Theorem 3.4.5]. □

6.2. **Reductive subgroup schemes of a parahoric group scheme.** In general, a parahoric group scheme $\mathcal{P}$ is not reductive over $\mathcal{A}$. In some recent work [McN18], we proved the following result:

Let T be a maximal $\mathcal{K}$-split torus, write $\mathscr{T}$ for "the" $\mathcal{A}$-split torus with generic fiber T, and let $\mathcal{P}$ be a parahoric group scheme containing $\mathscr{T}$ with generic fiber $G = \mathcal{P}_{\mathcal{K}}$. In some recent work, we obtained the following result:

Theorem 6.2.1. *Let $\mathcal{P}$ be a parahoric group scheme over $\mathcal{A}$ with generic fiber G, and assume that G splits over an unramified extension of $\mathcal{K}$. Then there is a subgroup scheme $\mathcal{M} \subset \mathcal{P}$ which is reductive over $\mathcal{A}$ with the following properties:*

(a) the special fiber $\mathcal{M}_k$ is a Levi factor of $\mathcal{P}_k$,
(b) the generic fiber $\mathcal{M}_{\mathcal{K}}$ is a reductive subgroup of G of type $C(\mu)$ – see section 6.1.

We now observe:

Proposition 6.2.2. *With notations as above, if $\mathcal{G}$ satisfies conditions (SG1) and (SG2) of section 4.1, then $\mathcal{M}$ does as well.*

Proof. It will suffice to argue that $\mathcal{M}$ is standard after passing to an unramified extension of $\mathcal{A}$. Hence we may suppose that $\mathcal{G}$ split reductive over $\mathcal{A}$. In that case, it was proved in [McN18, Proposition] that $\mathcal{M}_k$ is a subgroup of type $C(\mu)$ of $\mathcal{G}_k$, and $\mathcal{M}_{\mathcal{K}}$ is a subgroup of type $C(\mu)$ of $\mathcal{G}_{\mathcal{K}}$. Now the result follows, since it is clear from definitions that a reductive subgroup of type $C(\mu)$ of standard reductive group is again standard; see section 3.1. □

6.3. **A condition for a section to be balanced.** In this section, we formulate an "infinitesimal condition" that will permit us to recognize balanced nilpotent sections $X \in \mathrm{Lie}(\mathcal{P})$ for a parahoric group scheme $\mathcal{P}$.

Let $\mathcal{H}$ be a group scheme which is affine, smooth and of finite type over $\mathcal{A}$, and let $\mathcal{L}$ be an $\mathcal{H}$-module where $\mathcal{L}$ is free of finite rank as an $\mathcal{A}$-module.

Proposition 6.3.1. *Let $x \in \mathcal{L}$. Write $\mathfrak{h} = \mathrm{Lie}(\mathcal{H})$, and assume the following:*

(a) the $\mathcal{H}_{\mathcal{K}}$ orbit of $x_{\mathcal{K}}$ is smooth – i.e. $\dim \mathrm{Stab}_{\mathcal{H}_{\mathcal{K}}}(x_{\mathcal{K}}) = \dim_{\mathcal{K}} \mathfrak{c}_{\mathfrak{h}_{\mathcal{K}}}(x_{\mathcal{K}})$, and
(b) $\dim_{\mathcal{K}} \mathfrak{c}_{\mathfrak{h}_{\mathcal{K}}}(x_{\mathcal{K}}) = \dim_k \mathfrak{c}_{\mathfrak{h}_k}(x_k)$.

Then x is balanced for the action of $\mathcal{H}$.

[4] We will permit ourselves to write $C_G(\phi)$ for the centralizer of the image of ϕ, and $C_G^o(\phi)$ for the identity component of this centralizer.

Proof. Let $C = \mathrm{Stab}_{\mathcal{H}}(x)$. The group scheme $C_{\mathcal{K}}$ is smooth over $\mathcal{K}$ by assumption. It remains to argue that C_k is smooth over k and that $\dim C_{\mathcal{K}} = \dim C_k$.

It follows from Chevalley's upper semi-continuity theorem [EGAIV$_{III}$, §13.1.3] that

$$\dim C_{\mathcal{K}} \leqslant \dim C_k.$$

On the other hand, $\mathfrak{c}_{\mathfrak{g}_k}(x_k)$ is the Lie algebra of the group scheme $\mathrm{Stab}_{\mathcal{G}_k}(x_k) = C_k$. Thus $\dim C_k \leqslant \dim_k \mathfrak{c}_{\mathfrak{g}_k}(x_k)$ e.g. by [Knu+98, Lemma 21.8].

Combining these inequalities with our assumptions, we deduce that

$$\dim C_{\mathcal{K}} \leqslant \dim C_k \leqslant \dim_k \mathfrak{c}_{\mathfrak{g}_k}(x_k) = \dim_{\mathcal{K}} \mathfrak{c}_{\mathfrak{g}_{\mathcal{K}}}(x_{\mathcal{K}}) = \dim C_{\mathcal{K}}.$$

Thus equality holds everywhere, so indeed C_k is smooth, e.g. by [Knu+98, Prop. 21.9], and moreover $\dim C_k = \dim C_{\mathcal{K}}$. □

6.4. **Balanced nilpotent sections of parahoric group schemes.** In this section, we require that the reductive $\mathcal{A}$-group scheme $\mathcal{G}$ satisfy the conditions (SG1) and (SG2) of section 4.1.

Let $\mathcal{P}$ be a parahoric group scheme with generic fiber G, and consider a nilpotent element in the Lie algebra of the reductive quotient $\mathcal{P}_k/R_u\mathcal{P}_k$. Choose a reductive subgroup scheme $\mathcal{M} \subset \mathcal{P}$ as in Theorem 6.2.1, and identify this nilpotent element with an element $X_0 \in \mathrm{Lie}(\mathcal{M}_k)$. Application of Theorem 4.5.2 to the reductive group scheme $\mathcal{M}$ yields a balanced triple $(\mathcal{X}, \mathscr{S}, \phi)$ for $\mathcal{M}$ with $X_0 = \mathcal{X}_k$.

We are now going to argue that – under an additional assumption – the section X is already balanced in $\mathrm{Lie}(\mathcal{P})$ for the adjoint action of $\mathcal{P}$. Recall that $h = h(G)$ is the maximum value of the Coxeter number of an irreducible component of $G_{\overline{\mathcal{K}}}$ where $\overline{\mathcal{K}}$ is an algebraic closure of $\mathcal{K}$.

Proposition 6.4.1. *Assume that $p > 2h - 2$ where p is the characteristic of k. Then there is an $\mathcal{A}$-homomorphism $\Phi : \mathrm{SL}_{2,\mathcal{A}} \to \mathcal{M}$ such that $d\Phi(E) = X$ and $\Phi_{|\mathscr{S}} = \phi$. Moreover, $\mathrm{Lie}(\mathcal{P})(\phi; n) = 0 \implies |n| < p$. In particular, $\mathrm{Lie}(\mathcal{P})_{\mathcal{K}}$ is a restricted semisimple module for the adjoint action of $\mathrm{SL}_{2,\mathcal{K}}$ and $\mathrm{Lie}(\mathcal{P})_k$ is a restricted semisimple module for the adjoint action of $\mathrm{SL}_{2,k}$.*

Remark 6.4.2. In fact, the argument given below shows that the conclusion of the Proposition is valid provided that $\mathrm{Lie}(\mathcal{P}_{\mathcal{K}})(\phi_{\mathcal{K}}; n) \neq 0 \implies |n| < p$.

Proof. Since the characteristic of $\mathcal{K}$ is either p or 0, the condition $p > 2h - 2$ together with Proposition 3.4.2(a) implies that $(\mathcal{X}_{\mathcal{K}})^{[p]} = 0$, and Proposition 3.4.3 shows that

$$\mathrm{Lie}(G)(\phi_{\mathcal{K}}; n) \neq 0 \implies |n| < p.$$

It follows that

$$\mathrm{Lie}(\mathcal{P})(\phi; n) \neq 0 \implies |n| < p,$$

and in particular $\mathrm{Lie}(\mathcal{M}_k)(\phi_k; n) \neq 0 \implies |n| < p$. Now we may apply Proposition 3.4.2(b) to the nilpotent element $\mathcal{X}_k \in \mathrm{Lie}(\mathcal{M}_k)$ to learn that $(\mathcal{X}_k)^{[p]} = 0$. Thus, Theorem 5.2.2 yields an $\mathcal{A}$-homomorphism $\Phi : \mathrm{SL}_{2,\mathcal{A}} \to \mathcal{M}$ for which (in the notation of that Theorem) we have $d\Phi(E) = X$ and $\Phi_{|\mathscr{S}} = \phi$.

Consider the adjoint action of $\mathrm{SL}_{2,\mathcal{A}}$ on $\mathrm{Lie}(\mathcal{P})$ given by $\mathrm{Ad} \circ \Phi$. Proposition 5.3.2 shows that $\mathrm{Lie}(\mathcal{P}_k)$ is a semisimple $\mathrm{SL}_{2,k}$-module, that $\mathrm{Lie}(\mathcal{P}_{\mathcal{K}})$ is a semisimple $\mathrm{SL}_{2,\mathcal{K}}$-module. □

Theorem 6.4.3. *Suppose that $p > 2h - 2$. Then X is a balanced nilpotent section of $\mathrm{Lie}(\mathcal{P})$.*

Proof. Using Proposition 6.4.1, we find an $\mathcal{A}$-homomorphism $\Phi : \mathrm{SL}_{2,\mathcal{A}} \to \mathcal{P}$ for which $d\Phi(E) = X$ and $\Phi_{|\mathscr{S}} = \phi$. Moreover, Proposition 5.3.2 implies that

$$\dim_k \ker(\mathrm{ad}\, d\Phi E_k) = \dim_{\mathcal{K}} \ker(\mathrm{ad}\, d\Phi E_{\mathcal{K}}).$$

Since $\ker(\mathrm{ad}\, d\Phi E_k)$ identifies with the Lie algebraic centralizer $\mathfrak{c}_{\mathrm{Lie}\, \mathcal{P}_k}(\mathcal{X}_k)$, and since $\ker(\mathrm{ad}\, d\Phi E_{\mathcal{K}})$ identifies with the Lie algebraic centralizer $\mathfrak{c}_{\mathrm{Lie}(G)}(\mathcal{X}_{\mathcal{K}})$, we find that

$$\dim_k \mathfrak{c}_{\mathrm{Lie}\, \mathcal{P}_k}(\mathcal{X}_k) = \dim_{\mathcal{K}} \mathfrak{c}_{\mathrm{Lie}(G)}(\mathcal{X}_{\mathcal{K}}).$$

Since G is a geometrically standard reductive $\mathcal{K}$-group, the centralizer $C_G(\mathfrak{X}_{\mathcal{K}})$ is smooth over $\mathcal{K}$; see Proposition 3.1.4. Now Proposition 6.3.1 implies that X is balanced for the adjoint action of $\mathcal{P}$ on $\mathrm{Lie}(\mathcal{P})$ – i.e. X is a balanced nilpotent section, as required. □

7. Conjugacy of balanced nilpotent sections

Recall that $\mathcal{G}$ denotes a reductive group scheme over $\mathcal{A}$ with generic fiber $G = \mathcal{G}_{\mathcal{K}}$.

7.1. **Moy-Prasad filtrations.** Suppose in this section that $\mathcal{P} = \mathcal{P}_x$ is a parahoric group scheme with generic fiber G associated with the maximal torus T.

We are going to recall the *Moy-Prasad filtration* of the group $\mathcal{P}(\mathcal{A})$. This filtration was introduced in [MP94]; see also [Adl98, §1.4]. Since our reductive group G splits over an unramified extension of $\mathcal{K}$ some of the complexities of [Adl98, §1.4] don't occur.

Let $r \in \mathbf{Q}_{\geqslant 0}$. First suppose that T (and hence G) is split. Write $\mathscr{T}$ for the $\mathcal{A}$-form of the split torus T, and let

$$T_r = \ker(\mathscr{T}(\mathcal{A}) \to \mathscr{T}(\mathcal{A}/\pi^{\lceil r \rceil}\mathcal{A}))$$

where $\lceil r \rceil = \min\{n \in \mathbf{Z} \mid n \geqslant r\}$ is the "ceiling" function.

For a root α and $r \in \mathbf{Q}_{\geqslant 0}$, we define a subgroup $U_{\alpha,x,r} \subset \mathscr{U}_{\alpha,\langle x,\alpha\rangle}(\mathcal{A})$. Recall from section 2.5 that

$$\mathscr{U}_{\alpha,\langle x,\alpha\rangle}(\mathcal{A}) = \{u_\alpha(a) \mid \nu(a) \geqslant \langle x,\alpha\rangle\}$$

where ν is the valuation of $\mathcal{K}$ and where $u_\alpha : \mathbf{G}_a \to U_\alpha$ is determined by the choice of Chevalley system. We define

$$U_{\alpha,x,r} = \{u_\alpha(a) \mid \nu(a) \geqslant \langle x,\alpha\rangle + r\} = \mathscr{U}_{\alpha,\langle x,\alpha\rangle + r}(\mathcal{A}).$$

The r-th term of the Moy Prasad filtration of $\mathcal{P}(\mathcal{A})$ is the group

$$P_r = \langle T_r, U_{\alpha,x,r} \mid \alpha \in \Phi\rangle.$$

Now if G splits over the unramified extension $\mathcal{L}$ of $\mathcal{K}$ with ring of integers $\mathcal{B}$, let $\mathcal{Q} = \mathcal{P}_{\mathcal{B}}$ and let Q_r be the Moy-Prasad filtration of $\mathcal{Q}(\mathcal{B})$. Then the Moy-Prasad filtration of $\mathcal{P}(\mathcal{A})$ is given by

$$P_r = Q_r \cap \mathcal{P}(\mathcal{A}).$$

Let P_{m+} denote the union of the P_r for $r > m$, and let $P_+ = P_{0+}$.

Parallel to the Moy-Prasad filtration of $\mathcal{P}(\mathcal{A})$, there is an analogous Moy-Prasad filtration of the $\mathcal{A}$-Lie algebra $\mathrm{Lie}(\mathcal{P})$. When G is split over $\mathcal{K}$, it is given by

$$\mathrm{Lie}(\mathcal{P})_m = \pi^{\lceil m \rceil}\,\mathrm{Lie}(\mathscr{T}) \oplus \bigoplus_{\alpha\in\Phi} \mathrm{Lie}(\mathscr{U}_{\alpha,\langle x,\alpha\rangle + m})$$

If G splits over the unramified extension $\mathcal{L}$ of $\mathcal{K}$ with ring of integers $\mathcal{B}$, let $\mathcal{Q} = \mathcal{P}_{\mathcal{B}}$ and for $r \in \mathbf{Q}_{\geqslant 0}$ write $\mathrm{Lie}(\mathcal{Q})_r$ for the terms of the Moy-Prasad filtration of the $\mathcal{B}$-Lie algebra $\mathrm{Lie}(\mathcal{Q})$. Then the Moy-Prasad filtration of $\mathrm{Lie}(\mathcal{P})$ is defined by the rule

$$\mathrm{Lie}(\mathcal{P})_r = \mathrm{Lie}(\mathcal{P}) \cap \mathrm{Lie}(\mathcal{Q})_r.$$

Again, let $\mathrm{Lie}(\mathcal{P})_{m+}$ denote the union of the $\mathrm{Lie}(\mathcal{P})_r$ for $r > m$.

Proposition 7.1.1. *Suppose that* $\mathcal{P} = \mathcal{G}$ *is a* reductive *parahoric group scheme. Then the terms* $\mathrm{Lie}(\mathcal{G})_r \subset \mathrm{Lie}(\mathcal{G})$ *of the Moy-Prasad filtration are given by*

$$\mathrm{Lie}(\mathcal{G})_r = \ker\left(\mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(\mathcal{G})/\mathfrak{m}^{\lceil r \rceil}\,\mathrm{Lie}(\mathcal{G})\right).$$

Proof. It suffices to prove the result when $\mathcal{G}$ is *split* reductive, and in that case the result follows from definitions. □

Proposition 7.1.2. *Consider the quotient mapping* $(\flat) : \mathrm{Lie}(\mathcal{P}_k) \to \mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k)$. *Then* $\mathrm{Lie}(\mathcal{P})_+$ *is the kernel of the mapping* $\mathrm{Lie}(\mathcal{P}) \to \mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k)$ *obtained by composing the natural mapping* $\mathrm{Lie}(\mathcal{P}) \to \mathrm{Lie}(\mathcal{P}_k)$ *with* $(\flat)$.

Proof. First suppose that G is split over $\mathcal{K}$. It follows from [BT84, Prop. 4.6.10] that P_+ is contained in the indicated kernel. The reverse inclusion follows from [BT84, Cor. 4.6.7]. Thus, the Proposition holds for split G.

Now suppose that G splits over an unramified extension $\mathcal{L}$ of $\mathcal{K}$ with ring of integers $\mathcal{B}$. Then $\mathcal{P}$ arises by étale descent from a parahoric group scheme $\mathcal{Q}$ for $G_{\mathcal{L}}$. Since the residue field ℓ of $\mathcal{B}$ is a separable extension of k, the unipotent radical of $\mathcal{P}_k$ satisfies $(R_u\mathcal{P}_k)_\ell = R_u\mathcal{Q}_\ell$. Writing Q_m for the terms of the Moy-Prasad filtration, we have by definition $P_+ = Q_+ \cap \mathcal{P}(\mathcal{A})$. □

Proposition 7.1.3. *For* $r \in \mathbf{Q}_{\geqslant 0}$, $\mathrm{Lie}(\mathcal{P})_r$ *is a* $\mathcal{P}$*-submodule of* $\mathrm{Lie}(\mathcal{P})$ *for the adjoint action.*

Proof. This follows from [Adl98, Prop. 1.2.5]. □

7.2. **A condition for the conjugacy of balanced nilpotent sections.** Let $X \in \mathrm{Lie}(\mathcal{P})$ be a balanced nilpotent section, and let $\phi : \mathbf{G}_m \to \mathcal{P}$ be an $\mathcal{A}$-homomorphism for which ϕ_k is a cocharacter associated to $\mathcal{X}_k$ and $\phi_{\mathcal{K}}$ is a cocharacter associated to $\mathcal{X}_{\mathcal{K}}$.

We are going to prove a conjugacy result for balanced nilpotent sections under the following assumption.

Hypothesis 7.2.1. There is a $\mathcal{A}$-submodule $S \subset \mathrm{Lie}(\mathcal{P})$ with the following properies.

(a) For each $t \in \mathbf{Q}_{\geqslant 0}$ let $S_t = S \cap \mathrm{Lie}(\mathcal{P})_t$. Then
$$\mathrm{Lie}(\mathcal{P})_t = S_t + [X, \mathrm{Lie}(\mathcal{P})_t].$$
(b) S is invariant under the action of the image of the $\mathcal{A}$-homomorphism ϕ.

We will consider the validity of Hypothesis 7.2.1 below. Meanwhile, our target is the following result:

Theorem 7.2.2. *Assume that Hypothesis 7.2.1 holds for* X*, and let* $\mathcal{X}'$ *be a second balanced nilpotent section with* $\mathcal{X}'_k + \mathrm{Lie}(R_u\mathcal{P}_k) = \mathcal{X}_k + \mathrm{Lie}(R_u\mathcal{P}_k)$ *in* $\mathrm{Lie}(\mathcal{P}_k)$*. Then* $\mathcal{X}$ *and* $\mathcal{X}'$ *are conjugate by an element of* P_+*.*

Remark 7.2.3. The argument that will be given below is essentially that of [DeB02, §5.2]. If k has characteristic 0, or "large enough" characteristic for $\mathcal{G}_k$, one can find S by locating an $\mathfrak{sl}(2)$-triple (X, Y, H) containing X and taking S to be the centralizer of Y in $\mathrm{Lie}(\mathcal{G})$; compare e.g. with [DeB02, Lemma 5.2.1]. For completeness and for the reader's benefit, we are going to recapitulate the full argument for the Theorem, under the assumption of Hypothesis 7.2.1.

The first ingredient in the proof of Theorem 7.2.2 is the "mock exponential mapping" described by J. Adler:

Proposition 7.2.4. *Let* $s \in \mathbf{Q}_{>0}$ *and* $r \in \mathbf{Q}_{\geqslant 0}$*. There is a mapping* $\Phi : \mathrm{Lie}(\mathcal{P})_s \to P_s$ *which is continuous in the* $\mathcal{K}$*-analytic topologies such that for* $V \in \mathrm{Lie}(\mathcal{P})_s$ *and* $W \in \mathrm{Lie}(\mathcal{P})_r$*, we have*
$$\mathrm{Ad}(\Phi(V))W \equiv W + [V, W] \pmod{\mathrm{Lie}(\mathcal{P})_{(r+s)+}}.$$

Proof. [Adl98, §1.6]. □

On the generic fiber $G = \mathcal{P}_{\mathcal{K}}$, the assumption Hypothesis 7.2.1 gives a so-called "transvere slice" to the orbit of $\mathcal{X}_{\mathcal{K}}$, and we require some properties of this slice, including the following:

Proposition 7.2.5. *Let* G *be a geometrically standard reductive group over the field* $\mathcal{F}$*, let* $\mathcal{F}_{\mathrm{alg}}$ *be an algebraic closure of* $\mathcal{F}$*, let* $X \in \mathrm{Lie}(G)$ *be nilpotent, and let* ϕ *be a cocharacter of* G *associated to* X*. Suppose* $S \subset \mathrm{Lie}(G)$ *is a linear subspace invariant under the image of* ϕ*, and that* $\mathrm{Lie}(G)$ *is the direct sum* $S \oplus [X, \mathrm{Lie}(G)]$*. Then* $\mathrm{Ad}(G(\mathcal{F}_{\mathrm{alg}}))X \cap (X + S_{\mathcal{F}_{\mathrm{alg}}}) = \{X\}$*, where* $S_{\mathcal{F}_{\mathrm{alg}}} = S \otimes_{\mathcal{F}} \mathcal{F}_{\mathrm{alg}}$*.*

Proof. For the proof, we may and will suppose that $\mathcal{F} = \mathcal{F}_{\mathrm{alg}}$. Write Y for the G-orbit of X. Since the centralizer of X in G is smooth section 3.1, the tangent space of Y at X is $T_X Y = [X, \mathrm{Lie}(G)]$. Moreover, $S = T_X(X + S)$.

It follows that the subvarieties Y and $X + S$ are transversal at X, and in particular, there is an open subvariety $U \subset S$ such that $X \in U(\mathcal{F})$ and such that $(X + U) \cap Y = \{X\}$.

Now, it follows from [Jan04, §7.15] that the weights of the image of ϕ on S are *strictly negative*. Precisely as in the proof of [CG10, Prop. 3.7.15], the action of the image of ϕ determines an action of $\mathbf{G}_m$ on $X+S$ for which X is the only fixed point, and as in *loc. cit.* it follows that $(X+S)\cap Y = \{X\}$. □

Lemma 7.2.6. *Assume that Hypothesis 7.2.1 holds. Fix* $Z \in \mathrm{Lie}(\mathcal{P})_+$. *Let* $s > 0$ *and suppose that* $h \in P_s$ *and* $C \in S_+$ *have been chosen such that*

$$\mathrm{Ad}(h)(X+C) \equiv X+Z \pmod{\mathrm{Lie}(\mathcal{P})_{s+}}.$$

Let $t \in \mathbf{Q}$, $t > s$ *such that* $\mathrm{Lie}(\mathcal{P})_{t+} = \mathrm{Lie}(\mathcal{P})_s \neq \mathrm{Lie}(\mathcal{P})_{s+}$. *Then we may find* $h' \in P_+$ *and* $C' \in S_+$ *such that*

$$\mathrm{Ad}(h')(X+C') \equiv X+Z \pmod{\mathrm{Lie}(\mathcal{P})_{t+}}$$

and such that

$$h'h^{-1} \in P_t \quad \textit{and} \quad C - C' \in S_t.$$

Proof. Consider the element $Z_1 = \mathrm{Ad}(h)(X+C) - X - Z \in \mathrm{Lie}(\mathcal{P})_t$. In particular,

$$\mathrm{Ad}(h)(X+C) \equiv X+Z+Z_1 \pmod{\mathrm{Lie}(\mathcal{P})_{t+}}$$

According to Hypothesis 7.2.1, we may write

$$(\heartsuit) \quad Z_1 = D_1 + [X, Y_1]$$

for $D_1 \in S_t$ and $Y_1 \in \mathrm{Lie}(\mathcal{P})_t$.

Let $h_1 = \Phi(Y_1) \in P_t$ and let $C_1 = -\mathrm{Ad}(h^{-1})(D_1) \in \mathrm{Lie}(\mathcal{P})_t$.

We now compute, using Proposition 7.2.4 and ($\heartsuit$):

$$\begin{aligned}\mathrm{Ad}(h_1h)(X+C+C_1) &\equiv \mathrm{Ad}(h_1)(X+Z+Z_1-D_1) \pmod{\mathrm{Lie}(\mathcal{P})_{t+}}\\ &\equiv X+Z+Z_1-D_1+[Y_1, X+Z+Z_1-D_1] \pmod{\mathrm{Lie}(\mathcal{P})_{t+}}\\ &\equiv X+Z+[Y_1, Z+Z_1-D_1] \pmod{\mathrm{Lie}(\mathcal{P})_{t+}}\\ &\equiv X+Z \pmod{\mathrm{Lie}(\mathcal{P})_{t+}})\end{aligned}$$

where the final congruence is valid because $[Y_1, Z+Z_1-D_1] \in \mathrm{Lie}(\mathcal{P})_{t+}$. Now the result follows by taking $h' = h_1h$ and $C' = C + C_1$. □

Lemma 7.2.7. *Assume that Hypothesis 7.2.1 holds. Then*

$$\mathrm{Ad}(P_+)(X+S_+) = X + \mathrm{Lie}(\mathcal{P})_+.$$

Proof. Fix $Z \in \mathrm{Lie}(\mathcal{P})_+$. We may choose a strictly increasing sequence $t_0 = 0, t_1, t_2, \dots$ in $\mathbf{Q}_{\geqslant 0}$ such that

(a) $\lim_{i\to\infty} t_i = \infty$, and
(b) for $i \geqslant 0$, we have $\mathrm{Lie}(\mathcal{P})_{t_i+} = \mathrm{Lie}(\mathcal{P})_{t_{i+1}} \neq \mathrm{Lie}(\mathcal{P})_{t_{i+1}+}$.

Let $h_0 = 1$ and $C_0 = 0$. Using Lemma 7.2.6, we may find $h_1 \in P_+$ and $C_1 \in S_+$ such that $h_1h^{-1} \in P_{t_1}$, $C - C_1 \in \mathrm{Lie}(\mathcal{P})_{t_1}$, and $\mathrm{Ad}(h_1)(X+C_1) \equiv X+Z \pmod{\mathrm{Lie}(\mathcal{P})_{t_1+}}$. Inductive application of Lemma 7.2.6 now yields a sequence of elements $h_i \in P_+$ and $C_i \in S_+$ for $i \geqslant 0$ such that $h = \lim_{i\to\infty} h_i \in P_+$ and $C = \lim_{i\to\infty} C_i \in S_+$ exist and satisfy

$$\mathrm{Ad}(h)(X+C) \equiv X+Z \pmod{\mathrm{Lie}(\mathcal{P})_{t_i+}}$$

for all $i \geqslant 0$. It now follows that $\mathrm{Ad}(h)(X+C) = X+Z$, as required. □

Proof of Theorem 7.2.2. Since $\mathcal{X}, \mathcal{X}' \in \mathrm{Lie}(\mathcal{P})$ are two lifts of X_0, we have $\mathcal{X}' = \mathcal{X} + \mathcal{Z}$ for $\mathcal{Z} \in \mathrm{Lie}(\mathcal{P})_+$; see Proposition 7.1.2. Thus, it follows from Lemma 7.2.7 that $\mathcal{X}' = \mathrm{Ad}(g)(\mathcal{X}+c)$ for $c \in S_+$ and $g \in P_+$.

Now, the argument give in the proof of [DeB02, Cor. 5.2.4] shows that $\mathcal{X}'_{\mathcal{K}}$ is in the Zariski closure of the orbit of $\mathcal{X}_{\mathcal{K}}$. Since $\mathcal{X}$ and $\mathcal{X}'$ are both balanced, the dimensions of the orbits of $\mathcal{X}_{\mathcal{K}}$ and of $\mathcal{X}'_{\mathcal{K}}$ coincide. It follows that $\mathcal{X}_{\mathcal{K}}$ and $\mathcal{X}'_{\mathcal{K}}$ are geometrically conjugate for the action of $G = \mathcal{G}_{\mathcal{K}}$; see e.g. [Spr98, Lemma 2.3.3]. It follows that $\mathcal{X}$ and $\mathcal{X}+c$ are geometrically conjugate. Now Proposition 7.2.5 implies that $c = 0$, so indeed $\mathcal{X}' = \mathrm{Ad}(g)\mathcal{X}$ for $g \in P_+$ □

7.3. **Conjugacy of balanced liftings for a reductive group scheme.** Let $\mathcal{G}$ be a reductive group scheme over $\mathcal{A}$ with connected fibers, and suppose that the conditions (SG1) and (SG2) of Section 4.1 hold.

Fix a nilpotent element $X_0 \in \mathrm{Lie}(\mathcal{G}_k)$, and use Theorem 4.5.2 to find a balanced triple $(\mathcal{X}, \mathscr{S}, \phi)$ lifting X_0. In particular, $\mathcal{X}_k = X_0$, and $\phi : \mathbf{G}_m \to \mathcal{G}$ is an $\mathcal{A}$-homomorphism such that ϕ_k is associated with $X_0 = \mathcal{X}_k$ and $\phi_{\mathcal{K}}$ is associated with $\mathcal{X}_{\mathcal{K}}$.

For $r \in \mathbf{Q}_{\geqslant 0}$, write $\mathrm{Lie}(\mathcal{G})_r$ for the terms of the Moy-Prasad filtration section 7.1.

Theorem 7.3.1. *Hypothesis 7.2.1 holds for $\mathcal{P} = \mathcal{G}$ and $\mathcal{X}$; i.e. there is an $\mathcal{A}$-submodule $S \subset \mathrm{Lie}(\mathcal{G})$ such that*

(i) $\mathrm{Lie}(\mathcal{G})_r = S_r + [\mathcal{X}, \mathrm{Lie}(\mathcal{G})_r]$ *for each $r \in \mathbf{Q}_{\geqslant 0}$ where $S_r = S \cap \mathrm{Lie}(\mathcal{G})_r$, and*
(ii) S is stable under the action of the image of ϕ.

Proof. First choose a basis $Z_1^0, \dots, Z_r^0$ for a complement to $[X_0, \mathrm{Lie}(\mathcal{G}_k)]$ in $\mathrm{Lie}(\mathcal{G}_k)$ such that each Z_i^0 is a weight vector for the image of the cocharacter ϕ_k; i.e. $Z_i^0 \in \mathrm{Lie}(\mathcal{G}_k)(\phi_k; m_i)$ for suitable weights $m_1, \dots, m_r \in \mathbf{Z}$. Now choose for each i a lift $Z_i \in \mathrm{Lie}(\mathcal{G})(\phi; m_i)$ of Z_i^0, and take $S = \sum_{i=1}^r \mathcal{A} Z_i$.

Then by construction S is stable under the image of ϕ. Moreover, since $S + [\mathcal{X}, \mathrm{Lie}(\mathcal{G})] + \mathfrak{m}\,\mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}(\mathcal{G})$, the Nakayama Lemma implies that

$$(\clubsuit) \quad S + [\mathcal{X}, \mathrm{Lie}(\mathcal{G})] = \mathrm{Lie}(\mathcal{G}).$$

Since $\mathrm{Lie}(\mathcal{G})_r = \pi^{\lceil r \rceil} \mathrm{Lie}(\mathcal{G})$ for $r \in \mathbf{Q}_{\geqslant 0}$ by Proposition 7.1.1, so that $S_r = \pi^{\lceil r \rceil} S$, condition (i) now follows from ($\clubsuit$). ☐

Corollary 7.3.2. *If $\mathcal{X}$ and $\mathcal{X}'$ are two balanced nilpotent sections of* $\mathrm{Lie}(\mathcal{G})$ *with $\mathcal{X}_k = \mathcal{X}'_k = X_0$, then $\mathcal{X}$ and $\mathcal{X}'$ are conjugate by an element of $\mathcal{G}(\mathcal{A})$.*

Proof. In view of Theorem 7.3.1, the corollary follows from Theorem 7.2.2. ☐

7.4. **Conjugacy of balanced liftings for a parahoric group scheme.** Let $\mathcal{P}$ be a parahoric group scheme with generic fiber $\mathcal{P}_{\mathcal{K}} = G$. Choose a reductive subgroup scheme $\mathcal{M} \subset \mathcal{P}$ as in Theorem 6.2.1.

Fix a nilpotent element $X_0 \in \mathrm{Lie}(\mathcal{M}_k)$. Application of Theorem 4.5.2 yields a balanced triple $(\mathcal{X}, \mathscr{S}, \phi)$ in $\mathcal{M}$ with $X_0 = \mathcal{X}_k$.

Theorem 7.4.1. *Let p denote the characteristic of k, and assume that $p > 2h - 2$ where $h = h(G)$. Then Hypothesis 7.2.1 holds for $\mathcal{X}$ and ϕ; i.e. there is an $\mathcal{A}$-submodule $S \subset \mathrm{Lie}(\mathcal{P})$ such that*

(i) $\mathrm{Lie}(\mathcal{P})_r = S_r + [\mathcal{X}, \mathrm{Lie}(\mathcal{P})_r]$ *for each $r \in \mathbf{Q}_{\geqslant 0}$ where $S_r = S \cap \mathrm{Lie}(\mathcal{P})_r$, and*
(ii) S is stable under the action of the image of ϕ.

Proof. In view of our assumption on p, we may apply Proposition 6.4.1 to find an $\mathcal{A}$-homomorphism $\Phi : \mathrm{SL}_{2,\mathcal{A}} \to \mathcal{M}$ with $d\Phi(E) = \mathcal{X}$ and $\Phi_{|\mathscr{S}} = \phi$. Moreover, that Proposition shows that $\mathrm{Lie}(\mathcal{P})(\phi; n) \neq 0 \implies |n| < p$.

Let $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathrm{Lie}(\mathrm{SL}_{2,\mathcal{A}})$ and let $S = \ker(F : \mathcal{L} \to \mathcal{L})$. Then S is clearly stable under the action of ϕ. In view of what we have observed about the weights of ϕ on $\mathrm{Lie}(\mathcal{P})$, we may apply Proposition 5.3.2; part (c) of that result now shows that S satisfies condition (i), as required. ☐

Corollary 7.4.2. *Suppose that $p > 2h - 2$. If $\mathcal{Y}$ and $\mathcal{Y}'$ are balanced nilpotent sections of* $\mathrm{Lie}(\mathcal{P})$ *with* $\mathcal{Y}_k + \mathrm{Lie}(R_u\mathcal{P}_k) = \mathcal{Y}'_k + \mathrm{Lie}(R_u\mathcal{P}_k)$ *in $\mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k)$, then $\mathcal{Y}$ and $\mathcal{Y}'$ are conjugate by an element of $\mathcal{P}(\mathcal{A})$.*

Proof. Let $X_0 \in \mathrm{Lie}(\mathcal{M}_k)$ be an element whose image in $\mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k)$ coincides with the image of YY_k. Choose a balanced triple $(\mathcal{X}, \mathscr{S}, \phi)$ for $\mathcal{M}$ with $\mathcal{X}_k = X_0 \in \mathrm{Lie}(\mathcal{M}_k)$ as in the statement of Theorem 7.2.2. Then that Theorem together with Theorem 7.4.1 shows that X is $\mathcal{P}(\mathcal{A})$-conjugate to both $\mathcal{Y}$ and to $\mathcal{Y}'$. ☐

8. Rational nilpotent orbits

8.1. **Recollections.** In this section, let G be a connected and reductive group over the local field $\mathcal{K}$. For "large enough" residue characteristic, the paper [DeB02] relates the orbits of $G(\mathcal{K})$ on the nilpotent elements of $\mathrm{Lie}(G) = \mathrm{Lie}(G)(\mathcal{K})$ with the distinguished nilpotent orbits of the reductive quotients $\mathcal{P}_k/R_u\mathcal{P}_k$, for parahoric group schemes $\mathcal{P}$ having generic fiber $\mathcal{P}_{\mathcal{K}} = G$.

In turn, the work in [DeB02] was motivated by questions related to the harmonic analysis on $G(\mathcal{K})$ in case k is finite. Harish-Chandra and Roger Howe showed – see e.g. [HC99] – that the character of an irreducible smooth representation of $G(\mathcal{K})$ on a $\mathbf{C}$-vector space has a *local character expansion*: in some neighborhood of $0 \in \mathrm{Lie}(G) = \mathrm{Lie}(G)(\mathcal{K})$, this character can be expressed as a linear combination of the Fourier transforms of nilpotent orbital integrals. In particular, the $G(\mathcal{K})$-orbits of nilpotent elements in $\mathrm{Lie}(G) = \mathrm{Lie}(G)(\mathcal{K})$ play an important role in the harmonic analysis on the p-adic group $G(\mathcal{K})$.

The parametrization of nilpotent orbits obtained in [DeB02] contributed to DeBacker's proof of a conjecture of Hales, Moy and Prasad relating the depth of a smooth irreducible representation to the range of validity of its local character expansion. The interested reader may consult the introduction to [DeB02] for references and further details.

We note that DeBacker's parametrization of nilpotent orbits received some further study in [Nev11].

8.2. **Balanced elements and DeBacker's parametrization.** For a give parahoric $\mathcal{P}$ and a nilpotent element $X_0 \in \mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k)$, the parametrization of DeBacker mentioned in the preceding section is achieved – for "large enough" residue characteristic – by assigning to X_0 the nilpotent $G(\mathcal{K})$-orbit of minimal dimension meeting the coset $X_0 + \mathrm{Lie}(\mathcal{P})^+ \subset \mathrm{Lie}(\mathcal{P})$, where $\mathrm{Lie}(\mathcal{P})^+$ is the inverse image of $\mathrm{Lie}(R_u\mathcal{P}_k)$ under the quotient mapping $\mathrm{Lie}(\mathcal{P}) \to \mathrm{Lie}(\mathcal{P}_k) = \mathrm{Lie}(\mathcal{P}) \otimes_{\mathcal{A}} k$.

Under some additional assumptions, we are going to show that a balanced nilpotent element $\mathcal{X} \in \mathrm{Lie}(\mathcal{P})$ lifting X_0 lies in the orbit of smallest dimension meeting the coset $X_0 + \mathrm{Lie}(\mathcal{P})^+$. We begin with some preliminary results.

Let H be a linear algebraic group over a field $\mathcal{F}$. Suppose that the unipotent radical $R = R_uH$ is defined over $\mathcal{F}$ and that H has a Levi decomposition – i.e. there is a closed $\mathcal{F}$-subgroup $M \subset H$ such that the restriction of the quotient mapping $\pi : H \to H/R$ induces an isomorphism $\pi_{|M} : M \simeq H/R$. Let $X_0 \in \mathrm{Lie}(H/R_uH)$ and let $X \in \mathrm{Lie}(M)$ be the unique element with $d\pi(X) = X_0$.

Suppose that

$$\mathrm{Lie}(H) = L^0 \supset L^1 \supset L^2 \cdots \supset L^d = 0$$

is a filtration of $\mathrm{Lie}(H)$ by H-invariant subspaces for which R acts trivially on the quotient L^i/L^{i+1} for each $i = 1, \dots, d-1$.

Proposition 8.2.1. *Let* $X_0 \in \mathrm{Lie}(H/R_uH)$ *and let* $X \in \mathrm{Lie}(M)$ *be the unique element with* $d\pi(X) = X_0$. *For each* $Y \in d\pi^{-1}(X_0)$, *we have the inequality*

$$\sum_{i=0}^{d} \ker(\overline{\mathrm{ad}(X)} : L^i/L^{i+1} \to L^i/L^{i+1}) \geqslant \dim \mathfrak{c}_{\mathrm{Lie}(H)}Y,$$

where $\overline{\mathrm{ad}(X)}$ *is the mapping induced by* $\mathrm{ad}(X)$.

Proof. Let $Y \in d\pi^{-1}(X_0)$, and notice that $\mathrm{ad}(X)$ and $\mathrm{ad}(Y)$ induce the same mapping on the associated graded space $\mathrm{gr}(\mathrm{Lie}(H)) = \bigoplus_{i=0}^{d-1} L^i/L^{i+1}$.

Now the Proposition follows from the observation that the dimension of the kernel of the mapping $\mathrm{gr}(\mathrm{ad}(Y))$ exceeds the dimension of $\ker(\mathrm{ad}(Y)) = \mathfrak{c}_{\mathrm{Lie}(H)}(Y)$. □

Corollary 8.2.2. *Let* $X_0 \in \mathrm{Lie}(H/R_uH)$ *and let* $X \in \mathrm{Lie}(M)$ *be the unique element with* $d\pi(X) = X_0$. *Suppose that* $\Phi : \mathrm{SL}_{2,\mathcal{F}} \to M$ *is an* $\mathcal{F}$*-homomorphism, and that – in the notation of section 3.3 – we have* $X = d\phi(E_1)$. *If* $\mathrm{Lie}(H)$ *is a completely reducible* $\mathrm{SL}_{2,\mathcal{F}}$*-module, then*

$$\dim \mathfrak{c}_{\mathrm{Lie}(H)}(X) \geqslant \dim \mathfrak{c}_{\mathrm{Lie}(H)}(Y)$$

for all $Y \in d\pi^{-1}(X_0)$.

Proof. We *choose* a filtration

$$\mathrm{Lie}(H) = L^0 \supset L^1 \supset \cdots \supset L^d = 0$$

of $\mathrm{Lie}(H)$ as a module for H for which R acts trivially on each quotient L^i/L^{i+1}; such a filtration exists since the unipotence of R means that $V^R \neq \{0\}$ for any R-module $V \neq \{0\}$.

The assumption of complete reducibility implies that – as modules for $SL_{2,\mathcal{F}}$ –

$$\mathrm{Lie}(H) \simeq \bigoplus_{i=0}^{d-1} L^i/L^{i+1}.$$

Thus the dimension of the centralizer in $\mathrm{Lie}(H)$ of $X = d\phi(E_1)$ is precisely

$$\sum_{i=0}^{d-1} \dim\ker(\overline{\mathrm{ad}\, X} : L^i/L^{i+1} \to L^i/L^{i+1});$$

now the result follows from Proposition 8.2.1. □

Now suppose that G splits over an unramified extension of $\mathcal{K}$, and thus that $G = \mathcal{G}_{\mathcal{K}}$ for a reductive group scheme $\mathcal{G}$ over $\mathcal{A}$. Moreover, suppose that conditions (SG1) and (SG2) of section 4.1 hold for $\mathcal{G}$.

We now state the main result relating balanced nilpotent sections with DeBacker's description of nilpotent orbits.

Let $\mathcal{P}$ be a parahoric group scheme with generic fiber $\mathcal{P}_{\mathcal{K}} = G$, and let $X_0 \in \mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k)$ be a nilpotent element. Using Theorem 6.2.1 – i.e. the main result of [McN18] – we find a reductive $\mathcal{A}$-subgroup scheme $\mathcal{M} \subset \mathcal{P}$ such that $\mathcal{M}_k$ is a Levi factor of the special fiber $\mathcal{P}_k$. Since $\mathrm{Lie}(\mathcal{M}_k)$ may be identified with $\mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k)$, we may and will view X_0 as an element of $\mathrm{Lie}(\mathcal{M}_k)$. Now using Theorem 1.3.1 we find a balanced triple $(\mathcal{X}, \mathcal{T}, \phi)$ for $\mathcal{M}$; thus $\mathcal{X} \in \mathrm{Lie}(\mathcal{M})$ is a nilpotent section whose image in $\mathrm{Lie}(\mathcal{M}_k)$ is X_0 and the $\mathcal{A}$-homomorphism $\phi : \mathbf{G}_m \to \mathcal{M}$ has the property that ϕ_k is a cocharacter of $\mathcal{M}_k$ associated with $\mathcal{X}_k$ and $\phi_{\mathcal{K}}$ is a cocharacter of $\mathcal{M}_{\mathcal{K}}$ associated with $\mathcal{X}_{\mathcal{K}}$.

Recall from Proposition 7.1.2 that $\mathrm{Lie}(\mathcal{P})_+$ is the kernel of the natural mapping

$$\mathrm{Lie}(\mathcal{P}) \to \mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k).$$

Proposition 8.2.3. *Suppose that* $p > 2h-2$. *Then the* $G(\mathcal{K})$*-orbit of* $\mathcal{X}_{\mathcal{K}}$ *is the rational nilpotent orbit of smallest dimension which intersects the coset* $\mathcal{X} + \mathrm{Lie}(\mathcal{P})_+$ *non-trivially.*

Proof. Since G is a geometrically standard reductive group over $\mathcal{K}$, the centralizer in G of an element of $\mathrm{Lie}(G)$ is smooth. Thus, it suffices to argue for $\mathcal{Y} \in \mathcal{X} + \mathrm{Lie}(\mathcal{P})_+$ that

$$(\sharp) \quad \dim_{\mathcal{K}} \mathfrak{c}_{\mathrm{Lie}(G)}(\mathcal{Y}_{\mathcal{K}}) \leqslant \mathfrak{c}_{\mathrm{Lie}(G)}(\mathcal{X}_{\mathcal{K}})$$

.

First, Chevalley's upper semi-continuity theorem [EGAIV_{III}, §13.1.3] implies that

$$(\flat) \quad \dim_{\mathcal{K}} \mathfrak{c}_{\mathrm{Lie}(G)}(\mathcal{Y}_{\mathcal{K}}) \leqslant \dim \mathfrak{c}_{\mathrm{Lie}(\mathcal{P}_k)}(\mathcal{Y}_k).$$

Now, in view of our assumption on p, we may proceed as in the proof of Theorem 7.4.1 to find an $\mathcal{A}$-homomorphism $\Phi : SL_{2,\mathcal{A}} \to \mathcal{M}$ with $d\Phi(E) = \mathcal{X}$ and $\Phi_{|\mathscr{S}} = \phi$ such that $\mathrm{Lie}(\mathcal{P}_k)$ is a restricted semisimple module for $SL_{2,k}$.

Now Corollary 8.2.2 shows that

$$(*) \quad \dim_k \mathfrak{c}_{\mathrm{Lie}(\mathcal{P}_k)}(\mathcal{Y}_k) \leqslant \dim_k \mathfrak{c}_{\mathrm{Lie}(\mathcal{P}_k)}(\mathcal{X}_k).$$

Combining $(\flat)$ and $(*)$ now implies $(\sharp)$.

Finally, suppose that $\mathcal{Y} = \mathcal{X} + \mathcal{Z}$ for $\mathcal{Z} \in \mathrm{Lie}(\mathcal{P})_+$ and that equality holds in $(\sharp)$. Since G is a geometrically standard reductive group, the identity component of $C_G(\mathcal{Y}_{\mathcal{K}})$ is a smooth group scheme. Thus Proposition 6.3.1 together with $(\sharp)$ imply that $\mathcal{Y}$ is a balanced nilpotent section. Since the image of $\mathcal{Y}$ in $\mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k) = \mathrm{Lie}(\mathcal{P})/\mathrm{Lie}(\mathcal{P})_+$ coincides with that of $\mathcal{X}$, it follows from Corollary 7.4.2 that $\mathcal{X}$ and $\mathcal{Y}$ are conjugate by an element of $\mathcal{P}(\mathcal{A})$, and the result follows.

□

8.3. **Rational nilpotent elements on the generic fiber realized via balanced nilpotent sections.** In this section we prove under some additional conditions that each $\mathcal{K}$-rational nilpotent element $X_1 \in \mathrm{Lie}(G)$ may be written as $X_1 = \mathcal{X}_{\mathcal{K}}$ for a balanced nilpotent section $\mathcal{X} \in \mathrm{Lie}(\mathcal{P})$ for some parahoric group scheme $\mathcal{P}$ having generic fiber $\mathcal{P}_{\mathcal{K}} = G$. We give essentially the argument of [DeB02, Lemma 4.5.3]; Debacker attributes this argument to Gopal Prasad.

Here is the precise formulation of our result.

Theorem 8.3.1. *Write p for the characteristic of k, and suppose that $p > 2h - 2$. Let $X_1 \in \mathrm{Lie}(G)$ be nilpotent. Then there is a parahoric group scheme $\mathcal{P}$ with generic fiber $\mathcal{P}_{\mathcal{K}} = G$ and a nilpotent section $\mathcal{X} \in \mathrm{Lie}(\mathcal{P})$ balanced for the action of $\mathcal{P}$ with $\mathcal{X}_{\mathcal{K}} = X_1$.*

We first establish the following preliminary result:

Proposition 8.3.2. *Let $X_1 \in \mathrm{Lie}(G)$ be nilpotent, and let ϕ_1 be a cocharacter of G associated with X_1. Write p for the characteristic of $\mathcal{K}$. If $(X_1)^{[p]} = 0$* [5]*, then there is a parahoric group scheme $\mathcal{P}$ with generic fiber $\mathcal{P}_{\mathcal{K}} = G$ and an $\mathcal{A}$-homomorphism $\Phi : \mathrm{SL}_{2,\mathcal{A}} \to \mathcal{P}$ for which*

(a) If $\phi = \Phi_{|\mathscr{S}}$, then $\phi_{\mathcal{K}} = \phi_1$, and
(b) if $\mathcal{X} = d\Phi(E)$, then $\mathcal{X}_{\mathcal{K}} = X_1$.

Proof of Proposition 8.3.2. Let $\Phi_1 : \mathrm{SL}_{2,\mathcal{K}} \to G = \mathcal{G}_{\mathcal{K}}$ be the unique homomorphism with $d\Phi_1(E_1) = X_1$ and $\Phi_{1|\mathscr{S}_{\mathcal{K}}} = \phi_1$ as in Theorem 3.3.1.

Here we use the argument of [DeB02, Lemma 4.5.3]. Thus, we write $\mathcal{K}_u$ for the maximal unramified extension of $\mathcal{K}$, we consider the Bruhat-Tits building $\mathcal{B}_u$ of $G_{\mathcal{K}_u}$, and we write $\Gamma = \mathrm{Gal}(\mathcal{K}_u/\mathcal{K}) = \mathrm{Gal}(\bar{k}/k)$.

Now consider the action of $\mathrm{SL}_2(\mathcal{A}) \times \Gamma$ on $\mathcal{B}_u$, where the action of $\mathrm{SL}_2(\mathcal{A})$ is determined by Φ_1. According to [Tit79, §2.3.1], there is a fixed point x for this action. Since x is fixed by Γ, in fact x is in the Bruhat-Tits building of G, and hence its (connected) stabilizer in $G(\mathcal{K})$ is the group $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$-points of a parahoric group scheme $\mathcal{P}$ with generic fiber G. Since x is fixed by $\mathrm{SL}_2(\mathcal{A})$, we have $\Phi_1(\mathrm{SL}_2(\mathcal{A}_u)) \subset \mathcal{P}(\mathcal{A}_u)$ where $\mathcal{A}_u$ is the integral closure of $\mathcal{A}$ in $\mathcal{K}_u$.

Now, the group scheme SL_2 is étoffe [BT84, §1.7], hence it follows from [BT84, (1.7.1)] that there is an $\mathcal{A}_u$-homomorphism $\Psi : \mathrm{SL}_{2,\mathcal{A}_u} \to \mathcal{P}_{\mathcal{A}_u}$ such that compatible with the mapping Φ_1 on $\mathcal{A}_u$-points. Since x is fixed by Γ, it follows via étale descent that Ψ arises by base change from an $\mathcal{A}$-homomorphism $\Phi : \mathrm{SL}_{2,\mathcal{A}} \to \mathcal{P}$. It is immediate that Φ satisfies conditions (a) and (b). □

Proof of Theorem 8.3.1. According to Proposition 3.4.2(a), the assumption $p > 2h - 2$ implies that $(\mathcal{X}_{\mathcal{K}})^{[p]} = 0$. Now let $\Phi : \mathrm{SL}_{2,\mathcal{A}} \to \mathcal{P}$ as in the conclusion of Proposition 8.3.2, and let $\phi : \mathbf{G}_{m,\mathcal{A}} \to \mathcal{P}$ be the $\mathcal{A}$-homomorphism obtained by restricting Φ_1 to the diagonal torus of $\mathrm{SL}_{2,\mathcal{A}}$. Since $p > 2h - 2$, and Proposition 3.4.3 shows that $\mathrm{Lie}(G)(\phi_{\mathcal{K}}; i) \neq 0 \implies |i| < p$ and hence that $\mathrm{Lie}(\mathcal{P})(\phi; i) \neq 0 \implies |i| < p$. Thus, the hypothesis of Proposition 5.3.2 are satisfied; we deduce that $\mathrm{Lie}(\mathcal{P})_k$ is a semisimple module for $\mathrm{SL}_{2,k}$, that $\mathrm{Lie}(\mathcal{P})_{\mathcal{K}}$ is a semisimple module for $\mathrm{SL}_{2,\mathcal{K}}$, and that $\dim_k \ker(\mathcal{X}_k) = \dim_{\mathcal{K}}(\mathcal{X}_{\mathcal{K}})$. Arguing as in the proof Theorem 6.4.3, we see that $\mathcal{X}$ is balanced for the action of $\mathcal{P}$, as required. □

9. Smoothness of centralizers

Remark 9.0.1. Let $\mathcal{G}$ be a reductive group scheme over $\mathcal{A}$, and let $\mathcal{X} \in \mathrm{Lie}(\mathcal{G})$ be a balanced nilpotent section. In [McN08], we claimed that the centralizer $C = C_{\mathcal{G}}(X)$ is a smooth group scheme over $\mathcal{A}$. That claim is not supported by the given arguments.

[5] Recall that if the characteristic of $\mathcal{K}$ is zero, we simply define $(X_1)^{[p]} = 0$.

The paper [McN08] had two main results: Theorem A and Theorem B. Theorem B concerns a comparison of the component groups of the centralizers $C_{\mathcal{G}_{\mathcal{K}}}(X_{\mathcal{K}})$ and $C_{\mathcal{G}_k}(X_k)$; the proof of *loc. cit.* requires on the smoothness of C over $\mathcal{A}$, so the given proof of Theorem B is inadequate.

On the other hand, the proof of Theorem A given in [McN08] depended on the smoothness of the identity component C^0. While that smoothness is also not known, one may simply replace C^0 by the smooth group scheme obtained using the result Theorem 2.2.4 of Conrad - or more precisely, Theorem 2.2.5 - to obtain a proof of Theorem A by the arguments of *loc. cit.*

Since the present paper contains improvements to some of the intermediate results used in [McN08], we give another proof of [McN08, Theorem A] in the next section.

9.1. **Levi factors of the centralizer of a nilpotent element.** Let $\mathcal{G}$ be a reductive $\mathcal{A}$-group scheme and suppose that the conditions (SG1) and (SG2) of section 4.1 hold for $\mathcal{G}$.

Theorem 9.1.1. *Let $\mathcal{X} \in \mathrm{Lie}(\mathcal{G})$ be a balanced nilpotent section. Then the geometric root datum of the reductive quotient of the connected centralizer $C^0_{\mathcal{G}_{\mathcal{K}}}(\mathcal{X}_{\mathcal{K}})$ is the same as the geometric root datum of the reductive quotient of the connected centralizer $C^0_{\mathcal{G}_k}(\mathcal{X}_k)$.*

Proof. Let $X_0 = \mathcal{X}_k \in \mathrm{Lie}(\mathcal{G}_k)$. According to Theorem 4.5.2, we may choose a balanced triple $(\mathcal{Y}, \mathscr{S}, \psi)$ with $\mathcal{Y}_k = X_0$. It follows from Corollary 7.3.2 that $\mathcal{X}$ and $\mathcal{Y}$ are conjugate by an element of $\mathcal{G}(\mathcal{A})$. Thus a suitable $\mathcal{G}(\mathcal{A})$-conjugate of ψ yields an $\mathcal{A}$-homomorphism $\phi : \mathbf{G}_m \to \mathcal{G}$ such that ϕ_k is a cocharacter associated to $\mathcal{X}_k$ and $\phi_{\mathcal{K}}$ is a cocharacter associated to $\mathcal{X}_{\mathcal{K}}$.

Now let $C = C_{\mathcal{G}}(X)$ be the scheme-theoretic centralizer of the section $\mathcal{X}$ in $\mathcal{G}$, and let $\mathcal{H} \subset C$ be the locally closed subgroup scheme of Theorem 2.2.5. Recall that $\mathcal{H}$ is smooth and of finite type over $\mathcal{A}$, that $\mathcal{H}_{\mathcal{K}} = C^0_{\mathcal{G}_{\mathcal{K}}}(\mathcal{X}_{\mathcal{K}})$ and that $\mathcal{H}_k = C^0_{\mathcal{G}_k}(\mathcal{X}_k)$.

It is clear by construction of $\mathcal{H}$ that $\mathcal{H}$ is normalized by the action of the image of ϕ. Set $\mathcal{M} = C_{\mathcal{H}}(\phi)$ for the centralizer subgroup scheme of $\mathcal{H}$. According to [$\mathrm{SGA3}_{\mathrm{II}}$, Exp. XI, Cor 5.3], $\mathcal{M}$ is a closed subgroup scheme of $\mathcal{H}$ which is smooth over $\mathcal{A}$. Moreover, $\mathcal{M}$ is affine over $\mathcal{A}$, e.g. by Proposition 2.2.2.

Now, $\mathcal{M}_{\mathcal{K}}$ is the centralizer in $C^o_{\mathcal{G}_{\mathcal{K}}}(\mathcal{X}_{\mathcal{K}})$ of the image of $\phi_{\mathcal{K}}$ and $\mathcal{M}_k$ is the centralizer in $C^o_{\mathcal{G}_k}(\mathcal{X}_k)$ of the image of ϕ_k. Thus it follows from Proposition 3.2.2 that for $\mathcal{F} \in \{\mathcal{K}, k\}$, $\mathcal{M}_{\mathcal{F}}$ is a Levi factor of $C^0_{\mathcal{G}_{\mathcal{F}}}(\mathcal{X}_{\mathcal{F}})$.

In particular, $\mathcal{M}$ is a smooth and affine $\mathcal{A}$-group scheme with connected and reductive fibers, hence $\mathcal{M}$ is reductive [$\mathrm{SGA3}_{\mathrm{III}}$, Exp. XIX Defn 2.7]. Now [$\mathrm{SGA3}_{\mathrm{III}}$, Exp. XXII Prop. 2.8] shows that the root data of the fibers of $\mathcal{M}$ must be the same, at least geometrically. This completes the proof. □

We now quickly explain why the previous result provides a proof of [McN08, Theorem A]. Write $E = \overline{\mathcal{K}}$ and $F = \overline{k}$ for respective algebraic closures of $\mathcal{K}$ and k. Using the Bala-Carter Theorem, we may identify the set of nilpotent orbits of $\mathcal{G}_E$ in $\mathrm{Lie}(\mathcal{G}_E)$ with the set of nilpotent orbits of $\mathcal{G}_F$ in $\mathrm{Lie}(\mathcal{G}_F)$.

Suppose that the orbits of the nilpotent elements $X_1 \in \mathrm{Lie}(\mathcal{G}_E)$ and $X_0 \in \mathrm{Lie}(\mathcal{G}_F)$ are the same under the Bala-Carter identification, let C_E be the connected centralizer of X_E in G_E, and let C_F be the connected centralizer of X_F in G_F.

The statement of [McN08, Theorem A] was the following:

Corollary 9.1.2. *The root datum of the reductive quotient of C_E may be identified with the root datum of the reductive quotient of C_F.*

Proof. We may and will replace $\mathcal{A}$ by an unramified extension and hence suppose that the residue field $k = F$ is algebraically closed. Apply Theorem 4.5.2 to find a balanced triple $(\mathcal{Y}, \mathscr{S}, \phi)$. for which $X_0 = \mathcal{Y}_k$. Then $\mathcal{Y}_k$ is distinguished in $C_{\mathcal{G}}(\mathscr{S})_k$ and $\mathcal{Y}_{\mathcal{K}}$ is distinguished in $C_{\mathcal{G}}(\mathscr{S})_{\mathcal{K}}$ by condition (B4) in the definition of a *balanced triple*; see section 4.5.

Write $P = P(\phi)$ for the parabolic subgroup scheme of the centralizer $\mathcal{M} = C_{\mathcal{G}}(\mathscr{S})$ determined by the $\mathcal{A}$-morphism $\phi : \mathbf{G}_m \to \mathcal{M}$; see Proposition 2.4.1 and Proposition 2.4.3. Since the Bala-Carter datum of X_0 is the pair $(\mathcal{M}_k, P_k)$ and that of X_1 is the pair $(\mathcal{M}_{\mathcal{K}}, P_{\mathcal{K}})$, it follows that $\mathcal{Y}_{\mathcal{K}}$ is geometrically conjugate to X_1.

The Corollary now follows by applying Theorem 9.1.1 to the balanced nilpotent section Y. □

9.2. **Tori of centralizers.** Keep the assumptions of the preceding section; thus $\mathcal{G}$ is a reductive $\mathcal{A}$-group scheme and the conditions (SG1) and (SG2) of section 4.1 hold for $\mathcal{G}$.

The results of the previous section permit us to prove the following:

Proposition 9.2.1. *Let $\mathcal{X} \in \operatorname{Lie}(\mathcal{G})$ be a balanced nilpotent section. Then the centralizer $C_{\mathcal{G}_{\mathcal{K}}}(\mathcal{X}_{\mathcal{K}})$ has a maximal torus which splits over an unramified extension of $\mathcal{K}$.*

Proof. Indeed, let $\mathcal{H}$ be the locally closed, smooth subscheme of the centralizer $C_{\mathcal{G}}(\mathcal{X})$ as in Theorem 2.2.5. Then Theorem 9.1.1 shows that there is a closed subgroup scheme $\mathcal{M} \subset \mathcal{M}$ such that $\mathcal{M}_{\mathcal{K}}$ is a Levi factor of $\mathcal{H}_{\mathcal{K}} = C_{\mathcal{G}_{\mathcal{K}}}(\mathcal{X}_{\mathcal{K}})$ and $\mathcal{M}_k$ is a Levi factor of $\mathcal{H}_k = C_{\mathcal{G}_k}(\mathcal{X}_k)$.

In particular, $\mathcal{M}$ is reductive over $\mathcal{A}$ and so by Corollary 2.3.2 the group scheme $\mathcal{M}$ has a maximal torus over $\mathcal{A}$. It follows from Theorem 2.3.1 that this maximal torus splits over an unramified extension of $\mathcal{A}$, and the result follows. □

The Proposition shows that if $X_1 \in \operatorname{Lie}(\mathcal{G}_{\mathcal{K}})$ is a nilpotent element such that the centralizer $C_{\mathcal{G}_{\mathcal{K}}}(X_1)$ has no maximal torus which is unramified, then X_1 can not be realized as the value at the generic fiber of any balanced nilpotent section of $\operatorname{Lie}(\mathcal{G})$. On the other hand, Theorem 8.3.1 shows that – at least when $p > 2h-2$ – the nilpotent element X_1 is the value at the generic fiber of *some* balanced nilpotent section $\mathcal{X} \in \operatorname{Lie}(\mathcal{P})$ for *some* parahoric group scheme $\mathcal{P}$.

We conclude this section with an example demonstrating that - even when $\mathcal{G}_{\mathcal{K}}$ is split - there are nilpotent elements $X_1 \in \mathfrak{g}$ for which no maximal torus of $C_G(X_1)$ splits over an unramified extension of $\mathcal{K}$.

Recall that any separable field extension $\mathcal{K} \subset \mathcal{L}$ determines a quadratic form $Q = N_{\mathcal{L}/\mathcal{K}}$ over $\mathcal{K}$ on the 2 dimensional $\mathcal{K}$-vector space $W = \mathcal{L}$. For a field extension $\mathcal{K}_1$ of $\mathcal{K}$, the form $Q_{\mathcal{K}_1}$ is split if and only if $\mathcal{L} \subset \mathcal{K}_1$. In particular, the reductive group $O(W, Q)_{\mathcal{K}_1}$ has a $\mathcal{K}_1$-split torus only when $\mathcal{L} \subset \mathcal{K}_1$.

Proposition 9.2.2. *Suppose that the characteristic of K is not 2. Let $G = Sp_{4r}$ be the split symplectic group of rank $2r$ over $\mathcal{K}$ for some $r \geqslant 1$. For any 2 dimensional quadratic space (W, Q), there is a nilpotent element $Y = Y_Q \in \operatorname{Lie}(G)$ for which the reductive quotient of the centralizer $C^0_G(Y)$ is isomorphic to $SO(W, Q)$. In particular, there are nilpotent $Y \in \operatorname{Lie}(G)$ for which no maximal torus of $C_G(Y)$ splits over an unramified extension of $\mathcal{K}$.*

Sketch. Let (V_1, β) be a pair of a vector space with $\dim V_1 = 2r$, and a non-degenerate alternating form β. Write γ for the bilinear form on W determined by the quadratic form Q.

Form the vector space $V = V_1 \otimes W$ and consider the non-degenerate alternating form $\delta = \beta \otimes \gamma$ on V. Let $Y_1 \in \mathfrak{sp}(V_1, \beta)$ be a regular nilpotent element, and consider the element

$$Y = Y_1 \otimes 1_W \in \operatorname{Lie}(G) = \mathfrak{g} = \mathfrak{sp}(V, \delta).$$

We are going to describe a cocharacter associated to Y; see section 3.2. For this, let $\lambda_1 : \mathbf{G}_m \to Sp(V_1, \beta)$ be a cocharacter associated to Y_1. Recall that Y_1 is also regular when viewed as an element of $SL(V_1)$, and thus:

(♡) λ_1 is associated to Y_1 when viewed as a cocharacter of $SL(V_1)$.

Consider the cocharacter $\lambda : \mathbf{G}_m \to Sp(V, \delta)$ for which $\lambda(t) = \lambda_1(t) \otimes 1_W$. We are going to argue that λ is associated with Y. It will suffice to check this after extending scalars, so assume for the moment that the quadratic space (W, Q) is *split* (i.e. is *hyperbolic*). Fix a hyperbolic basis e, f of W, and consider the $\mathbf{Z}/2\mathbf{Z}$-grading of V for which $V_{0+2\mathbf{Z}} = V_1 \otimes e$ and $V_{1+2\mathbf{Z}} = V_1 \otimes f$. Since the $V_{i+2\mathbf{Z}}$

are isotropic for δ, this grading amounts to a homomorphism $\phi : \mu_2 \to G = \mathrm{Sp}(V, \delta)$. Write M for the connected centralizer $C^0_G(\phi)$; then

$$M = \{(g, h) \in \mathrm{GL}(V_1) \times \mathrm{GL}(V_1) \mid \det g \cdot \det h = 1\}.$$

Evidently λ takes values in the derived group $\mathrm{SL}(V_1) \times \mathrm{SL}(V_1)$ of M. According to ($\heartsuit$) above, λ is associated to Y in M; it now follows from Theorem A.5 that λ is associated to Y in G.

Now return to the general case (in which (W, Q) need not by split). If $H = C^0_G(Y)$ is the connected centralizer of Y, one knows that H has a Levi decomposition, with Levi factor given by the centralizer $C_H(\lambda)$ of the image of λ.

It is easy to see that $C_H(\lambda)$ identifies with $\mathrm{SO}(W, \gamma) = \mathrm{SO}(W, Q)$, proving the first assertion of the Proposition. The remaining assertion now follows by taking $Y = Y_Q$ where Q is the norm form $N_{\mathcal{L}/\mathcal{K}}$ of some totally ramified quadratic field extension $\mathcal{L}$ of $\mathcal{K}$. □

Appendix A. Nilpotent elements and subgroups of type $C(\mu)$

Let G be a connected and reductive group over the field $\mathcal{F}$, and let $M \subset G$ be a subgroup of G of type $C(\mu)$ – see section 6.1. By definition, $M = C^o_G(\psi)$ where $\psi : \mu_n \to G$ is an $\mathcal{F}$-homomorphism.

In this appendix, we consider the cocharacters of M and of G which are associated to a nilpotent elment $X \in \mathrm{Lie}(M)$. Since $X \in \mathrm{Lie}(M)$, the image of the μ-homomorphism ψ is contained in $C_G(X)$.

Proposition A.1. *Suppose that $\mathcal{F}$ is algebraically closed. If H is linear algebraic group for which H^0 is a reductive group and if $\psi : \mu_n \to H$ is a homomorphism, there is a maximal torus T of H normalized by the image of ψ.*

Proof. Write $n = p^a \cdot m$ with $\gcd(p, m) = 1$; thus $\mu_n \simeq \mu_{p^a} \times \mu_m$. Since μ_{p^a} is connected, the image $\psi(\mu_{p^a})$ is contained in H^0. It was proved in [McN18, Theorem 3.4.1] that the image $S = \psi(\mu_{p^a})$ lies in some maximal torus T of H^0. Note that $C^0_H(S)$ is reductive and contains a maximal torus of H; thus it suffices to complete the proof after replacing H by $C_H(S)$. Since now the image S is contained in every maximal torus of H, it is enough to argue that the image $S_1 = \psi(\mu_m)$ normalizes some maximal torus of H. Since $\mathcal{F}$ is algebraically closed, that now follows from [Ste68, Theorem 7.5]. □

Proposition A.2. *Let H be a linear algebraic group over $\mathcal{F}$ satisfying the following two conditions:*

(a) the unipotent radical $R_u H$ is defined and split over $\mathcal{F}$, and
(b) H has a Levi decomposition; i.e. there is a reductive $\mathcal{F}$-subgroup $M \subset H$ for which $\pi_{|M} : M \to H/R_u H$ is an isomorphism, where $\pi : H \to H/R_u H$ is the quotient mapping.

If $D \subset H$ is a subgroup scheme of multiplicative type, then there is $u \in (R_u H)(\mathcal{F})$ for which $uDu^{-1} \subset M$.

Proof. Write $U = R_u H$, and write $\overline{D} \subset H/U$ for the image of D. It follows from [$\mathrm{SGA3}_{II}$, Exp. XVII Prop 4.3.1] that the restriction of the quotient mapping $\pi : H \to H/U$ determines an isomorphism $\pi_{|D} : D \xrightarrow{\sim} \overline{D}$. In particular, the group scheme $E = \pi^{-1}(\overline{D})$ is an *extension* of the group scheme $\overline{D}$ of multiplicative type by the connected and $\mathcal{F}$-split unipotent group U.

Write $\gamma : R/U \to M$ for the inverse of the isomorphism $\pi_{|M} : M \to H/U$, and let $D_1 = \gamma(\overline{D})$. Then $D_1 \subset E$ and $D_1 \subset M$. It now follows from [$\mathrm{SGA3}_{II}$, Exp. XVII Thm 5.1.1] applied to the extension E that D and D_1 are conjugate by an element of $U(\mathcal{F})$, as required. □

Proposition A.3. *Let X as above.*

(a) The image of ψ centralizes some cocharacter ϕ associated with X in G.
(b) If $\mathcal{F}$ is algebraically, the image of ψ normalizes some maximal torus of $C^o_G(X)$.

Proof. Write $C = C_G(X)$ and note that the image of ψ is contained in C. Recall that the unipotent radical U of C is defined and split over $\mathcal{F}$, and if ϕ is a cocharacter associated with X in G, the centralizer C_ϕ in C of the image of ϕ is a Levi factor of C. Now, after replacing ϕ by a $U(\mathcal{F})$-conjugate, Proposition A.2 shows that the image of ψ is contained in C_ϕ. Assertion (a) follows at once.

Since C_ϕ is a Levi factor, it contains a maximal torus of C. Now (b) follows from Proposition A.1. □

We require the following technical result, which is a slight generalization of [MS03, Lemma 24]. For the completeness and for the convenience of the reader, we repeat the proof.

Proposition A.4. *Suppose that $\mathcal{F}$ is algebraically closed, and let* H *be a linear algebra group over $\mathcal{F}$. Assume that H^0 is reductive, and let $\psi : \mu_n \to H$ be an $\mathcal{F}$-homomorphism. Assume that $S \subset H$ is a central torus in* H *which is normalized by the image of ψ and that $C_S^0(\psi)$ is a* maximal *central torus of $C_H^0(\psi)$. Then*

$$(C_H^0(\psi), C_H^0(\psi)) = C_{(H,H)}^0(\psi).$$

In particular, the identity component of the centralizer in the derived group $\mathrm{der}(H) = (H, H)$ *of the image of* ψ *is* semisimple.

Proof. Write $N = C_{(H,H)}^0(\psi)$. It is clear that $(C_H^0(\psi), C_H^0(\psi)) \subset N$, and it remains to argue the reverse inclusion. For that, it is enough to argue that N is semisimple. Indeed, we then have $N = (N, N)$, and since $N \subset C_H^0(\psi)$, we may deduce the required inclusion $N \subset (C_H^0(\psi), C_H^0(\psi))$.

Choose a maximal torus $T \subset H$ normalized by the image of ψ. The adjoint action of ψ yields $\mathbf{Z}/n\mathbf{Z}$-gradings

$$\mathrm{Lie}(H) = \bigoplus_{i \in \mathbf{Z}/n\mathbf{Z}} \mathrm{Lie}(H)(i), \quad \mathrm{Lie}((H,H)) = \bigoplus_{i \in \mathbf{Z}/n\mathbf{Z}} \mathrm{Lie}((H,H))(i) \quad \text{and} \quad \mathrm{Lie}(T) = \bigoplus_{i \in \mathbf{Z}/n\mathbf{Z}} \mathrm{Lie}(T)(i),$$

and we have $\mathrm{Lie}(C_T^0(\psi)) = \mathrm{Lie}(T)(0)$, $\mathrm{Lie}(C_H^0(\psi)) = \mathrm{Lie}(H)(0)$ and $\mathrm{Lie}(C_{(H,H)}^0(\psi)) = \mathrm{Lie}((H,H))(0)$.

It follows from [Spr98, (4.4.12)] that the differential at $(1,1)$ of the product mapping $\mu : T \times (H,H) \to H$ is surjective. Moreover, since $\mathrm{Lie}((H,H))$ contains each non-zero T-weight space of $\mathrm{Lie}(H)$, $\mathrm{Lie}(H)$ is the sum of $\mathrm{Lie}(T)$ and $\mathrm{Lie}((H,H))$. Since this product map respects the action of the image of ψ, we find that $d\mu_{(1,1)} : \mathrm{Lie}(T)(i) \oplus \mathrm{Lie}((H,H))(i) \to \mathrm{Lie}(H)(i)$ is surjective for each $i \in \mathbf{Z}/n\mathbf{Z}$. In particular,

$$d\mu_{(1,1)} : \mathrm{Lie}(T)(0) \oplus \mathrm{Lie}((H,H))(0) \to \mathrm{Lie}(H)(0)$$

is surjective. This surjectivity implies that μ restricts to a dominant morphism

$$\widetilde{\mu} : C_T^0(\psi) \times N \to C_H^0(\psi).$$

Since $C_T^0(\psi)$ normalizes N, the image is a subgroup. Since $C_H^0(\psi)$ is connected, $\widetilde{\mu}$ is surjective; thus $C_H^0(\psi) = C_T^0(\psi)N$.

The group N is reductive; let R denote its maximal central torus. Now, R is contained in each maximal torus of N; in particular R is contained in $C_{T_1}(\psi)$ for some maximal torus T_1 of H normalized by the image of ψ. Choosing $T = T_1$ in the preceding discussion, we find that $C_H^0(\psi) = C_{T_1}^0(\psi) \cdot N$. Thus we find that R is moreover central in $C_H^0(\psi)$. But we have assumed that $C_S^0(\psi)$ to be the maximal central torus of $C_H^0(\psi)$, so we find that $R \subset C_S^0(\psi) \cap N$.

Finally, $C_S^0(\psi)$ is contained in the center Z of H. Since $Z \cap (H, H)$ is finite – see [Spr98, (8.1.6)] – it follows that $C_S^0(\psi) \cap (H, H)$ is finite, hence also $C_S^0(\psi) \cap N$ is finite, as well. This proves that $R = 1$ so indeed N is semisimple, as required. □

Theorem A.5. *A cocharacter of* M *is associated with the nilpotent element* X *in* M *if and only if it is associated to* X *in* G*.*

Proof. In view of the conjugacy of associated cocharacters Proposition 3.2.2, the Theorem will follow if we argue that there is a cocharacter of M which is associated to X in both M and G.

When $M = L$ is a Levi factor of a parabolic of G, this conclusion is immediate from definitions, since we can find a reductive subgroup L_1 for which $X \in \mathrm{Lie}(L_1)$ is distinguished, and for which L_1 is a Levi factor of a parabolic of G and L_1 is a Levi factor of a parabolic of L.

If ϕ is a cocharacter of M, the condition that ϕ is associated to X in either G or M is unaffected by extension of scalars. Thus, to prove the Theorem, we may and will suppose that $\mathcal{F}$ is algebraically closed.

Recall that $M = C^o_G(\psi)$ for a homomorphism $\psi : \mu_n \to G$ for some $n \geqslant 2$. Fix a maximal torus S_0 of $C_M(X)$. If we now set $G_1 = C_G(S_0)$ and $M_1 = C_M(S_0)$, then G_1 is a Levi factor of a parabolic of G, M_1 is a Levi factor of a parabolic of M, $M_1 = C^0_{G_1}(\psi)$ is a subgroup of G_1 of type $C(\mu)$, and X is distinguished in $\mathrm{Lie}(M_1)$.

Since the conclusion of the Theorem is valid for Levi factors of parabolic subgroups, a cocharacter of M_1 associated to X in G_1 is associated to X in G, and a similar statement holds for M_1 and M. Thus in giving the proof, we may and shall replace G by G_1 and M by M_1 and so we suppose that X is distinguished in $\mathrm{Lie}(M)$.

According to Proposition A.3, we may choose a cocharacter ϕ associated to X which is centralized by the image of ψ. In particular, ϕ is a cocharacter of M. We are going to argue that ϕ is associated to X in M. In view of the conjugacy of associated cocharacters Proposition 3.2.2, this will complete the proof of the Theorem. Since X is distinguished in $\mathrm{Lie}(M)$, and since evidently $X \in \mathrm{Lie}(M)(\phi; 2)$, in order to argue that ϕ is associated to X, we only must argue that the image of ϕ lies in the derived group M.

For this, use Proposition A.3 to choose a maximal torus S of $C_G(X)$ which is normalized by the image of ψ. Now, X is distinguished in the Lie algebra of $H = C_G(S)$, so the image of ϕ is contained in $C^0_{(H,H)}(\psi)$. Since X is distinguished in $\mathrm{Lie}(M)$, and since the image of ψ normalizes S, it follows that $C_S(\psi)$ coincides with the connected center of M.

Now Proposition A.4 implies that $(C^0_H(\psi), C^0_H(\psi)) = C^0_{(H,H)}(\psi)$, so the image of ϕ lies in

$$(C^0_H(\psi), C^0_H(\psi)) \subset (M, M)$$

as required. □

Remark A.6. A result similar to Theorem A.5 was obtained in [MS03, Prop. 23] for "pseudo-Levi subgroups" M of G, though the result was only stated in *loc. cit.* for distinguished X. In general, the class of subgroups of type $C(\mu)$ is strictly larger than the class of pseudo-Levi subgroups – see the discussion in the introduction to [McN18]. The proof we gave here is basically that given in [MS03], except that we have used here the result Proposition A.2 for diagonalizable group schemes deduced from [$\mathrm{SGA3}_{\mathrm{II}}$, Exp. XVII Thm 5.1.1] rather than the result [Jan04, (11.24)], which is only valid for smooth linearly reductive groups.

References

[Adl98] Jeffrey D. Adler. "Refined anisotropic K-types and supercuspidal representations". In: *Pacific J. Math.* 185.1 (1998), pp. 1–32.

[BT84] F. Bruhat and J. Tits. "Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée". In: *Inst. Hautes Études Sci. Publ. Math.* 60 (1984), pp. 197–376.

[Car93] Roger W. Carter. *Finite groups of Lie type*. Wiley Classics Library. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication. John Wiley & Sons, Ltd., Chichester, 1993, p. 544.

[CG10] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry*. Modern Birkhäuser Classics. Reprint of the 1997 edition. Birkhäuser Boston, 2010, p. 495.

[DeB02] Stephen DeBacker. "Parametrizing nilpotent orbits via Bruhat-Tits theory". In: *Ann. of Math. (2)* 156.1 (2002), pp. 295–332.

[Don93] Stephen Donkin. "On tilting modules for algebraic groups". In: *Math. Z.* 212.1 (1993), pp. 39–60.

[$EGAIV_{III}$] A. Grothendieck. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III". In: *Inst. Hautes Études Sci. Publ. Math.* 28 (1966), p. 255.

[GY03] Wee Teck Gan and Jiu-Kang Yu. "Schémas en groupes et immeubles des groupes exceptionnels sur un corps local. I. Le groupe G_2". In: *Bull. Soc. Math. France* 131.3 (2003), pp. 307–358.

[HC99] Harish-Chandra. *Admissible invariant distributions on reductive* p*-adic groups*. Vol. 16. University Lecture Series. Preface and notes by Stephen DeBacker and Paul J. Sally, Jr. American Mathematical Society, Providence, RI, 1999, pp. xiv+97.

[Her13] Sebastian Herpel. "On the smoothness of centralizers in reductive groups". In: *Trans. Amer. Math. Soc.* 365.7 (2013), pp. 3753–3774.

[Jan03] Jens Carsten Jantzen. *Representations of algebraic groups*. 2nd. Vol. 107. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003, p. 576.

[Jan04] Jens Carsten Jantzen. "Nilpotent orbits in representation theory". In: *Lie theory*. Vol. 228. Progr. Math. Birkhäuser Boston, Boston, MA, 2004, pp. 1–211.

[Knu+98] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The book of involutions*. Vol. 44. American Mathematical Society Colloquium Publications. With a preface in French by J. Tits. American Mathematical Society, Providence, RI, 1998, p. 593.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*. Vol. 6. Oxford Graduate Texts in Mathematics. Translated from the French by Reinie Erné, Oxford Science Publications. Oxford University Press, Oxford, 2002, p. 576.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*. 2nd. Vol. 8. Cambridge Studies in Advanced Mathematics. Translated from the Japanese by M. Reid. Cambridge University Press, Cambridge, 1989, pp. xiv+320.

[McN00] George McNinch. "Filtrations and positive characteristic Howe duality". In: *Math. Z.* 235.4 (2000), pp. 651–685.

[McN03a] George McNinch. *Adjoint Jordan Blocks*. 2003. arXiv: 0207001 [math.RT].

[McN03b] George McNinch. "Sub-principal homomorphisms in positive characteristic". In: *Math. Z.* 244.2 (2003), pp. 433–455.

[McN04] George McNinch. "Nilpotent orbits over ground fields of good characteristic". In: *Math. Ann.* 329.1 (2004), pp. 49–85.

[McN05] George McNinch. "Optimal SL(2) homomorphisms". In: *Comment. Math. Helv.* 80.2 (2005), pp. 391–426.

[McN08] George McNinch. "The centralizer of a nilpotent section". In: *Nagoya Math. J.* 190 (2008), pp. 129–181.

[McN10] George McNinch. "Levi decompositions of a linear algebraic group". In: *Transform. Groups* 15.4 (2010), pp. 937–964.

[McN13] George McNinch. "On the descent of Levi factors". In: *Arch. Math. (Basel)* 100.1 (2013), pp. 7–24.

[McN14] George McNinch. "Levi factors of the special fiber of a parahoric group scheme and tame ramification". In: *Algebr. Represent. Theory* 17.2 (2014), pp. 469–479.

[McN18] George McNinch. "Reductive subgroup schemes of a parahoric group scheme". In: *Transf. Groups* to appear (2018).

[MP94] Allen Moy and Gopal Prasad. "Unrefined minimal K-types for p-adic groups". In: *Invent. Math.* 116.1-3 (1994), pp. 393–408.

[MS03] George McNinch and Eric Sommers. "Component groups of unipotent centralizers in good characteristic". In: *J. Algebra* 260.1 (2003). Special issue celebrating the 80th birthday of Robert Steinberg, pp. 323–337.

[MT07] George McNinch and Donna M. Testerman. "Completely reducible SL(2) homomorphisms". In: *Trans. Amer. Math. Soc.* 359.9 (2007), 4489–4510 (electronic).

[MT09] George McNinch and Donna M. Testerman. "Nilpotent centralizers and Springer isomorphisms". In: *J. Pure Appl. Algebra* 213.7 (2009), pp. 1346–1363.

[MT16] George McNinch and Donna M. Testerman. "Central subalgebras of the centralizer of a nilpotent element". In: *Proc. Amer. Math. Soc.* 144.6 (2016), pp. 2383–2397.

[Nev11] Monica Nevins. "On nilpotent orbits of SL_n and Sp_{2n} over a local non-Archimedean field". In: *Algebr. Represent. Theory* 14.1 (2011), pp. 161–190.

[Pom77] Klaus Pommerening. "Über die unipotenten Klassen reduktiver Gruppen". In: *J. Algebra* 49.2 (1977), pp. 525–536.

[Pom80] Klaus Pommerening. "Über die unipotenten Klassen reduktiver Gruppen. II". In: *J. Algebra* 65.2 (1980), pp. 373–398.

[Pre03] Alexander Premet. "Nilpotent orbits in good characteristic and the Kempf-Rousseau theory". In: *J. Algebra* 260.1 (2003). Special issue celebrating the 80th birthday of Robert Steinberg, pp. 338–366.

[PY06] Gopal Prasad and Jiu-Kang Yu. "On quasi-reductive group schemes". In: *J. Algebraic Geom.* 15.3 (2006). With an appendix by Brian Conrad, pp. 507–549.

[Sei00] Gary M. Seitz. "Unipotent elements, tilting modules, and saturation". In: *Invent. Math.* 141.3 (2000), pp. 467–502.

[Ser02] Jean-Pierre Serre. *Galois cohomology*. English. Springer Monographs in Mathematics. Translated from the French by Patrick Ion and revised by the author. Springer-Verlag, Berlin, 2002, p. 210.

[Ser96] Jean-Pierre Serre. "Exemples de plongements des groupes $PSL_2(\mathbf{F}_p)$ dans des groupes de Lie simples". In: *Invent. Math.* 124.1-3 (1996), pp. 525–562.

[$SGA3_I$] M. Demazure and A. Grothendieck. *Schémas en groupes (SGA 3). Tome I. Propriétés générales des schémas en groupes*. Ed. by P. Gille and P. Polo. Documents Mathématiques (Paris), 7. Séminaire de Géométrie Algébrique du Bois Marie 1962–64. Revised and annotated edition of the 1970 French original. Société Mathématique de France, Paris, 2011, p. 610.

[$SGA3_{II}$] M. Demazure and A. Grothendieck. *Schémas en groupes (SGA 3). Tome II. Groupes de type multiplicatif, et structure des schémas en groupes généraux.* Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153. Springer-Verlag, Berlin-New York, 1970, p. 654.

[$SGA3_{III}$] M. Demazure and A. Grothendieck. *Schémas en groupes (SGA 3). Tome III. Structure des schémas en groupes réductifs*. Ed. by P. Gille and P. Polo. Documents Mathématiques (Paris), 8. Séminaire de Géométrie Algébrique du Bois Marie 1962–64. Revised and annotated edition of the 1970 French original. Société Mathématique de France, Paris, 2011, p. l337.

[Spr98] T. A. Springer. *Linear algebraic groups*. 2nd. Modern Birkhäuser Classics. Birkhäuser, Boston, MA, 1998, p. 334.

[SS70] T. A. Springer and R. Steinberg. "Conjugacy classes". In: *Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)*. Lecture Notes in Mathematics, Vol. 131. Springer, Berlin, 1970, pp. 167–266.

[Ste68] Robert Steinberg. *Endomorphisms of linear algebraic groups*. Memoirs of the American Mathematical Society, No. 80. American Mathematical Society, Providence, R.I., 1968, p. 108.

[Tit79] J. Tits. "Reductive groups over local fields". In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*. Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., 1979, pp. 29–69.

Department of Mathematics, Tufts University, 503 Boston Avenue, Medford, MA 02155, USA
E-mail address: george.mcninch@tufts.edu, mcninchg@member.ams.org