

# REDUCTIVE SUBGROUP SCHEMES OF A PARAHORIC GROUP SCHEME

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ABSTRACT. Let  $K$  be the field of fractions of a complete discrete valuation ring  $\mathcal{A}$  with residue field  $k$ , and let  $G$  be a connected reductive algebraic group over  $K$ . Suppose  $\mathcal{P}$  is a parahoric group scheme attached to  $G$ . In particular,  $\mathcal{P}$  is a smooth affine  $\mathcal{A}$ -group scheme having generic fiber  $\mathcal{P}_K = G$ ; the group scheme  $\mathcal{P}$  is in general not reductive over  $\mathcal{A}$ .

If  $G$  splits over an unramified extension of  $K$ , we find in this paper a closed and reductive  $\mathcal{A}$ -subgroup scheme  $\mathcal{M} \subset \mathcal{P}$  for which the special fiber  $\mathcal{M}_k$  is a Levi factor of  $\mathcal{P}_k$ . Moreover, we show that the generic fiber  $M = \mathcal{M}_K$  is a subgroup of  $G$  which is geometrically of type  $C(\mu)$  – i.e. after a separable field extension,  $M$  is the identity component  $M = C_G^\circ(\phi)$  of the centralizer of the image of a homomorphism  $\phi : \mu_n \rightarrow H$ , where  $\mu_n$  is the group scheme of  $n$ -th roots of unity for some  $n \geq 2$ . For a connected and split reductive group  $H$  over any field  $\mathcal{F}$ , the paper describes those subgroups of  $H$  which are of type  $C(\mu)$ .

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## 1. INTRODUCTION

**1.1. The connected centralizer of a  $\mu$ -homomorphism.** Consider a field  $\mathcal{F}$  and a connected and reductive algebraic group  $G$  over  $\mathcal{F}$ . In sections 2 and 3 of this paper, we are going to describe a certain class of connected and reductive subgroups of  $G$  which we call the subgroups of type  $C(\mu)$ .

For  $n \geq 2$ , denote by  $\mu_n$  the group scheme of  $n$ -th roots of unity; over the field  $\mathcal{F}$ ,  $\mu_n$  is represented by the  $\mathcal{F}$ -algebra  $\mathcal{F}[T]/\langle T^n - 1 \rangle$ . If  $\mathcal{F}$  has characteristic 0,  $\mu_n$  is always a smooth group scheme, but  $\mu_n$  is not smooth if  $\mathcal{F}$  has characteristic  $p > 0$  and  $n \equiv 0 \pmod{p}$ .

We say that a connected subgroup  $M$  of  $G$  is of type  $C(\mu)$  if  $M$  is the identity component of the centralizer in  $G$  of the image of a homomorphism  $\phi : \mu_n \rightarrow G$ <sup>1</sup>. Moreover,  $M$  is *geometrically* of type  $C(\mu)$  if  $M_L$  is of type  $C(\mu)$  for some finite, separable field extension  $K \subset L$ ; the example of Proposition 3.6.3 exhibits a group which is geometrically of type  $C(\mu)$  but not itself of type  $C(\mu)$ .

If  $\mu_n$  is smooth, then  $M$  is the centralizer of some (semisimple) element  $\zeta \in G(L)$  in the image of  $\phi$  for some finite separable field extension  $L$  of  $K$ , but for example if  $n = p$  where  $p > 0$  is the characteristic of  $K$ , the group  $\mu_p(\mathcal{F}_{\text{alg}})$  of points over an algebraic closure of  $\mathcal{F}$  is trivial, and instead  $M$  is the centralizer of a semisimple element  $X \in \text{Lie}(G)$ .

In section 3 of this paper, we examine such connected centralizers. Among other things, we show that the image of  $\phi$  lies in a maximal torus  $T$  of  $G$ ; see Proposition 3.4.1. In particular, this allows us to deduce that a subgroup of type  $C(\mu)$  is reductive and contains a maximal torus of  $G$ ; if  $M$  (and thus

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<sup>1</sup>We will permit ourselves to write  $C_G(\phi)$  for the centralizer of the image of  $\phi$ , and  $C_G^\circ(\phi)$  for the identity component of this centralizer.

also  $G$ ) are moreover split, it follows that  $M$  is determined by a subsystem  $\Psi$  of the root system  $\Phi$  of  $G$  (relative to a split maximal torus  $T$ ).

The manuscript [MS03] used the terminology *pseudo-Levi subgroups* of  $G$  to describe the subgroups of the form  $L = C_G^o(s)$  for semisimple elements  $s \in G$ <sup>2</sup>. Any pseudo-Levi subgroup is of type  $C(\mu)$ , and if  $\mathcal{F}$  has characteristic zero, a subgroup  $M$  is pseudo-Levi if and only if it is of type  $C(\mu)$ . But in general there are subgroups of type  $C(\mu)$  which are not pseudo-Levi subgroups in the sense of this older paper; see e.g. the examples in Section 3.6.

Now suppose that  $G$  is split and that  $T$  is a maximal  $k$ -split torus. We introduce an equivalence relation on homomorphisms  $\phi : \mu_n \rightarrow T$  for varying  $n$ , and we call the classes “ $\mu$ -homomorphisms”. Then the group of  $\mu$ -homomorphisms with values in  $T$  may be identified with  $Y \otimes \mathbf{Q}/\mathbf{Z}$  where  $Y = X_*(T)$  is the group of cocharacters of  $T$ . Let  $x \in Y \otimes \mathbf{Q}$ , write  $\bar{x}$  for its class in  $Y \otimes \mathbf{Q}/\mathbf{Z}$ , let  $\phi_{\bar{x}}$  denote the equivalence class of a homomorphism  $\psi : \mu_n \rightarrow T$ , and let  $M = C_G^o(\phi_{\bar{x}}) = C_G^o(\psi)$  the corresponding subgroup of  $G$  of type  $C(\mu)$ .

In section 2, we describe a subsystem  $\Phi_x$  of the root system  $\Phi$  determined in a natural way by the point  $x \in Y \otimes \mathbf{Q}$  – see section 2.5. In fact,  $\Phi_x$  can be characterized as the root system whose Dynkin diagram is obtained by “removing certain nodes (determined by  $x$ ) from the extended Dynkin diagram of  $G$ ” – see Remark 2.5.4. We show that  $M = C_G^o(\phi_{\bar{x}})$  has root system  $\Phi_x$ ; see Theorem 3.4.6.

**1.2. Levi decomposition of the special fiber of a parahoric group scheme.** Let  $H$  be a linear algebraic group over any field  $\mathcal{F}$ . We say that the unipotent radical of  $H$  is defined over  $\mathcal{F}$  if there is an  $\mathcal{F}$ -subgroup  $R \subset H$  such that  $R_{\overline{\mathcal{F}}}$  coincides with the *geometric* unipotent radical  $R_u(H_{\overline{\mathcal{F}}})$ ; if moreover  $R$  is an  $\mathcal{F}$ -split unipotent group, we say that the unipotent radical of  $H$  is defined and split over  $\mathcal{F}$ . If  $\mathcal{F}$  is imperfect, there are examples of groups  $H$  whose unipotent radical is not defined over  $\mathcal{F}$ .

If the unipotent radical of  $H$  is defined over  $\mathcal{F}$ , write  $\pi : H \rightarrow H/R_u H$  for the quotient mapping. To say that  $H$  has a *Levi decomposition* (over  $\mathcal{F}$ ) means that there is a (necessarily reductive)  $\mathcal{F}$ -subgroup  $M$  of  $H$  such that the restriction  $\pi|_M : M \rightarrow H/R_u H$  is an isomorphism<sup>3</sup>

Our interest here is in linear algebraic groups which arise when considering reductive groups over *local field*. Thus, suppose that  $K$  is a local field, by which we mean the field of fractions of a complete discrete valuation ring  $\mathcal{A}$ . Write  $k$  for the residue field of  $\mathcal{A}$ . We make no assumptions on  $k$  – in particular, we do not require  $k$  to be perfect.

Consider an extension field  $L \supset K$  of finite degree, and let  $\mathcal{B}$  denote the integral closure of  $\mathcal{A}$  in  $L$ ; then  $\mathcal{B}$  is also a discrete valuation ring, say with residue field  $\ell$ . Recall that  $L$  is *unramified* over  $K$  provided that  $\ell$  is a *separable* extension of  $k$  with  $[L : K] = [\ell : k]$ .

We now consider a reductive algebraic group  $G$  over the field  $K$ . Following the work of F. Bruhat and J. Tits, one can view  $G$  as the generic fiber  $G = \mathcal{P}_K$  of various smooth affine group schemes  $\mathcal{P}$  over  $\mathcal{A}$  which we will refer to as *parahoric group schemes*; see Definitions 4.5.3 and 4.3.4. In this manuscript, we only consider parahoric group schemes  $\mathcal{P}$  for which the generic fiber  $G = \mathcal{P}_K$  splits over an unramified extension of  $K$ ; under this assumption, the fibers of  $\mathcal{P}$  are *connected* – see Theorem 4.2.2.

In fact, the group of  $K$ -rational points  $G(K)$  of  $G$  acts on the Bruhat-Tits building of  $G$ , and the subgroup of  $\mathcal{A}$ -rational points  $\mathcal{P}(\mathcal{A}) \subset G(K)$  – known as a *parahoric subgroup* – is very closely related to the stabilizer of a point for this action. If the residue field  $k$  is finite, the  $\mathcal{P}(\mathcal{A})$  are compact open subgroups of the locally compact group  $G(K)$ .

The special fiber  $\mathcal{P}_k$  is a connected linear algebraic group over  $k$  which in general is not reductive. Under the additional assumptions that the residue field  $k$  is perfect and that  $G$  splits over an unramified extension of  $k$ , we proved in [McN10] that (\*) the identity component  $\mathcal{P}_k$  of the special fiber has a *Levi decomposition*.

In the present work, we improve the result (\*) with no assumption on the residue field  $k$ , under the assumption that  $G$  splits over an unramified extension. We show under these assumptions that that

<sup>2</sup>This class of subgroups was viewed as a generalization of the class of all Levi factors of parabolic subgroups of  $G$

<sup>3</sup>If the unipotent radical of  $H$  fails to be defined over  $\mathcal{F}$ ,  $H$  may still possess a *Levi factor*  $M$  – i.e. an  $\mathcal{F}$ -subgroup  $M$  for which  $\pi : M_{\overline{\mathcal{F}}} \rightarrow H_{\overline{\mathcal{F}}}/R_u(H_{\overline{\mathcal{F}}})$  is an isomorphism where  $\overline{\mathcal{F}}$  is an algebraic closure of  $\mathcal{F}$ . In this paper, we only consider Levi decompositions of groups whose unipotent radicals are defined over the ground field.

the unipotent radical of  $\mathcal{P}_k$  is defined and split over  $k$  – see Proposition 4.3.7(a) and Proposition 4.5.5 –, and that a Levi factor of  $\mathcal{P}_k$  can be realized as the special fiber of a reductive subgroup scheme of  $\mathcal{P}$ . Let  $T$  be a maximal  $K$ -split torus, write  $\mathcal{T}$  for “the”  $\mathcal{A}$ -split torus with generic fiber  $T$ , and let  $\mathcal{P}$  be a parahoric group scheme containing  $\mathcal{T}$ . In the final sections of this paper – see Section 4.4 and Section 4.5 – we are going to prove:

**Theorem 1.** *Assume that  $G$  splits over an unramified extension of  $K$ . There is a reductive  $\mathcal{A}$ -subgroup scheme  $\mathcal{M} \subset \mathcal{P}$  containing  $\mathcal{T}$  such that*

- (a) *The special fiber  $\mathcal{P}_k$  has a Levi decomposition with Levi factor  $\mathcal{M}_k$ , and*
- (b) *The generic fiber  $\mathcal{M}_K$  is a connected reductive subgroup of  $G$  of type  $C(\mu)$  which contains  $T$ .*

When  $G$  is split over  $K$ , a parahoric subgroup  $\mathcal{P} = \mathcal{P}_x$  containing  $\mathcal{T}$  is determined by the choice of an element  $x \in V = Y \otimes \mathbf{Q}$ . If  $\bar{x}$  is the class of  $x$  in  $Y \otimes \mathbf{Q}/\mathbf{Z}$ , the reductive subgroup scheme  $\mathcal{M}$  in the Theorem is precisely the connected centralizer  $C_{\mathcal{P}}^0(\phi_{\bar{x}})$  where  $\phi_{\bar{x}} : \mu \rightarrow \mathcal{T}$  is the  $\mu$ -homomorphism determined by  $\bar{x}$ ; see Theorem 4.4.2.

We remark that Theorem 1 plays an important role in our recent manuscript [McN16] which relates nilpotent orbits on the reductive quotient of the special fiber of a parahoric group scheme  $\mathcal{P}$  with the nilpotent orbits on the generic fiber  $\mathcal{P}_k$ .

Finally, we remark that after completion of this paper, an anonymous referee pointed out to us that in the manuscript [Tit90], Jacques Tits considers split reductive groups  $G$  over the field  $k((t))$ ; when  $\mathcal{P}$  is a maximal parahoric, Tits finds in that paper the group scheme  $\mathcal{M}$  of Theorem 1 and shows that  $\mathcal{M}_k$  is a Levi factor of  $\mathcal{P}_k$ .

**1.3. Notation and terminology.** By a *linear algebraic group*  $H$  over a field  $\mathcal{F}$ , we mean an affine group scheme which is smooth and of finite type over  $\mathcal{F}$ ; this amounts to a the same thing as a (reduced) linear algebraic group defined over  $\mathcal{F}$  as in [Bor91] or [Spr98]. Unless otherwise indicated, by a subgroup of  $H$  we mean a closed  $\mathcal{F}$ -subgroup scheme.

Suppose that  $\mathcal{A}$  is a local integral domain with fractions  $K$  and residue field  $k$ . If  $X$  is a separated scheme of finite type over  $\mathcal{A}$ , the *generic fiber* of  $X$  is the  $K$ -variety  $X_K = X \times_{\mathrm{Spec}(\mathcal{A})} \mathrm{Spec}(K)$  obtained by base change, and the *special fiber* of  $X$  is the  $k$ -variety  $X_k = X \times_{\mathcal{A}} K = X \times_{\mathrm{Spec}(\mathcal{A})} \mathrm{Spec}(k)$  obtained by base change.

If  $X$  is a group scheme over  $\mathcal{A}$ , then  $X_K$  is a  $K$ -group scheme and  $X_k$  is a  $k$ -group scheme. If  $X$  is smooth, affine, and of finite type over  $\mathcal{A}$ , then  $X_K$  and  $X_k$  are *linear algebraic groups*.

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## 2. CERTAIN SUB-SYSTEMS OF A ROOT SYSTEM

The goal of this section is to describe certain subsystems  $\Psi$  of a root system  $\Phi$ . Probably most of the material in this section is well-known, though I don’t really know of a concise account for some of the results. Some of this material appears e.g. in the papers [Ree10] and [Ree+12].

The subsystems  $\Psi$  depend on the choice of a point  $x$  in the linear space affording the *reflection representation* of the Weyl group of  $\Psi$ . They are described in section 2.5 – see especially Theorem 2.5.3. More precisely,  $\Psi$  depends only on the facet  $F$  containing  $x$ , where the facets are defined with respect to the affine root system. As perhaps a justification for the level of detail given here, note that our exposition confirms that the subsystems  $\Psi$  of interest can all be obtained using points  $x$  taken from a  $\mathbf{Q}$ -form of the reflection representation; see especially Proposition 2.4.2.

Suppose that  $G$  is a split reductive group over a field with root system  $\Phi$ . In the terminology used in the introduction, the subsystems we describe will be precisely the root systems of the split reductive subgroups  $M$  of  $G$  which are of type  $C(\mu)$ ; these subgroups will be studied in section 3.

Throughout this section, we will adopt the following notations:  $V$  will denote a finite dimensional vector space over the field  $\mathbf{Q}$  of rational numbers, and  $q$  will denote a positive definite quadratic form  $q$  on  $V$ . We write  $\langle v, w \rangle \in \mathbf{Q}$  for the value of the associated bilinear form at  $v, w \in V$ ; thus  $q(v) = \langle v, v \rangle$ . The bilinear form provides an identification of  $V$  with its dual space  $V^\vee$ ; in what follows, we freely apply this identification when making reference to results from [Bou02].

Given any field extension  $\mathbf{Q} \subset E$ , we form the tensor product  $V_E = V \otimes_{\mathbf{Q}} E$ . The quadratic form  $q$  determines by extension of scalars a non-degenerate form on  $V_E$  which we also denote by  $q$ . If  $E$  has a real embedding, then  $q$  remains positive definite on  $V_E$ . In particular, the form  $q$  determines the Euclidean metric topology on  $V_{\mathbf{R}}$ .

**2.1. Simplicial cones and simplices.** Let  $w_0 \in V$  and let  $v_1, \dots, v_d \in V$  be  $\mathbf{Q}$ -linearly independent vectors in  $V$ . For a field  $E$  with an embedding in  $\mathbf{R}$ , the *rational simplicial E-cone* based at  $w_0$  determined by the vectors  $\vec{v}$  is the subset

$$C_F = C_F(w_0, v_1, \dots, v_d) = \left\{ w_0 + \sum_{j=1}^d a_j v_j \mid a_j \in E_{>0} \quad \forall j = 1, \dots, d \right\}$$

**Lemma 2.1.1.** *For  $w_0$  and  $v_1, \dots, v_d$  as above,  $C_{\mathbf{Q}} = C_{\mathbf{R}} \cap V$ , and  $C_{\mathbf{Q}}$  is dense in  $C_{\mathbf{R}}$  for the Euclidean topology on  $V_{\mathbf{R}}$ .*

Now fix points  $w_0, w_1, \dots, w_d \in V$  which are *affinely independent*. A convex combination  $\sum_{i=0}^d t_i w_i$  of these points is determined by positive scalars  $t_0, t_1, \dots, t_d$  for which  $\sum_{i=0}^d t_i = 1$ .

If  $E$  is a field with a real embedding, the *E-simplex* determined by the points  $w_0, \dots, w_d$  is the set

$$S_E = S_E(w_0, \dots, w_d) = \left\{ \sum_{i=0}^d t_i w_i \mid t_i \in E_{>0}, \quad \sum_{i=0}^d t_i = 1 \right\}.$$

**Lemma 2.1.2.** *Let  $w_0, \dots, w_d$  affinely independent. Then  $S_{\mathbf{Q}} = S_{\mathbf{R}} \cap V$  and  $S_{\mathbf{Q}}$  is dense in  $S_{\mathbf{R}}$  for the Euclidean topology on  $V_{\mathbf{R}}$ .*

Now, the results Lemmas 2.1.1 and 2.1.2 are both immediate consequences of the fact that  $\mathbf{Q}$  is dense in  $\mathbf{R}$  for the Euclidean topology.

**2.2. Root systems and the finite Weyl group.** Let  $\Phi \subset V$  be a (reduced) root system in the span  $V_e = \mathbf{Q}\Phi \subset V$ ; see [Bou02, p. VI.1.1] for definitions. For  $\alpha \in \Phi$ , write  $\alpha^\vee = 2\alpha/q(\alpha) \in V_e$ . Recall [Bou02, VI §1] that:

- (i) the reflection  $s_\alpha \in O(V, q)$  given by the rule  $s_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha$  is the orthogonal reflection  $s_H$  in the hyperplane  $H = H_\alpha = \alpha^\perp$ , and
- (ii)  $\langle \Phi, \alpha^\vee \rangle \subset \mathbf{Z}$ .

Write

$$\mathcal{H} = \{H_\alpha = \alpha^\perp \mid \alpha \in \Phi\}$$

for the system of linear hyperplanes in  $V$  determined by  $\Phi$ . The *Weyl group* of  $\Phi$  is the finite reflection group  $W = W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle = \langle s_H \mid H \in \mathcal{H} \rangle \subset O(V, q)$ .

**Proposition 2.2.1.** [Bou02, VI §1.2] *Write  $V = \bigoplus_{i=0}^d V_i$  where  $V_0 = V^W$  and  $V_1, \dots, V_d$  are non-trivial irreducible  $\mathbf{Q}W$ -representations. Then  $V_e = \bigoplus_{i=1}^d V_i$ , and if  $1 \leq i \leq d$ , then:*

- (a)  $V_i$  is an absolutely irreducible  $\mathbf{Q}W$ -module.
- (b)  $\Phi_i = \Phi \cap V_i$  is a root system in  $V_i$ .

(c) If  $W(\Phi_i)$  denotes the Weyl group of  $\Phi_i$ , and if  $W_i$  is the subgroup of  $W$  acting trivially on  $V_i^\perp$ , then  $W_i$  identifies with  $W(\Phi_i)$  and  $W$  is the direct product  $W = W_1 \times \cdots \times W_p$ .

The root systems  $\Phi_i \subset \Phi$  of the preceding Proposition are the *irreducible components* of the root system  $\Phi$ .

*Remark 2.2.2.* Let  $\mathcal{F}$  be a field, and let  $G$  be a split reductive group over  $\mathcal{F}$ . The choice of a maximal split torus  $T$ , determines the *root datum*  $(X, Y, \Phi, \Phi^\vee)$  of  $G$  relative to  $T$ . Here,  $X = X^*(T)$  is the character group of  $T$ ,  $Y = X_*(T)$  is the cocharacter group of  $T$ ,  $\Phi \subset X$  is the set of roots, and  $\Phi^\vee \subset Y$  is the set of co-roots. In this situation, the Weyl group acts on both  $X$  and on  $Y$ . We take  $V = Y \otimes_{\mathbf{Z}} \mathbf{Q}$  and fix a  $W$ -invariant positive definite bilinear form on  $V$ . Then  $X$  may be identified with the lattice  $\{x \in V \mid \langle x, Y \rangle \subset \mathbf{Z}\}$ . In particular, with these identifications, the roots  $\Phi$  are contained in  $V$ , and in this way the results described in this section can be applied to the root system of the group  $G$ .

**2.3. Affine hyperplanes and affine Weyl group.** Consider the group  $\text{Aff}(V)$  of all *affine displacements* of  $V$ ; it is the semidirect product of  $\text{GL}(V)$  and the normal subgroup of “translations by  $V$ ”:

$$\text{Aff}(V) \simeq V \rtimes \text{GL}(V).$$

A root  $\alpha \in \Phi$  and an integer  $\ell$  together determine an affine function

$$a(\alpha, \ell) : V \rightarrow \mathbf{Q} \quad \text{by the rule} \quad a(\alpha, \ell)(w) = \langle \alpha, w \rangle - \ell,$$

and an affine hyperplane

$$H_{\alpha, \ell} = a(\alpha, \ell)^{-1}(0) = \{w \in V \mid \langle \alpha, w \rangle = \ell\} \subset V.$$

Write

$$\tilde{\mathcal{H}} = \{H_{\alpha, \ell} \mid \alpha \in \Phi, \ell \in \mathbf{Z}\}$$

for the collection of these hyperplanes in  $V$ .

For  $H \in \tilde{\mathcal{H}}$ , write  $s_H \in \text{Aff}(V)$  for the orthogonal (affine) reflection in the hyperplane  $H$ . The affine Weyl group is the subgroup

$$W_{\text{aff}} = \langle s_H \mid H \in \tilde{\mathcal{H}} \rangle \subset \text{Aff}(V).$$

The action by conjugation of  $W_{\text{aff}}$  on its normal subgroup  $V$  determines a linear representation  $U : W_{\text{aff}} \rightarrow \text{GL}(V)$ .

**Proposition 2.3.1.** [Bou02, VI §2.1 Prop. 1] *The image of  $U$  is the Weyl group  $W$  of the root system  $\Phi$ , the kernel of  $U$  is the subgroup  $\mathbf{Z}\Phi^\vee$  of the translation subgroup  $V \subset \text{Aff}(V)$ , and  $W_{\text{aff}}$  is the semidirect product of  $W$  and the subgroup  $\mathbf{Z}\Phi^\vee$ .*

Note that  $\mathcal{H} \subset \tilde{\mathcal{H}}$ . For  $\alpha \in \Phi$  and  $\ell \in \mathbf{Z}$ , the affine function  $a(\alpha, \ell)$  has a unique extension to an affine function on  $V_{\mathbf{R}}$ , and its zero-locus is a hyperplane  $H_{\alpha, \ell, \mathbf{R}}$  in  $V_{\mathbf{R}}$ . We thus find systems of hyperplanes  $\mathcal{H}_{\mathbf{R}}$  and  $\tilde{\mathcal{H}}_{\mathbf{R}}$  in  $V_{\mathbf{R}}$  determined by  $\Phi$ , and bijective mappings  $\mathcal{H} \rightarrow \mathcal{H}_{\mathbf{R}}$  and  $\tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}_{\mathbf{R}}$  given by  $H \mapsto H_{\mathbf{R}}$ .

**2.4. Facets, chambers and alcoves.** As in [Bou02, V §1], we may speak of the *facets*  $\mathbf{F}$  in  $V_{\mathbf{R}}$  for the system of affine hyperplanes  $\tilde{\mathcal{H}}_{\mathbf{R}} = \{H_{\alpha, \ell, \mathbf{R}}\}$  in  $V_{\mathbf{R}}$ , and for the system of linear hyperplanes  $\mathcal{H}_{\mathbf{R}} = \{H_{\alpha, \mathbf{R}}\}$  in  $V_{\mathbf{R}}$ .

By a *facet*  $F$  in  $V$  relative to  $\tilde{\mathcal{H}}$  resp. relative to  $\mathcal{H}$ , we shall mean the intersection  $F = \mathbf{F} \cap V$  of  $V$  with a facet  $\mathbf{F}$  in  $V_{\mathbf{R}}$  relative to  $\tilde{\mathcal{H}}_{\mathbf{R}}$  resp. relative to  $\mathcal{H}_{\mathbf{R}}$ . If  $F = \mathbf{F} \cap V$ , the *closure* of  $F$  is by definition the set  $\bar{F} = \bar{\mathbf{F}} \cap V$  where  $\bar{\mathbf{F}}$  is the closure of  $\mathbf{F}$  for the Euclidean topology on  $V_{\mathbf{R}}$ .

A *chamber* of  $V$  is a facet  $C$  relative to  $\mathcal{H}$  which is contained in no hyperplane  $H \in \mathcal{H}$ . An *alcove* of  $V$  is a facet  $A$  relative to  $\tilde{\mathcal{H}}$  which is contained in no hyperplane  $H \in \tilde{\mathcal{H}}$ .

**Proposition 2.4.1.** *Let  $F = V \cap \mathbf{F}$  be a facet in  $V$  relative to  $\mathcal{H}$  resp.  $\tilde{\mathcal{H}}$ , where  $\mathbf{F}$  is a facet in  $V_{\mathbf{R}}$  relative to  $\mathcal{H}_{\mathbf{R}}$  resp.  $\tilde{\mathcal{H}}_{\mathbf{R}}$ . Then:*

(a)  *$F$  is dense in  $\mathbf{F}$  for the Euclidean topology on  $V_{\mathbf{R}}$ , and*

(b) If  $F'$  is another facet in  $V$  relative to  $\mathcal{H}$  resp.  $\tilde{\mathcal{H}}$ , then  $F = F'$  if and only if  $\mathbf{F} = \mathbf{F}'$ .

*Proof.* Of course, (b) is a consequence of (a).

To prove (a), first recall the decomposition  $V = V_0 \oplus V_1 \oplus \cdots \oplus V_d$  of Proposition 2.2.1. Write  $\Phi$  as the union of its irreducible subsystems  $\Phi_i \subset V_i$ . For  $1 \leq i \leq d$ , consider the systems of hyperplanes  $\mathcal{H}_i = \{H_\alpha \mid \alpha \in \Phi_i\}$  and  $\tilde{\mathcal{H}}_i = \{H_{\alpha,\ell} \mid \alpha \in \Phi_i, \ell \in \mathbf{Z}\}$  in  $V$ ; each  $H$  in either  $\tilde{\mathcal{H}}_i$  or  $\mathcal{H}_i$  determines a hyperplane  $H \cap V_i$  in  $V_i$ .

It follows from [Bou02, V §3.8 Prop. 6] that  $F_{\mathbf{R}} = V_{0,\mathbf{R}} \times \mathbf{F}_1 \times \cdots \times \mathbf{F}_d$  where  $\mathbf{F}_i$  is a facet in  $V_{i,\mathbf{R}}$  for the system of hyperplanes determined by  $\mathcal{H}_{i,\mathbf{R}}$  resp.  $\tilde{\mathcal{H}}_{i,\mathbf{R}}$ .

If  $F$  is a facet relative to  $\tilde{\mathcal{H}}$ , then  $\mathbf{F}_i$  is a facet relative to  $\tilde{\mathcal{H}}_{i,\mathbf{R}}$ , and according to [Bou02, VI §2.2 Corollary to Prop. 5], there is an affinely independent collection of points  $w_0, \dots, w_d \in V$  such that  $\mathbf{F}_i$  is the real simplex  $S_{\mathbf{R}}(w_0, \dots, w_d)$  determined by these points as in section 2.1. Now Lemma 2.1.2 shows that  $F_i$  is the  $\mathbf{Q}$ -simplex  $S_{\mathbf{Q}}(w_0, \dots, w_d)$  and that  $F_i$  is dense in  $\mathbf{F}_i$  as required.

If instead  $F$  is a facet relative to  $\mathcal{H}$ , then  $\mathbf{F}_i$  is a facet relative to  $\mathcal{H}_{i,\mathbf{R}}$  and according to [Bou02, VI §1.5 Theorem 2] there is a linearly independent subset  $v_1, \dots, v_d$  of  $V$  such that  $\mathbf{F}_i$  is the real simplicial cone  $C_{\mathbf{R}}(0, v_1, \dots, v_d)$  as in section 2.1. Now, Lemma 2.1.1 shows that  $F_i$  is the  $\mathbf{Q}$ -simplicial cone  $C_{\mathbf{Q}}(0, v_1, \dots, v_d)$  and that  $F_i$  is dense in  $\mathbf{F}_i$  as required.  $\square$

**Proposition 2.4.2.** (a) The Weyl group  $W$  acts on the set of all facets in  $V$  relative to  $\mathcal{H}$  and preserves the subset of all chambers. Moreover, the closure of a chamber is a fundamental domain for the action of  $W$  on  $V$ .

(b) The affine Weyl group  $W_{\text{aff}}$  acts on the set of all facets in  $V$  relative to  $\tilde{\mathcal{H}}$  and preserves the subset of all alcoves. Moreover, the closure of an alcove is a fundamental domain for the action of  $W_{\text{aff}}$  on  $V$ .

*Proof.* The analogous statements are known to hold for the actions of  $W$  and  $W_{\text{aff}}$  on the Euclidean space  $V_{\mathbf{R}}$ ; see [Bou02, V §3.3 Theorem 2]. Since by Proposition 2.4.1 a facet  $F = V \cap \mathbf{F}$  is dense in  $\mathbf{F}$  for the Euclidean topology, and since the action of an element of  $W$  or  $W_{\text{aff}}$  on  $V_{\mathbf{R}}$  is continuous, the conclusion follows at once.  $\square$

**2.5. The root subsystem associated to a point in  $V$ .** The subset  $\Psi \subset \Phi$  is said to be *closed* if whenever  $\alpha, \beta \in \Psi$  and  $\alpha + \beta \in \Phi$ , then  $\alpha + \beta \in \Psi$ . The subset  $\Psi$  is *symmetric* if  $\Psi = -\Psi = \{-\alpha \mid \alpha \in \Psi\}$ . If  $\Psi$  is closed and symmetric, it is again a root system.

We now fix a point  $x \in V$ , and we put

$$\Phi_x = \{\alpha \in \Phi \mid \langle \alpha, x \rangle \in \mathbf{Z}\}.$$

Note that  $\Phi_x$  is evidently a *closed* and *symmetric* subset of  $\Phi$ ; hence  $\Phi_x$  is a root system.

Verification of the following Proposition is immediate:

**Proposition 2.5.1.** (a)  $\Phi_x$  depends only on the  $W_{\text{aff}}$  orbit of  $x$ ; i.e.  $\Phi_{wx} = \Phi_x$  for all  $w \in W_{\text{aff}}$ .

(b) If  $\Phi = \Phi_1 \cup \cdots \cup \Phi_d$  is the decomposition of  $\Phi$  into irreducible components, then  $\Phi_x$  is the disjoint union of the subsystems  $\Phi_{i,x}$  for  $1 \leq i \leq d$ .

In view of this Proposition, in order to describe  $\Phi_x$  it is sufficient to suppose that  $\Phi$  is irreducible. Moreover, we may even choose an “optimal” representative for the  $W_{\text{aff}}$ -orbit of  $x$ . The following Lemma provides a useful representative from our point of view.

Let  $A$  be an alcove in  $V$  relative to  $\tilde{\mathcal{H}}$ . A *face* of  $A$  is a facet  $F$  contained in  $\bar{A}$  for which  $F \subset H$  for precisely one hyperplane  $H \in \tilde{\mathcal{H}}$ ; if  $F$  is a face of  $A$  and  $F \subset H \in \tilde{\mathcal{H}}$ , we say that the hyperplane  $H$  is a *wall* of  $A$ .

**Proposition 2.5.2.** Let  $\Phi$  be an irreducible root system, and recall that  $V_e = V_1$  is the  $\mathbf{Q}$ -span of  $\Phi$  in  $V$ . Let  $\alpha_1, \dots, \alpha_\ell$  be a system of simple roots, and let  $\omega_1, \dots, \omega_\ell \in V_1$  be elements with  $\langle \alpha_i, \omega_j \rangle = \delta_{i,j}$ . Write  $\alpha_0 = -\tilde{\alpha}$  where

$$\tilde{\alpha} = \sum_{i=1}^{\ell} n_i \alpha_i$$

is the highest root in  $\Phi$  for this choice of simple roots, and let  $\omega_0 = 0$ .

- (a) The simplex  $A = C(\omega_0, \omega_1/n_1, \dots, \omega_\ell/n_\ell)$  is an alcove for the root system  $\tilde{\mathcal{H}}$ .  
 (b) The walls of  $A$  are the hyperplanes defined by the affine functions

$$\{a(\alpha_0, 1)\} \cup \{a(\alpha_i, 0) \mid 1 \leq i \leq \ell.\}$$

*Proof.* (a) follows from [Bou02, VI.2.2 Corollary]; (b) then follows from definitions.  $\square$

If the affine hyperplane  $H \in \tilde{\mathcal{H}}$  is the zero locus of the affine function  $a(\alpha, d)$ , we say that  $H$  is labeled by  $\alpha$ .

**Theorem 2.5.3.** *Suppose that  $\Phi$  is irreducible, and that  $\alpha_1, \dots, \alpha_\ell$  is a system of simple roots. Let  $A$  be the alcove of Proposition 2.5.2; thus  $A$  is the simplex determined by 0 and the elements  $\omega_i/n_i$ . Now suppose that  $x \in \bar{A}$ . Define*

$$J = \{i \in \{0, 1, \dots, \ell\} \mid x \text{ lies on the wall of } A \text{ labeled by } \alpha_i\}.$$

Then  $\{\alpha_j \mid j \in J\}$  is a system of simple roots for  $\Phi_x$ .

*Proof.* Since any proper subset of the roots  $\{\alpha_0, \alpha_1, \dots, \alpha_\ell\}$  is  $\mathbf{Q}$ -linearly independent in  $V$ , we only must argue that any  $\alpha \in \Phi_x$  is a  $\mathbf{Z}$ -linear combination of the indicated roots.

Fix  $\alpha \in \Phi_x$ , let  $n = \langle \alpha, x \rangle \in \mathbf{Z}$ , and consider the affine reflection  $s_{\alpha, n}$  – i.e. the reflection in the hyperplane  $H = \{y \in V \mid \alpha(y) = n\}$ . Then  $s_{\alpha, n}$  fixes  $x$ .

It follows from [Bou02, (V.3.3)] that the stabilizer  $W_x$  of  $x$  in  $W_{\text{aff}}$  is the subgroup  $W_x = W_J = \langle s_{H_j} \rangle$  where  $H_j$  is the wall of  $A$  labeled by the root  $\alpha_j$ .

Since  $s_{\alpha, n}$  is a reflection in the Coxeter group  $W_x$ , it follows from [Bou02, V.3.2 Corollary to Theorem 1] that  $s_{\alpha, n}$  is  $W_x$ -conjugate to  $s_{H_j}$  for some  $j \in J$ ; it now follows that  $\alpha$  is a  $\mathbf{Z}$ -linear combination of the roots  $\{\alpha_j \mid j \in J\}$ , as required.  $\square$

*Remark 2.5.4.* The irreducible root system  $\Phi$  is determined by its *Dynkin diagram*; see [Bou02, (VI.4.2)]. With notation as before, the roots  $\{\alpha_0, \dots, \alpha_\ell\}$  label the nodes of the *completed Dynkin diagram loc. cit.* (VI.4.3). Now, the Theorem implies that the (ordinary) Dynkin diagram of the – in general, reducible – root system  $\Phi_x$  is obtained by discarding those vertices of this completed diagram labeled by the simple roots  $\alpha_t$  for  $t \notin J$  together with any edges connected to such a discarded vertex, where  $J \subset \{0, \dots, \ell\}$  is the subset found in the statement of Theorem 2.5.3.

### 3. $\mu$ -HOMOMORPHISMS AND THEIR CENTRALIZERS

Fix throughout Section 3 a connected and reductive algebraic group  $G$  over the ground field  $\mathcal{F}$ .

Consider an  $\mathcal{F}$ -homomorphism  $\phi : \mu_n \rightarrow G$ , where  $\mu_n$  is the finite group scheme of  $n$ -th roots of unity. The main goal of this section is to describe the connected centralizer  $C_G^o(\phi)$  of the image of  $\phi$ . Recall from the introduction that  $M$  is a *subgroup of type*  $C(\mu)$  if  $M = C_G^o(\phi)$  for some  $\phi$  as above.

We are going to show that the image of  $\phi$  always lies in some maximal torus  $T$  of  $G$  – see Proposition 3.4.1. As a consequence, we see that  $M$  is a reductive subgroup containing  $T$ ; see Proposition 3.1.2.

We define in section 3.4 an equivalence relation on the collection of homomorphisms  $\psi : \mu_n \rightarrow T$  for varying  $n$ ; the equivalence classes are called  $\mu$ -homomorphisms. If  $T$  is a split torus, the collection of  $\mu$ -homomorphisms with values in  $T$  is in bijection with the group  $Y \otimes \mathbf{Q}/\mathbf{Z}$ , where  $Y = X_*(T)$  is the group of cocharacters of  $T$ . Suppose that the  $\mu$ -homomorphism  $\phi = \phi_{\bar{x}}$  is determined by the class  $\bar{x}$  in  $Y \otimes \mathbf{Q}/\mathbf{Z}$  of the point  $x \in Y \otimes \mathbf{Q}$ . We will show – see Theorem 3.4.6 – that the group  $M = C_G^o(\phi)$  of type  $C(\mu)$  has root system  $\Phi_x$  as described in Theorem 2.5.3.

We recall that the so-called Borel-de Siebenthal procedure – first described in [BS49] – can be used to obtain the root systems of *all* split reductive subgroups containing a maximal torus; it amounts to recursive application of the procedure described in Remark 2.5.4. The subgroups of type  $C(\mu)$  do not exhaust all such subgroups.

Related results and descriptions of centralizers are studied e.g. in [Hum95, Theorem 2.15], in [Lus95] – see especially Lemma 5.4 in that citation –, and in [MS03]. In [Ser06], J-P. Serre suggested to

replace the “semisimple elements of finite order” by  $\mu$ -homomorphisms. Finally, note that the recent paper [PL15] has some overlap with the point of view taken here.

**3.1. The centralizer of a diagonalizable subgroup scheme.** The affine group scheme  $M$  is *diagonalizable* if the Abelian group of its characters  $X^*(M) = \text{Hom}_{\mathcal{F}\text{-gp}}(M, \mathbf{G}_m)$  forms a basis of the coordinate algebra  $\mathcal{F}[M]$  as a linear space. More generally,  $M$  is of *multiplicative type* if there is a finite separable extension  $E/\mathcal{F}$  for which the  $E$ -group  $M_E$  obtained by base change is diagonalizable. For example, a torus  $T$  is of multiplicative type, and  $T$  is diagonalizable if and only if it is split.

Let  $M$  be a group scheme of multiplicative type, which is of finite type over  $\mathcal{F}$ . Suppose given a morphism  $f : M \rightarrow G$  over  $\mathcal{F}$ . Then we have:

**Theorem 3.1.1.** *The centralizer  $H = C_G(f)$  of the image of  $f$  is a closed, smooth, reductive  $\mathcal{F}$ -subgroup scheme of  $G$ .*

*Proof.* This result is originally due to Richardson; see [Ric82, Prop 10.1.5]. It can also be obtained as follows: [SGA3<sub>II</sub>, Exp. XI, Cor. 5.3] shows that  $H$  is a closed and smooth subgroup scheme<sup>4</sup>. Since  $M$  is of multiplicative type,  $M_{\overline{\mathcal{F}}}$  is diagonalizable, where  $\overline{\mathcal{F}}$  is an algebraic closure of  $\mathcal{F}$ . It follows from [Jan03, (I.2.11)] that all linear representations of  $M_{\overline{\mathcal{F}}}$  are completely reducible; now [CGP15, Theorem A.8.12] shows that the identity component of the centralizer  $H$  of  $M$  is indeed reductive.  $\square$

It is more straightforward to see that the centralizer of the image of  $f$  is reductive when the image of  $f$  lies in a maximal torus. Indeed, we have the following:

**Proposition 3.1.2.** *Suppose that  $f$  factors through a maximal torus  $T$  of  $G$ . Then:*

- (a) *The centralizer  $H = C_G(f(M))$  of the image of  $f$  is a reductive subgroup of  $G$ .*
- (b) *Suppose that  $T$  (and hence  $G$ ) is split, write  $\Phi$  for the set of roots of  $T$  in  $\text{Lie}(G)$ , and for  $\alpha \in \Phi$ , write  $U_\alpha$  for the root subgroup. Then the set of roots of  $H$  is  $\Phi' = \{\alpha \in \Phi \mid f^*\alpha = 0\}$ .*

*Proof.* It suffices to prove the result after extending the ground field; thus, to prove (a) we may and will suppose that  $T$  is a split torus and that  $M$  is diagonalizable. As in the statement of (b), let  $\Phi \subset X^*(T)$  be the roots of  $T$  in  $\text{Lie}(G)$ . Write  $f^* : X^*(T) \rightarrow X^*(M)$  for the mapping determined on characters by  $f$ .

Put  $\Phi' = \{\alpha \in R \mid f^*\alpha = 0\}$  as in (b). Since  $M$  is diagonalizable,  $\text{Lie}(H) = \text{Lie}(G)^M$ . For  $\alpha \in R$ , it follows that  $\text{Lie}(G)_\alpha \subset \text{Lie}(H)$  if and only if  $\alpha \in \Phi'$ .

Now,  $T$  is contained in  $H$ . Thus, in the terminology of [SGA3<sub>III</sub>, Exp. XXII, §5],  $H$  is a subgroup of type  $(R)$ . Now [SGA3<sub>III</sub>, Exp. XXII Prop. 5.10.1] implies that  $H$  is reductive once we observe that  $\Phi' = -\Phi'$ . But by definition,

$$\alpha \in \Phi' \iff j^*(\alpha) = 0 \iff -\alpha \in \Phi',$$

whence the Theorem.  $\square$

**3.2. The group scheme  $\mu_n$ .** Fix an integer  $n \geq 1$  and let  $\mu_n$  be the group scheme with coordinate ring  $\mathcal{F}[\mu_n] = \mathcal{F}[T]/\langle T^n - 1 \rangle$ . We may view  $\mu_n$  as the scheme theoretic kernel of the mapping  $(t \mapsto t^n) : \mathbf{G}_m \rightarrow \mathbf{G}_m$ , or alternately as the *Cartier dual*  $\mu_n = (\mathbf{Z}/n\mathbf{Z})^D$  of the cyclic group  $\mathbf{Z}/n\mathbf{Z}$  viewed as a “constant group scheme” over  $\mathcal{F}$  – see [Knu+98, §20]. It is a finite and commutative group scheme over  $\mathcal{F}$ , and it is smooth over  $\mathcal{F}$  if and only if  $n$  is invertible in  $\mathcal{F}$ ; see e.g. Example (21.11) and Example (21.5)(4) of [Knu+98].

Let  $n, m \in \mathbf{Z}_{>0}$  with  $m \mid n$  and consider the homomorphisms

$$\mathbf{Z}/m\mathbf{Z} \rightarrow \mathbf{Z}/n\mathbf{Z} \quad \text{via} \quad (a + m\mathbf{Z} \mapsto a \cdot \frac{n}{m} + n\mathbf{Z})$$

and

$$\mathbf{Z}/n\mathbf{Z} \rightarrow \mathbf{Z}/m\mathbf{Z} \quad \text{via} \quad (a + n\mathbf{Z} \mapsto a + m\mathbf{Z}).$$

<sup>4</sup>In fact,  $G$  need not be reductive for this claim;  $H$  is closed and smooth provided only that  $G$  is smooth



Via Cartier duality, these mappings respectively induce homomorphisms of group schemes

$$\tau_{n,m} : \mu_n \rightarrow \mu_m \quad \text{and} \quad \iota_{m,n} : \mu_m \rightarrow \mu_n.$$

It follows from the formalism of Cartier duality that  $\iota_{m,n}$  is a closed embedding, and that  $\ker(\tau_{n,m}) \simeq \mu_{n/m}$  is the image of  $\iota_{n/m,n}$ .

**3.3. The infinitesimal group scheme  $\mu_q$  with  $q$  a power of  $p$ .** Suppose now that the characteristic of  $\mathcal{F}$  is  $p > 0$ , and let  $q = p^s$  be a power of  $p$  for some  $s \geq 1$ . In this section, we investigate homomorphisms  $\phi : \mu_q \rightarrow H$  for a smooth affine  $\mathcal{F}$ -group scheme  $H$ .

We first recall the following:

**Proposition 3.3.1.** [SGA3<sub>III</sub>, Cor. 4.1.7] *Let  $M$  be a connected and reductive group over  $\mathcal{F}$ . Then the center  $Z(M)$  of  $M$  (as a group scheme) is contained in each maximal torus of  $M$ .*

The main result we require is the following *folk result*:

**Proposition 3.3.2.** *If  $\phi : \mu_q \rightarrow G$  is a homomorphism of group schemes, the image of  $\phi$  is contained in a torus of  $G$ .*

We now deduce the result in two different ways; see also Remark 3.3.6 for further discussion.

*First proof of Proposition 3.3.2.* It is enough to argue that the image of  $\phi$  is contained in *some* torus in  $G$ . Let  $M = C_G^o(\phi)$  be the identity component of the centralizer of the image of  $\phi$ . According to Theorem 3.1.1, one knows that  $M$  is reductive. Since  $\mu_q$  is connected, evidently the image of  $\phi$  is contained in the *center* of the connected and reductive group  $M$ . Now the result follows from Proposition 3.3.1, and the proof is complete.  $\square$

We now outline a proof that *avoids* use of the conclusion from Theorem 3.1.1 that the centralizer  $M$  is reductive. To this end, we will first establish that when  $s = 1$ , such a homomorphism  $\phi : \mu_p \rightarrow H$  is completely determined by its tangent mapping.

Recall that a finite dimensional  $p$ -Lie algebra  $L$  over  $\mathcal{F}$  has a restricted enveloping algebra  $U^{[p]}(L)$  which is a co-commutative Hopf algebra over  $\mathcal{F}$  with dimension  $p^{\dim L}$  as  $\mathcal{F}$ -vector space.

**Proposition 3.3.3.** *The coordinate algebra  $\mathcal{F}[\mu_p]$  is isomorphic to the dual Hopf algebra of  $U^{[p]}(\text{Lie}(\mu_p))$ .*

*Proof.* This follows from [SGA3<sub>I</sub>, II §4 no. 7 and II §7 no. 4].

Since the argument is straightforward, for the benefit of the reader we present the following sketch. Write  $\mathfrak{m} \subset \mathcal{F}[\mu_p]$  for the *augmentation ideal* – i.e. the kernel of the augmentation mapping  $\mathcal{F}[\mu_p] \rightarrow \mathcal{F}$ . As explained in [Jan03, (I.7)], the algebra of distributions  $\text{Dist}(\mu_p)$  consists of all linear functionals  $\mu \in \mathcal{F}[\mu_p]^\vee = \text{Hom}_{\mathcal{F}}(\mathcal{F}[\mu_p], \mathcal{F})$  for which  $\mu$  vanishes on some power of  $\mathfrak{m}$ ; since the ideal  $\mathfrak{m}$  is nilpotent,  $\text{Dist}(\mu_p)$  coincides with the dual Hopf algebra  $\mathcal{F}[\mu_p]^\vee$ . Now, the Lie algebra  $\text{Lie}(\mu)$  identifies with the space of linear functionals  $\mu$  for which  $\mu(1) = 0$  and  $\mu(\mathfrak{m}^2) = 0$ . The inclusion  $\text{Lie}(\mu_p) \rightarrow \text{Dist}(\mu_p)$  induces an algebra homomorphism  $U^{[p]}(\text{Lie}(\mu_p)) \rightarrow \text{Dist}(\mu_p)$ , and it is straightforward to see that this mapping is injective. Since  $\dim_{\mathcal{F}} U^{[p]}(\text{Lie}(\mu_p)) = p = \dim_{\mathcal{F}} \mathcal{F}[\mu_p]$ , the conclusion of the Proposition follows.  $\square$

We are now going to produce a basis vector  $\delta = \delta_1$  for  $\text{Lie}(\mu_p)$ , as follows. Note that  $\mathcal{F}[\mu_p] = \mathcal{F}[T]/\langle T^p - 1 \rangle = \mathcal{F}[t]$ . The elements  $(t - 1)^i$  form a  $\mathcal{F}$ -basis for  $\mathcal{F}[\mu_p]$  for  $0 \leq i \leq p - 1$ . For each  $0 \leq i \leq p - 1$ , there is an element  $\delta_i \in \mathcal{F}[\mu_p]^\vee$  for which

$$\langle \delta_i, (t - 1)^j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

and  $\{\delta_i\}$  form a  $\mathcal{F}$ -basis for  $\mathcal{F}[\mu_p]^\vee$ . In fact,  $\delta_0$  is the mapping  $f \mapsto f(1)$ , and  $\delta_1$  is the “point-derivation” given by  $f \mapsto \left( t \frac{df}{dt} \right) \Big|_{t=1}$ ; the element  $\delta = \delta_1$  spans the Lie algebra  $\text{Lie}(\mu_p)$  as a  $\mathcal{F}$ -vector space, and a simple calculation shows that  $\delta^{[p]} = \delta_1$ .

**Proposition 3.3.4.** *Let  $H$  be an affine group scheme over  $\mathcal{F}$ . Then the assignment  $\phi \mapsto d\phi(\delta_1)$  determines a bijection*

$$\mathrm{Mor}_{\mathrm{gp\ scheme}}(\mu_p, H) \rightarrow \{Y \in \mathrm{Lie}(H) \mid Y = Y^{[p]}\}$$

where  $\mathrm{Mor}_{\mathrm{gp\ scheme}}(\mu_p, H)$  denotes the set of homomorphisms of  $\mathcal{F}$ -group schemes from  $\mu_p$  to  $H$ .

*Proof.* This follows from the result proved in [SGA3<sub>I</sub>, Exp. VII-A, 7.2 and 7.4] and also in [SGA3<sub>I</sub>, II §7 4.2(v)]; see [CGP15, A.7.14], as well.

However, in the relatively simple situation at hand, we can give a simple proof, as follows. First, since  $H$  and  $\mu_p$  are affine, the group scheme homomorphisms  $\mu_p \rightarrow H$  are in one-to-one correspondence with the Hopf algebra homomorphisms  $\mathcal{F}[H] \rightarrow \mathcal{F}[\mu_p]$ .

Let  $\mathfrak{p} \subset \mathcal{F}[H]$  and  $\mathfrak{m} \subset \mathcal{F}[\mu_p]$  be the kernels of the respective augmentation mapping. Write  $I = \langle f^p \mid f \in \mathfrak{p} \rangle \subset \mathcal{F}[H]$ . It follows from [Jan03, I.9.6(1)] that  $\mathcal{F}[H_1] = \mathcal{F}[H]/I$  is a Hopf algebra which determines an infinitesimal subgroup scheme  $H_1 \subset H$ .

Since  $\mathfrak{m}^p = 0$ , any Hopf algebra homomorphism  $\mathcal{F}[H] \rightarrow \mathcal{F}[\mu_p]$  must vanish on  $I$  and hence factors through a Hopf algebra homomorphism  $\mathcal{F}[H_1] = \mathcal{F}[H]/I \rightarrow \mathcal{F}[\mu_p]$ . It follows from [Jan03, I.9.6(2)] that  $\mathcal{F}[H_1]$  identifies with the dual Hopf algebra to  $U^{[p]}(\mathrm{Lie}(H))$ .

Now, in view of Proposition 3.3.3, taking duals gives a natural bijection between Hopf algebra mappings

$$U^{[p]}(\mathrm{Lie}(\mu_p)) \rightarrow U^{[p]}(\mathrm{Lie}(H))$$

and Hopf algebra mappings

$$U^{[p]}(\mathrm{Lie}(H))^\vee = \mathcal{F}[H_1] \rightarrow U^{[p]}(\mathrm{Lie}(\mu_p))^\vee = \mathcal{F}[\mu_p].$$

Finally, an algebra homomorphism  $U^{[p]}(\mathrm{Lie}(\mu_p)) \rightarrow U^{[p]}(\mathrm{Lie}(H))$  is completely determined by a homomorphism of  $p$ -Lie algebras  $\mathrm{Lie}(\mu_p) \rightarrow \mathrm{Lie}(H)$ ; since  $\delta_1$  is a basis vector for  $\mathrm{Lie}(\mu_p)$ , such homomorphisms of  $p$ -Lie algebras correspond bijectively with those elements  $Y \in \mathrm{Lie}(H)$  for which  $Y^{[p]} = Y$ .  $\square$

**Proposition 3.3.5.** (a) *If  $1 \leq m < n$ , the tangent mapping  $d\tau_{p^n, p^m} : \mathrm{Lie}(\mu_{p^n}) \rightarrow \mathrm{Lie}(\mu_{p^m})$  is zero.*  
 (b) *If  $1 \leq m \leq n$ , the tangent mapping  $d\iota_{p^m, p^n} : \mathrm{Lie}(\mu_{p^m}) \rightarrow \mathrm{Lie}(\mu_{p^n})$  is an isomorphism.*  
 (c) *Suppose that  $H$  is a smooth and affine  $\mathcal{F}$ -group scheme, that  $\phi : \mu_{p^m} \rightarrow H$  is a homomorphism of group schemes, and that  $d\phi = 0$ . Then  $\phi$  factors through the homomorphism  $\tau_{p^m, p^{m-1}} : \mu_{p^m} \rightarrow \mu_{p^{m-1}}$ .*

*Proof.* We first observe that for all  $n \geq 1$ , the inclusion  $\mu_{p^n} \subset \mathbf{G}_m$  induces an isomorphism

$$\mathrm{Lie}(\mu_{p^n}) \rightarrow \mathrm{Lie}(\mathbf{G}_m).$$

Now, the mapping  $\mu_{p^n} \rightarrow \mu_{p^m}$  in (a) is induced by the mapping  $t \mapsto t^{p^{n-m}} : \mathbf{G}_m \rightarrow \mathbf{G}_m$ , whose tangent mapping is indeed zero. Moreover, the mapping  $\mu_{p^m} \rightarrow \mu_{p^n}$  in (b) is induced by the *identity mapping*  $\mathbf{G}_m \rightarrow \mathbf{G}_m$ , whose tangent mapping is indeed an isomorphism.

As to (c), recall that we may identify  $\mu_p$  as the image of  $\iota_{p, p^m}$  in  $\mu_{p^m}$ , and as the kernel of the mapping  $\tau_{p^m, p^{m-1}}$ . Since  $d\phi = 0$ , it follows from Proposition 3.3.4 that the restriction of  $\phi$  to  $\mu_p$  is trivial, hence  $\phi$  factors through the quotient  $\mu_{p^m}/\mu_p \simeq \mu_{p^{m-1}}$ , as required.  $\square$

We can now give the

*Second proof of Proposition 3.3.2.* Writing  $q = p^s$ , we are going to show by induction on  $s \geq 1$  that the image of  $\phi$  is contained in a maximal torus of  $G$ .

Let us first treat the case  $s = 1$ . Let  $\phi : \mu_p \rightarrow G$  be a homomorphism of group schemes. Let  $X = d\phi(\delta) \in \mathrm{Lie}(G)$ . Since  $X^{[p]} = X$ , we see for each linear representation  $(\rho, V)$  of  $G$  that the minimal polynomial for the action of  $d\rho(X)$  on  $V$  divides the separable polynomial  $T^p - T$ , and [Spr98, Theorem 4.4.20] shows that  $X$  is a *semisimple* element of  $\mathrm{Lie}(G)$ .

Moreover [Bor91, Prop. 11.8] show that there is an  $\mathcal{F}$ -torus  $T \subset G$  with  $X \in \mathrm{Lie}(T)$ . It now follows from Proposition 3.3.4 that the image of  $\phi$  is contained in  $T$ , as required. This completes the proof when  $p = q$ .

We remark that we now know  $M = C_G^o(\phi)$  to be a reductive subgroup containing a maximal torus of  $G$  when  $q = p$ , using the more elementary result Proposition 3.1.2.

Now suppose that  $q = p^s$  for  $s > 1$  and that the result is known by induction for homomorphisms  $\mu_{p^t}$  with  $t < s$ . In view of Proposition 3.3.5(c), we may suppose that  $d\phi \neq 0$ .

Now let  $X_0 = d\phi(\delta_1)$  where  $\delta_1 \in \text{Lie}(\mu_p) = \text{Lie}(\mu_q)$  is the basis element fixed in the remarks preceding Proposition 3.3.4. According to that proposition,  $X_0$  determines a homomorphism  $\phi_0 : \mu_p \rightarrow G$  which clearly centralizes the image of  $\phi$ . By the induction hypothesis (or just the case  $q = p$ ), we find a maximal torus  $T$  of  $G$  containing the image of  $\phi_0$ . Now let  $M_0 = C_G^o(\phi_0)$ ; we've remarked already that  $M_0$  is a reductive subgroup of  $G$ . Since  $T$  and  $\mu_q$  are connected,  $M_0$  contains  $T$  and the image of  $\phi$ . Moreover, the image of  $\phi_0$  is contained in the center  $Z_0$  of  $M_0$ .

Since  $X_0 \in \text{Lie}(Z_0)$ , the composite homomorphism

$$\tilde{\phi} : \mu_q \xrightarrow{\phi} M_0 \rightarrow M_0/Z_0$$

has  $d\tilde{\phi} = 0$ , so Proposition 3.3.5 implies that  $\tilde{\phi}$  factors through the homomorphism  $\tau_{p^m, p^{m-1}}$ . It now follows by induction on  $m$  that  $\tilde{\phi}$  takes values in a maximal torus  $S$  of  $M_0/Z_0$ . But then the pre-image in  $M_0$  of a maximal torus of  $M_0/Z_0$  is a maximal torus of  $M_0$  - and hence of  $G$  - containing the image of  $\phi$ , and the proof of the Theorem is complete.  $\square$

*Remark 3.3.6.* (a) After completing an initial version of this manuscript, I learned that Brian Conrad recently gave a proof of Proposition 3.3.2 – see the appendix to [MT16]; the argument given in the “Second proof of Proposition 3.3.2” is similar to the one given by Conrad.

(b) In an email communication in 2007, J-P. Serre communicated to me a proof of Proposition 3.3.2 similar to the above “second proof”.

(c) Still another proof of the proposition is given in the recent paper [Prop. 6.2 PL15].

**3.4.  $\mu$ -homomorphisms with values in a reductive group.** In this section, we introduce the notion of a  $\mu$ -homomorphism. We begin by extending Proposition 3.3.2 to cover all  $\mu_n$ . More precisely:

**Proposition 3.4.1.** *Let  $n \in \mathbf{Z}_{\geq 1}$ . If  $\phi : \mu_n \rightarrow G$  is a homomorphism of group schemes, then the image of  $\phi$  lies in a torus of  $G$ .*

*Proof.* Write  $n = q \cdot n_0$  where  $q = p^m$  is a power of the characteristic  $p$ , and where  $\gcd(p, n_0) = 1$ . Write  $\phi_0 = \phi|_{\mu_q}$  and  $\phi_1 = \phi|_{\mu_{n_0}}$ . We have seen in Proposition 3.3.2 that the image of  $\phi_0$  is contained in a maximal torus of  $G$ ; in particular Proposition 3.1.2 shows that  $M_0 = C_G^o(\phi_0)$  is reductive. Proposition 3.3.1 now shows that the image of  $\phi_0$  is contained in each maximal torus of  $M_0$ . Thus the theorem will follow provided we prove that the image of  $\phi_1$  lies in a maximal torus of  $M_0$ .

It is therefore enough to prove the Theorem in the case of a homomorphism  $\phi : \mu_n \rightarrow G$  where  $n$  satisfies  $\gcd(n, p) = 1$ . Let  $E \supset \mathcal{F}$  be a Galois extension containing a primitive  $n$ -th root of unity  $\zeta$ . It follows from [Spr98, Theorem 6.4.5] that  $s = \phi(\zeta)$  lies in a maximal torus of  $G_E$ . Then  $C_{G_E}^o(s)$  is reductive by Proposition 3.1.2. In particular, the image of  $\phi_E$  lies in each maximal torus of  $C_{G_E}^o(s)$ . Since  $C_{G_E}^o(s) = (C_G^o(\phi))_E$ , it follows that the image of  $\phi$  is contained in each maximal torus of  $C_G^o(\phi)$ , and this completes the proof.  $\square$

Continue to suppose that  $G$  is a reductive group over  $\mathcal{F}$ . Given homomorphisms  $\phi : \mu_n \rightarrow G$  and  $\psi : \mu_m \rightarrow G$ , we regard  $\phi$  and  $\psi$  as *equivalent* provided that for some  $d \in \mathbf{Z}$  with  $m \mid d$  and  $n \mid d$ , the mappings  $\phi \circ \tau_{d,n}$  and  $\psi \circ \tau_{d,m}$  coincide.

**Definition 3.4.2.** By a  $\mu$ -homomorphism with values in a reductive group  $G$ , we mean an equivalence class of a homomorphism  $\phi_0 : \mu_m \rightarrow G$  of group schemes. We will denote such an equivalence class symbolically by  $\phi : \mu \rightarrow G$ .

The following is an immediate consequence of Proposition 3.4.1:

**Corollary 3.4.3.** *If  $\phi : \mu \rightarrow G$  is a  $\mu$ -homomorphism, there is a maximal torus  $T$  of  $G$  such that any homomorphism  $\mu_n \rightarrow G$  representing  $\phi$  has image in  $T$ .*

For each integer  $N$ , we identify  $\mathbf{Z}/N\mathbf{Z}$  with the subgroup  $\frac{1}{N}\mathbf{Z}/\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z}$  by the assignment  $a + N\mathbf{Z} \mapsto \frac{a}{N} + \mathbf{Z}$ . If  $T$  is a split torus, we write  $\text{Hom}(\mu, T)$  for the group of all  $\mu$ -homomorphisms with values in  $T$ . The next result describes this group.

**Proposition 3.4.4.** *Let  $T$  be a split torus. The assignment*

$$\text{Hom}(\mu, T) \rightarrow \text{Hom}_{\mathbf{Z}}(X, \mathbf{Q}/\mathbf{Z}) \simeq Y \otimes \mathbf{Q}/\mathbf{Z}$$

*given by  $\phi \mapsto \phi_0^*$  is bijective, where  $\phi_0 : \mu_n \rightarrow T$  represents  $\phi$ , where  $\phi_0^* : X \rightarrow \mathbf{Z}/n\mathbf{Z} = \frac{1}{n}\mathbf{Z}/\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z}$  is the map on character groups that  $\phi_0$  determines via Cartier duality, where  $X = X^*(T)$  is the character group of  $T$ , and where  $Y = X_*(T) \simeq X^\vee$  is the cocharacter group of  $T$ .*

*Proof.* Let  $\phi, \psi : \mu \rightarrow T$  be  $\mu$ -homomorphisms, represented respectively by the homomorphism  $\phi_0 : \mu_n \rightarrow T$  and  $\psi_0 : \mu_m \rightarrow T$ . Recall that  $\tau_{an,n}^* : \mathbf{Z}/n\mathbf{Z} = \frac{1}{n}\mathbf{Z}/\mathbf{Z} \rightarrow \mathbf{Z}/an\mathbf{Z} = \frac{1}{an}\mathbf{Z}/\mathbf{Z}$  is given by  $\frac{t}{n} + \mathbf{Z} \mapsto \frac{at}{an} + \mathbf{Z}$ . It is thus easy to see that  $\phi_0$  and  $\psi_0$  are equivalent if and only if  $\phi_0^*$  and  $\psi_0^*$  coincide as homomorphisms  $X \rightarrow \mathbf{Q}/\mathbf{Z}$ ; this shows that the indicated mapping is well-defined and injective.

If now  $f : X \rightarrow \mathbf{Q}/\mathbf{Z}$  is any group homomorphism, the image of  $f$  is finitely generated hence contained in the subgroup  $\frac{1}{N}\mathbf{Z}/\mathbf{Z} = \mathbf{Z}/N\mathbf{Z}$  for some  $N$ . Then  $f$  determines a homomorphism  $\phi_f : \mu_N \rightarrow T$  via Cartier duality, and it is clear that the above mapping assigns  $f$  to the  $\mu$ -homomorphism represented by  $\phi_f$ ; thus the indicated mapping is surjective.  $\square$

*Remark 3.4.5.* Let  $\mathcal{A}$  denote an integral domain with field of fractions  $K$ , and let  $T$  be a split torus over  $K$ . Write  $\mathcal{T}$  for the *canonical*  $\mathcal{A}$ -group scheme associated to  $T$  as in [BT84, p. I.2.11]. Thus  $\mathcal{T}$  is a split torus over  $\mathcal{A}$  and  $X^*(\mathcal{T}) = X^*(T)$ . Using Proposition 3.4.4, the discussion in [BT84, p. I.2.11] shows that the assignment  $\phi \mapsto \phi_0^*$  determines a bijection between the collection  $\text{Hom}_{\mathcal{A}}(\mu, \mathcal{T})$  of  $\mu$ -homomorphisms  $\phi : \mu_{\mathcal{A}} \rightarrow \mathcal{T}$  over  $\mathcal{A}$  and the group  $\text{Hom}_{\mathbf{Z}}(X^*(T), \mathbf{Q}\mathbf{Z})$

If  $\phi$  is a  $\mu$ -homomorphism with values in the split torus  $T$ , we write  $\phi^* = \phi_0^*$  for the corresponding element of  $Y \otimes \mathbf{Q}/\mathbf{Z}$ , with notation as in Proposition 3.4.4. Conversely, if  $x \in V = Y \otimes \mathbf{Q}$  and  $\bar{x} \in Y \otimes \mathbf{Q}/\mathbf{Z}$  the image of  $x$ , we write  $\phi_x = \phi_{\bar{x}}$  for the corresponding homomorphism  $\mu \rightarrow T$ .

We now give a description of the centralizer of a  $\mu$ -homomorphism with values in a maximal split torus of  $G$ , and thus describes the subgroups  $M$  of  $G$  of type  $C(\mu)$  as discussed in the introduction to this paper.

**Theorem 3.4.6.** *Assume that  $G$  is split reductive over  $\mathcal{F}$ , and write  $T$  for a maximal split  $\mathcal{F}$ -torus. Suppose that  $\psi$  be a  $\mu$ -homomorphism with values in  $T$ , and write  $M = C_G^o(\psi)$ .*

- (a)  *$M$  is a reductive subgroup of  $G$  containing the maximal torus  $T$ .*
- (b) *Let  $\Phi \subset X^*(T)$  be the set of roots of  $G$  with respect to  $T$ , and let  $Y = X_*(T)$  be the cocharacter group of  $T$ . For  $\alpha \in \Phi$ , write  $U_\alpha$  for the corresponding root subgroup of  $G$ . Choose  $y \in Y \otimes \mathbf{Q}$  representing the element  $\psi^* \in Y \otimes \mathbf{Q}/\mathbf{Z}$ , and let  $\Phi_y$  denote the corresponding closed and symmetric system of roots*

$$\Phi_y = \{\alpha \in \Phi \mid \langle \alpha, y \rangle \in \mathbf{Z}\}$$

*as in section 2.5. Then*

$$M = \langle T, U_\alpha \mid \alpha \in \Phi_y \rangle.$$

*In particular,  $\Phi_y$  is the root system of  $M$ .*

*Proof.* (a) just restates proposition 3.4.1(b).

Now for  $\alpha \in \Phi$ , it is easy to see that  $\mu$  acts trivially on  $U_\alpha$  if and only if  $\langle \alpha, x \rangle \in \mathbf{Z}$ . Thus the characterization of  $M$  in (b) follows immediately from Proposition 3.1.2.  $\square$

*Remark 3.4.7.* (a) The Theorem now shows that section 2.5 describes the root systems  $\Phi_x$  for the connected centralizer  $M$  of the image of a  $\mu$ -homomorphism. In particular, Theorem 2.5.3 describes a simple system of roots for  $\Phi_x$ , and thus Remark 2.5.4 describes the Dynkin diagram of  $M$ .

- (b) The root system  $\Phi_y$  is of course independent of the choice of representative  $y \in Y \otimes \mathbf{Q}$  representing  $\psi^* \in Y \otimes \mathbf{Q}/\mathbf{Z}$ . Moreover, the above description shows that subgroup  $M$  only depends on the ( $W_{\text{aff}}$ -orbit of the) *facet* of  $V$  containing  $y$ .
- (c) Following [Ser06], one can describe  $\mu$ -homomorphisms using *Kac coordinates*, as follows. Suppose that  $G$  is simple and that  $\{\alpha_1, \dots, \alpha_r\}$  is a system of simple roots. After replacing  $y \in Y \otimes \mathbf{Q}$  by an  $W_{\text{aff}}$ -conjugate point and thus we may suppose - in the terminology and notation of section 2.5 - that  $y$  is contained in the  $\mathbf{Q}$ -simplex with vertices  $0 = \omega_0, \omega_1/n_1, \dots, \omega_r/n_r$ . In particular,  $y = \sum_{i=1}^r a_i \omega_i/n_i$  for certain  $a_i \in \mathbf{Q}$  with  $0 \leq a_i \leq 1$  for each  $i$ , and  $0 \leq \sum a_i \leq 1$ . The rational numbers  $(a_1, \dots, a_r)$  are known as the Kac coordinates of the  $\mu$ -homomorphism.

If  $\mathcal{F}$  is algebraically closed, the Kac coordinates determine the  $\mu$ -homomorphism up to conjugation in  $G(\mathcal{F})$ ; see [Theorem 6.3 PL15].

**3.5. Some automorphisms of subgroups of type  $C(\mu)$ .** Keep the notations of section 3.4; thus  $G$  is a split reductive group over  $\mathcal{F}$ ,  $T$  is a maximal split torus of  $G$ ,  $X = X^*(T)$ , and  $Y = X_*(T)$ .

Fix  $y \in Y \otimes \mathbf{Q}$ , and let  $\phi = \phi_y : \mu \rightarrow T$  be the  $\mu$ -homomorphism determined by  $\bar{y} \in Y \otimes (\mathbf{Q}/\mathbf{Z})$ . As before, consider the identity component  $M = M_y$  of the centralizer in  $G$  of the image of  $\phi_y$ .

Let  $Z$  denote the center of  $M$ , and write  $M_{\text{ad}} = M/Z$  for the *adjoint quotient* of  $M$ . Then the image  $T_1 = T/Z$  of  $T$  in  $M_{\text{ad}}$  is a maximal torus of  $M_{\text{ad}}$ .

**Proposition 3.5.1.** (a) *The character group of  $T_1$  is given by  $X^*(T_1) = \mathbf{Z}\Phi_y$ .*

(b)  *$y$  determines a cocharacter  $\psi_y \in X_*(T_1)$  with*

$$\langle \gamma, \psi_y \rangle = \langle \gamma, y \rangle \quad \text{for } \gamma \in \Phi_y.$$

(c) *For any  $t \in \mathcal{F}^\times$ ,  $t_y = \text{Int}(\psi_y(t))$  determines a  $\mathcal{F}$ -automorphism of  $M$ . Moreover, if  $\alpha \in \Phi_y$  and  $x_\alpha : \mathbf{G}_a \rightarrow U_\alpha$  is any isomorphism such that  $\text{Ad}(s)x_\alpha(u) = x_\alpha(\alpha(s)u)$  for  $s \in T(\mathcal{F})$  and  $u \in \mathcal{F}$ , then  $t_y x_\alpha(u) = x_\alpha(t^{(\alpha, y)}u)$  for  $u \in \mathcal{F}$*

*Proof.* Indeed, (a) follows since  $M/Z$  is an adjoint semisimple group with root system  $\Phi_y$ . Since  $X_*(T_1)$  is dual to  $X^*(T_1) = \mathbf{Z}\Phi_y$ , the existence of the cocharacter  $\psi_y$  in (b) follows from (a). Now (c) follows from definitions.  $\square$

**3.6. Examples of subgroups of type  $C(\mu)$ .** In this section, we present some examples to illustrate some features of subgroups of type  $C(\mu)$ . We begin by pointing out that there are subgroups of type  $C(\mu)$  which are not the centralizer of any semisimple element of  $G$ . Note the following:

**Proposition 3.6.1.** *Assume that  $G$  is  $\mathcal{F}$ -split, absolutely simple of adjoint type with split maximal torus  $T$ , let  $\alpha_1, \dots, \alpha_\ell \in \Phi \subset X^*(T)$  be a system of simple roots, let  $\omega_i^\vee \in X_*(T)$  be the fundamental dominant co-roots, and let  $\tilde{\alpha}$  be the highest root. Write*

$$\tilde{\alpha} = \sum_{i=1}^{\ell} n_i \alpha_i \quad \text{for } n_i \in \mathbf{Z}_{>0}.$$

*For  $1 \leq i \leq \ell$ ,  $\omega_i/n_i \in Y \otimes \mathbf{Q}$  determines a  $\mu$ -homomorphism, and we write  $M_i$  for its connected centralizer. If  $n_i = p$ , the center of  $M_i$  is an infinitesimal group scheme, so that  $M_i$  is not the connected centralizer of any semisimple element of  $G$ .*

*Proof.* For each  $i$ , the root system of  $M_i$  with respect to  $T$  has  $\Delta_i = \{\alpha_0 = -\tilde{\alpha}\} \cup \{\alpha_j \mid j \neq i\}$  as a system of simple roots. It follows from [SS70, §4: 4.5] that if  $n_i$  is prime,  $\mathbf{Z}\Delta/\mathbf{Z}\Delta_i \simeq \mathbf{Z}/n_i\mathbf{Z}$ . Now the Proposition follows from the fact that the center of  $M_i$  is the  $\mathcal{F}$ -group scheme which is the *Cartier dual* - see Section 3.2 - of  $\mathbf{Z}\Delta/\mathbf{Z}\Delta_i$ .  $\square$

**Remark 3.6.2.** Keep the notations of Proposition 3.6.1.

- (a) When  $\Phi = G_2$ , then  $\tilde{\alpha} = 3\alpha_1 + 2\alpha_2$ . And indeed, the group  $M_1$  with root system  $A_2$ , has infinitesimal center in characteristic 3, and the group  $M_2$  with root system  $A_1 \times A_1$  has infinitesimal center in characteristic 2.
- (b) When  $\Phi = E_8$ , the coefficient of  $\alpha_5$  in  $\tilde{\alpha}$  is 5. The subgroup  $M_5$  with root system  $A_4 \times A_4$  indeed has infinitesimal center in characteristic 5.

We now demonstrate that a reductive subgroup  $H$  of  $G$  may have the property that  $H_E$  is of type  $C(\mu)$  for some finite separable field extension, but that  $H$  is not of type  $C(\mu)$ .

**Proposition 3.6.3.** *Let  $G = \mathrm{Sp}_4(\mathcal{F})$  be the split symplectic group of rank 2. There is a reductive  $\mathcal{F}$ -subgroup  $H \subset G$  containing a maximal torus of  $G$  with the following properties:*

- (a)  $H$  is an  $\mathcal{F}$ -form of  $\mathrm{SL}_2 \times \mathrm{SL}_2$ .
- (b) There is a separable quadratic field extension  $E$  such that  $H_E$  is of type  $C(\mu)$ .
- (c)  $H$  is not the centralizer of the image of any homomorphism  $\mu \rightarrow G$  defined over  $\mathcal{F}$ .

*Sketch.* Let  $\mathcal{F} \subset E$  be a separable quadratic field extension. and write  $x \mapsto \bar{x}$  for the action of the non-trivial element of  $\mathrm{Gal}(E/\mathcal{F})$ . Consider the alternating form  $\beta$  on the  $\mathcal{F}$ -vector space  $V = E \oplus E$  given by the formula  $\beta(v, w) = \mathrm{tr}_{E/\mathcal{F}}(v_1 \bar{w}_2 - w_1 \bar{v}_2)$  where  $\mathrm{tr}_{E/\mathcal{F}}$  is the trace mapping. Since  $E$  is a separable extension of  $\mathcal{F}$ , the alternating pairing  $\beta$  is non-degenerate.

Thus we may identify  $G$  with  $\mathrm{Sp}(V, \beta)$ . If we write  $A = \mathrm{End}_{\mathcal{F}}(V) \simeq \mathrm{Mat}_4(\mathcal{F})$ , there is a symplectic involution  $\sigma$  on  $A$  determined by the property  $\beta(Xv, w) = \beta(v, \sigma(X)w)$  for  $v, w \in V$ ; then  $G = \mathrm{Iso}(A, \sigma)$  and we may describe  $G$  “functorially” by the rule

$$G(\Lambda) = \{X \in A \otimes_{\mathcal{F}} \Lambda \mid X \cdot \sigma(X) = 1\}$$

for a commutative  $\mathcal{F}$ -algebra  $\Lambda$ .

Viewing  $V$  as an  $E$ -vector space, we find the  $\mathcal{F}$ -subalgebra  $B = \mathrm{End}_E(V) \subset A = \mathrm{End}_{\mathcal{F}}(V)$ . Evidently  $B \simeq \mathrm{Mat}_2(E)$  and one readily checks that  $B$  is  $\sigma$ -invariant; in fact,  $\sigma|_B$  is the “standard” involution of the quaternion  $E$ -algebra  $B$ . The algebra  $B$  determines a subgroup  $H$  of  $G$  defined functorially by the rule

$$H(\Lambda) = \{b \in B \otimes_{\mathcal{F}} \Lambda \mid b \cdot \sigma(b) = 1\} \subset G(\Lambda);$$

thus  $H = \mathrm{Iso}(B, \sigma|_B)$  is a semisimple subgroup of  $G$ ; moreover,  $H \simeq R_{E/\mathcal{F}} \mathrm{SL}_2$  is a  $\mathcal{F}$ -form of  $\mathrm{SL}_2 \times \mathrm{SL}_2$ . This proves (a)

Now, the center of  $H_E$  is  $\mu_2 \times \mu_2$ . Since  $V_E = V \otimes_{\mathcal{F}} E$  is the direct sum of two copies of the “natural” four dimensional representation of  $\mathrm{SL}_2 \times \mathrm{SL}_2$  and is a faithful representation of  $G_E$ , it follows that  $H_E$  is the centralizer of any homomorphism  $\mu \rightarrow Z(H_E) = \mu_2 \times \mu_2$  whose image is not central in  $G$ . There are precisely two such  $\mu$ -homomorphisms, and they are interchanged by the action of the non-trivial element of the Galois group  $\Gamma = \mathrm{Gal}(E/\mathcal{F})$ .  $\square$

*Remark 3.6.4.* With  $H \subset G$  as in the proof of Proposition 3.6.3, the given arguments show also the following:

- If the characteristic of  $\mathcal{F}$  is not 2, then  $H$  is the centralizer of a semisimple element in  $G(E)$ .
- If the characteristic of  $\mathcal{F}$  is 2, the  $H$  is the centralizer of a semisimple element

$$X \in \mathrm{Lie}(G)(E) = \mathrm{Lie}(G_E) = \mathrm{Lie}(G) \otimes_{\mathcal{F}} E,$$

and if  $Z = Z(H)$  denotes the center of  $H$ , then the group of points  $Z(\mathcal{F}_{\mathrm{alg}})$  is trivial where  $\mathcal{F}_{\mathrm{alg}}$  is an algebraic closure of  $\mathcal{F}$ .

We finally demonstrate that the property that a reductive subgroup  $H \subset G$  is of type  $C(\mu)$  is in general affected by isogeny.

**Proposition 3.6.5.** *Keep the notations of Proposition 3.6.3, let  $G_1 = \mathrm{PSp}_{4, \mathcal{F}}$ , and write*

$$\pi : G = \mathrm{Sp}_4 \rightarrow G_1 = \mathrm{PSp}_4$$

*for the isogeny. There is a semisimple subgroup  $H \subset G$  such that  $H$  is not of type  $C(\mu)$  while  $H_1 = \pi(H) \subset G_1$  is of type  $C(\mu)$ .*

*Sketch.* Let  $H$  be as in the proof of Proposition 3.6.3. The center of  $H = R_{E/\mathcal{F}} \mathrm{SL}_2$  is  $R_{E/\mathcal{F}} \mu_2$ , which is an  $\mathcal{F}$ -form of  $\mu_2 \times \mu_2$ . In fact,  $\mathcal{F}$ -homomorphisms  $\mu_2 \rightarrow R_{E/\mathcal{F}} \mu_2$  correspond bijectively to  $\mathrm{Gal}(E/\mathcal{F})$ -equivariant maps  $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \rightarrow \mathbf{Z}/2\mathbf{Z}$  where  $\mathrm{Gal}(E/\mathcal{F})$  acts trivially on the target, and the non-trivial element of  $\mathrm{Gal}(E/\mathcal{F})$  acts on the domain group by switching the factors.

Thus there are precisely two  $\mathcal{F}$ -homomorphisms  $\mu_2 \rightarrow R_{E/\mathcal{F}}\mu_2$ , and the non-trivial such homomorphism corresponds to the inclusion of the center of  $G = \mathrm{Sp}_4$  in the center of  $H = R_{E/\mathcal{F}}\mathrm{SL}_2$ . It follows that the center of  $H_1$  is  $\mathcal{F}$ -isomorphic to  $(R_{E/\mathcal{F}}\mu_2/\mu_2) \simeq \mu_2$ . Since  $H$  – and hence also  $H_1$  – is the connected centralizer of its center, the result follows.  $\square$

#### 4. REDUCTIVE GROUPS OVER A LOCAL FIELD

Throughout this section,  $\mathcal{A}$  will denote a complete discrete valuation ring with field of fractions  $K$  and residue field  $k$ . Consider a connected and reductive algebraic group  $G$  over  $K$ . We are going to consider the parahoric group schemes  $\mathcal{P}$  attached to  $G$ . In particular,  $\mathcal{P}$  is a smooth, affine  $\mathcal{A}$ -group scheme, and its generic fiber  $\mathcal{P}_K$  coincides with the given group  $G$ .

In [BT84], the authors describe the parahoric group schemes for quasisplit  $G$  using *schematic root data*; we begin by recalling this notion in Section 4.1 and the associated construction in Theorem 4.3.1. In case  $G$  is split over  $K$  with maximal split torus  $T$ , those parahoric group schemes  $\mathcal{P} = \mathcal{P}_x$  which “contain  $T$ ” (in a suitable sense) arise from points  $x$  in  $V = X_*(T) \otimes \mathbf{Q}$ ; we recall this description in Section 4.3.

Suppose that  $G$  is split with root system  $\Phi \subset X^*(T)$ , and consider the parahoric group scheme  $\mathcal{P}_x$  corresponding to  $x \in V$ . Then  $x$  determines a subsystem  $\Phi_x$  as in Section 2.5; in Theorem 4.4.2 we show that  $\Phi_x$  determines a “Chevalley schematic root datum” for the subgroup  $M = C_G^o(\phi_{\bar{x}})$  of type  $C(\mu)$  where  $\phi_{\bar{x}}$  is the  $\mu$ -homomorphism determined by the image  $\bar{x}$  of  $x$  in  $X_*(T) \otimes \mathbf{Q}/\mathbf{Z}$ ; see Theorem 3.4.6. In particular, we find a Chevalley group scheme  $\mathcal{M}$  with generic fiber  $M$ , and we see  $\mathcal{M}$  as a closed subscheme of  $\mathcal{P}_x$ .

Existence of this subgroup scheme  $\mathcal{M}$  settles the proof of our main result – Theorem 1 – in case  $G$  is split. Finally, when  $G$  splits over an unramified extension, the result is obtained by descent in Section 4.5.

**4.1. A Chevalley system for a split reductive group.** Suppose that the group  $G$  is split over  $K$ . Fix a maximal split torus  $T$  of  $G$ , and let  $(X, Y, \Phi, \Phi^\vee)$  denote the root datum of  $G$  with respect to  $T$ .

Thus  $\Phi \subset X = X^*(T)$  is the set of roots of  $T$  in  $\mathrm{Lie}(G)$ . For each  $\alpha \in \Phi$ , there is a corresponding 1-dimensional  $K$ -subgroup  $U_\alpha$  – the *root subgroup* – normalized by  $T$ . Write  $G_\alpha = \langle U_{\pm\alpha} \rangle$ .

Let us fix an isomorphism between the diagonal maximal torus  $S$  of  $\mathrm{SL}_2$  and the 1-dimensional split torus  $\mathbf{G}_m$ , and write  $U^+ \subset \mathrm{SL}_2$  for the upper triangular unipotent subgroup. For a root  $\alpha$ , a *pinning* of  $\alpha$  will mean a central  $K$ -isogeny  $\psi_\alpha : \mathrm{SL}_2 \rightarrow G_\alpha$  such that:

- (P1)  $\psi_\alpha$  maps  $S$  to  $T$  and the restriction of  $\psi_\alpha$  to  $S$  identifies with the co-root  $\alpha^\vee \in X_*(T)$  (for the fixed identification of  $S$  and  $\mathbf{G}_m$ ), and
- (P2)  $\psi_\alpha$  maps  $U^+$  isomorphically to  $U_\alpha$ .

We fix an identification  $U^+ \simeq \mathbf{G}_a$ . Then upon restriction to the root subgroups of  $\mathrm{SL}_2$ ,  $\psi_\alpha$  determines  $K$ -isomorphisms  $\psi_{\alpha,\pm} : \mathbf{G}_a \rightarrow U_{\pm\alpha}$ .

For a fixed system of simple roots  $\Delta \subset \Phi$ , a pinning of  $G$  relative to  $T$  is a collection of pinnings  $(\psi_\alpha)_{\alpha \in \Delta}$ .

**Proposition 4.1.1.** [SGA3<sub>III</sub>, XXIII Prop. 6.2] *A pinning  $(\psi_\alpha)_{\alpha \in \Delta}$  of  $G$  relative to  $T$  determines a Chevalley system  $(\psi_\alpha)_{\alpha \in \Phi^+}$  for  $G$  prolonging it.*

- Remark 4.1.2.* (a) Recall [BT84, (3.2.2)] that a Chevalley system is a collection of pinnings  $(\psi_\alpha)$  for each root  $\alpha \in \Phi^+$  satisfying an additional “compatibility” property spelled out e.g. in [BT84, (3.2.2) condition (Ch 2)].
- (b) See the discussion following [CGP15, Defn A.4.12] to see that the definitions we have given of “pinning” and “Chevalley system” encode the same information as the definitions given in [SGA3<sub>III</sub>, XXIII Defn 1.1 and 6.1].
- (c) If  $(\psi_\alpha)_{\alpha \in \Phi^+}$  is a Chevalley system, then  $(X_\alpha = d\psi_\alpha(1))_\alpha$  is a *Chevalley basis* for  $\mathrm{Lie}(G)$ ; see e.g. [Hum78, §25] for the definition.

**4.2. Schematic root data for a split reductive group  $G$ .** Following [BT84, §3.1], we consider the notion of a *schematic root datum*

$$\mathcal{D} = (\mathcal{T}, (\mathcal{U}_a)_{a \in \Phi})$$

for  $G$ . In *loc. cit.* the case of quasi-split  $G$  is considered; we have simplified the definition somewhat since we only consider this notion when  $G$  is split over  $K$ .

According to [BT84, (1.2.11)], there is an essentially unique  $\mathcal{A}$ -split torus  $\mathcal{T}$  with generic fiber  $T = \mathcal{T}_K$ . By a *schematic root datum*, we mean a collection  $(\mathcal{U}_a)_{a \in \Phi}$  where  $\mathcal{U}_a$  is a  $\mathcal{A}$ -group scheme for each  $a \in \Phi$ , such that

- ( $\diamond$ 1)  $\mathcal{U}_a$  is affine, flat and of finite type over  $\mathcal{A}$  for each  $a$ ,
- ( $\diamond$ 2)  $U_a = \mathcal{U}_{a,K}$  for each  $a$ ,
- ( $\diamond$ 3) for  $a, b \in \Phi$  with  $a \neq -b$ , for a suitable ordering on  $\Phi \cap (a, b)$  the commutator mapping

$$\gamma_{a,b} : U_a \times U_b \rightarrow \prod_{c \in \Phi \cap (a,b)} U_c$$

arises by base change from an  $\mathcal{A}$ -morphism  $\mathcal{U}_a \times \mathcal{U}_b \rightarrow \prod_{c \in \Phi \cap (a,b)} \mathcal{U}_c$ .

- ( $\diamond$ 4) for  $a \in \Phi$ , the mapping  $((z, u) \rightarrow zuz^{-1}) : T \times U_a \rightarrow U_a$  arises by base change from an  $\mathcal{A}$ -morphism  $\mathcal{T} \times \mathcal{U}_a \rightarrow \mathcal{U}_a$ .

*Remark 4.2.1.* In condition ( $\diamond$ 3) we have been vague about the notion “suitable ordering”; see [BT84, p. 3.1.2] for details. In this paper, we will not have occasion to verify the conditions ( $\diamond$ 1)–( $\diamond$ 4) directly.

A schematic root data  $\mathcal{D}$  determines a group scheme with generic fiber  $G$ , according to the following important result:

**Theorem 4.2.2.** [BT84, Théorèmes 3.8.1 and 3.8.3] *Let  $\mathcal{D} = (\mathcal{U}_a)_{a \in \Phi}$  be a schematic root datum for  $G$ , and suppose that all  $\mathcal{U}_a$  are smooth over  $\mathcal{A}$ . Then there is an group scheme  $\mathcal{P} = \mathcal{P}_{\mathcal{D}}$  which is affine, smooth and of finite type over  $\mathcal{A}$  with connected fibers such that the following hold:*

- (S1) *the inclusion  $T \rightarrow G$  (resp.  $U_a \rightarrow G$  for  $a \in \Phi$ ) prolongs to a isomorphism from  $\mathcal{T}$  (resp.  $\mathcal{U}_a$ ) to a closed  $\mathcal{A}$ -subgroup scheme of  $\mathcal{P}$ ;*
- (S2) *for each system of positive roots  $\Phi^+$  of  $\Phi$  and for each order of  $\Phi^+$  (resp. of  $\Phi^- = -\Phi^+$ ), the product mapping is an isomorphism of schemes from  $\prod_{a \in \Phi^+} \mathcal{U}_a$  to a closed subgroup scheme  $\mathcal{U}^+$  (resp.  $\mathcal{U}^-$ ) of  $\mathcal{P}$ ;*
- (S3) *The product mapping determines an isomorphism of schemes from  $\mathcal{U}^- \times \mathcal{T} \times \mathcal{U}^+$  to an open sub-scheme of  $\mathcal{P}$ .*

Moreover,  $\mathcal{P}$  is up to isomorphism the unique  $\mathcal{A}$ -group scheme with generic fiber  $G = \mathcal{P}_K$  having the properties (S1), (S2) and (S3).

Let us fix a root  $\alpha \in \Phi$ , and consider the rank 1 split reductive  $K$ -subgroup  $G_\alpha = \langle T, U_{\pm\alpha} \rangle$ . We observe:

**Proposition 4.2.3.** *If  $\mathcal{D}$  is a schematic root datum for  $G$  as above, then  $\mathcal{D}_\alpha = (\mathcal{U}_\alpha, \mathcal{U}_{-\alpha})$  is a schematic root datum for  $G_\alpha$ . If  $\mathcal{P}_\alpha$  is the  $\mathcal{A}$ -group scheme determined by this schematic root datum, then  $\mathcal{P}_\alpha$  is a closed subgroup scheme of  $\mathcal{P}$ .*

*Proof.* It is immediate from definitions that  $\mathcal{D}_\alpha$  is a schematic root datum. It only remains to argue that  $\mathcal{P}_\alpha$  is a closed subgroup scheme of  $\mathcal{P}$ . For this, we choose a faithful linear representation  $\mathcal{L}$  of  $\mathcal{P}$ , where  $\mathcal{L}$  is a free  $\mathcal{A}$ -module of finite rank; see [BT84, §1.4.5].

Since  $\mathcal{L}$  is a faithful  $\mathcal{P}$ -module, it is clear that  $\mathcal{L}$  is a  $\mathcal{D}$ -module in the sense of [BT84, §3.1.4] and hence also  $\mathcal{L}$  is a  $\mathcal{D}_\alpha$ -module. We apply [BT84, §3.8.1] to find a closed embedding  $\mathcal{P}_\alpha$  as a subgroup scheme of  $\text{GL}(\mathcal{L})$ . Now our assertion follows since it is immediate that this embedding factors through the embedding  $\mathcal{P} \subset \text{GL}(\mathcal{L})$ .  $\square$



**4.3. Schematic root data and parahoric group schemes.** In this section, we first describe how one obtains split reductive groups over  $\mathcal{A}$  using Chevalley schematic root data, essentially constructed from a Chevalley system for  $G$  as in Proposition 4.1.1 and Remark 4.1.2.

The parahoric group schemes are parameterized by points in the affine building associated with  $G$ . Since the affine building is the union of apartments each determined by a maximal split torus of  $G$  and since each maximal split torus is conjugate by an element of  $G(K)$ , it suffices to describe the parahoric subgroups associated with such an apartment. In the remainder of this section, we describe the groups schemes.

Recall that a finitely generated free  $\mathcal{A}$ -module  $M$  determines an  $\mathcal{A}$ -group scheme  $M_{\text{add}}$  whose generic fiber  $M_{\text{add},K}$  is the vector group determined by the  $K$ -vector space  $M \otimes_{\mathcal{A}} K$ .

Let  $U$  be a 1 dimensional commutative unipotent group scheme over  $K$ , and let  $\mathbf{G}_a \xrightarrow{\psi} U$  be a fixed isomorphism. For  $n \in \mathbf{Z}$ , consider the fractional ideal  $I = \pi^n \mathcal{A} \subset K$ . Then  $I_{\text{add}}$  is a smooth  $\mathcal{A}$ -group scheme with generic fiber  $\mathbf{G}_{a,K}$ , and by transport of structure along  $\psi$ , one finds a smooth  $\mathcal{A}$ -group scheme  $\mathcal{U}_n$  with an identification  $\mathcal{U}_{n,K} = U$ . Moreover, the isomorphism  $\psi$  determines an isomorphism  $I = \pi^n \mathcal{A} \xrightarrow{\psi} \mathcal{U}_n(\mathcal{A})$ . Compare [BT84, (4.3.2)]

More generally, for  $r \in \mathbf{Q}$ , write  $\lceil r \rceil = \min\{n \in \mathbf{Z} \mid n \geq r\}$  for the ‘‘ceiling function’’, and write  $\mathcal{U}_r = \mathcal{U}_{\lceil r \rceil}$ . Thus  $\mathcal{U}_r(\mathcal{A}) = \{\psi(a) \mid v(a) \geq r\}$  where  $v : K^\times \rightarrow \mathbf{Z}$  is the (normalized) valuation of  $K$ , and  $\mathcal{U}_r = (\pi^{\lceil r \rceil} \mathcal{A})_{\text{add}}$ .<sup>5</sup>

Consider now a Chevalley system  $(\psi_\alpha)$  for the group  $G$  and the split torus  $T$ , as in Proposition 4.1.1. For each  $\alpha \in \Phi$ , recall that  $\psi_\alpha$  determines isomorphisms  $\psi_{\alpha,\pm} : \mathbf{G}_a \rightarrow U_{\pm\alpha}$ . We write  $\mathcal{U}_{\pm\alpha} = \mathcal{U}_{\pm\alpha,0}$  for the  $\mathcal{A}$ -group schemes obtained as above by transport of structure along  $\psi_{\alpha,\pm}$  from the unit ideal  $\mathcal{A}$ .

**Theorem 4.3.1.** (a) *The collection  $(\mathcal{U}_\alpha)_{\alpha \in \Phi}$  determines a schematic root datum  $\mathcal{D}_{\text{Chev}(\Phi)}$ , called a **Chevalley schematic root datum**.*  
 (b) *The group scheme  $\mathcal{G} = \mathcal{P}_{\mathcal{D}}$  determined by the Chevalley schematic root datum  $\mathcal{D} = \mathcal{D}_{\text{Chev}(\Phi)}$  as in Theorem 4.2.2 is a split reductive group scheme over  $\mathcal{A}$ .*

*Proof.* The result amounts to the description of a split reductive group scheme over  $\mathcal{A}$ , and thus essentially follows from [SGA3<sub>III</sub>]. Using the language of schematic root data, (a) follows from [BT84, Prop. 3.2.7], and (b) follows from [BT84, (3.2.13)].  $\square$

We immediately obtain:

**Proposition 4.3.2.** *Let  $\alpha \in \Phi$  be a root, and let  $\mathcal{G}_\alpha$  be the group scheme determined by the schematic root datum  $(\mathcal{U}_\alpha, \mathcal{U}_{-\alpha})$  as in Proposition 4.2.3. Then  $\mathcal{G}_\alpha \subset \mathcal{G}$  is a split reductive group with  $\mathcal{G}_{\alpha,K} = G_\alpha$ .*

Now recall that we write  $V = X_*(T) \otimes \mathbf{Q}$ . We point out that  $V \otimes_{\mathbf{Q}} \mathbf{R}$  is the apartment of the affine building of  $G$  determined by  $T$ . We are going to describe the parahoric group schemes determined by points in  $V$ .

Fix  $x \in V$ . For  $\alpha \in \Phi$ , consider the group scheme  $\mathcal{U}_{\alpha,x} = (\mathcal{U}_\alpha)_{\langle \alpha, x \rangle}$ . Thus  $\mathcal{U}_{\alpha,x}$  is obtained via  $\psi_{\alpha,+}$  from the  $\mathcal{A}$ -group scheme  $I_{\text{add}}$  where  $I$  denotes the ideal  $\pi^{\lceil \langle \alpha, x \rangle \rceil} \mathcal{A}$ .

**Theorem 4.3.3.** *The collection  $\mathcal{D}_x = (\mathcal{U}_{\alpha,x})_{\alpha \in \Phi}$  is a schematic root datum.*

*Proof.* Use [BT84, (4.6.26)] to see that the function  $\Phi \rightarrow \mathbf{R}$  given by  $\alpha \mapsto \langle \alpha, x \rangle$  is *concave*. Now the assertion follows from [BT84, Theorem 4.5.4].  $\square$

We write  $\mathcal{P} = \mathcal{P}_x$  for the group scheme obtained via Theorem 4.2.2 from the schematic root datum  $\mathcal{D}_x$  for  $G$ . For a facet  $F$  in  $V$ , write  $\mathcal{P}_F = \mathcal{P}_x$  for some (any) point  $x \in F$ .

**Definition 4.3.4.** When  $G$  is  $K$ -split, the *parahoric group schemes* attached to  $G$  are the  $G(K)$ -conjugates of the group schemes  $\mathcal{P}_F$  of Theorem 4.3.3 for  $F$  a facet in  $V$ .

<sup>5</sup>Of course, the notation  $\mathcal{U}_r$  can be made meaningful for  $r \in \mathbf{R}$ , but we only require this notion for rational  $r$ .

*Remark 4.3.5.* We have assumed  $G = \mathcal{P}_K$  to be connected. Since  $G$  is split, the special fiber  $\mathcal{P}_k$  is always connected; see Theorem 4.2.2.

In [BT84], the *parahoric subgroups* of  $G(K)$  are the “connected stabilizers” in  $G(K)$  of facets of the *building* of  $G$ . It follows from [BT84, (5.2.6)], the parahoric subgroups are precisely the  $G(K)$ -conjugates of the subgroups  $\mathcal{P}_x(\mathcal{A})$  for some  $x$  as above. See also [BT84, Prop.4.6.28]. Note that from the point of view of the action of  $G(K)$  on its affine building, when  $G$  is simply connected, the subgroup  $\mathcal{P}_x(\mathcal{A})$  is precisely the stabilizer of the point  $x$ , but not in general – see e.g. the example  $G = \mathrm{PGL}_n$  in [Tit79, §3.10].

The main results of this paper apply only to reductive groups  $G$  which split over an unramified extension of  $K$ . As we note below in Proposition 4.5.2, the parahoric group schemes in this setting – see Definition 4.5.3 – are obtained by descent from the split case. On the other hand, for a quasisplit group  $G$  which splits only over a ramified extension of  $K$ , in general a group scheme arising from the appropriate notion of schematic root datum for  $G$  need not have connected fibers; we don’t consider these group schemes in this manuscript.

In Section 4.4, we are going to obtain a Levi decomposition of the special fiber of  $\mathcal{P}_x$ . In order to do so, we require results about the reductive quotient of this special fiber. Recall that we introduced in section 2.5 the subsystem  $\Phi_x$ . Here is an alternative description which will be useful below.

**Lemma 4.3.6.**  $\Phi_x = \{\alpha \in \Phi \mid [\langle \alpha, x \rangle] + [\langle -\alpha, x \rangle] = 0\}$ .

*Proof.* Indeed, for any rational (or real) number  $r$ ,  $[r] + [-r] = 0$  if and only if  $r \in \mathbf{Z}$ . □

**Proposition 4.3.7.** (a) *The unipotent radical of the special fiber  $\mathcal{P}_{x,k}$  – a linear algebraic  $k$ -group – is defined and split over  $k$ .*

(b) *The reductive quotient of the special fiber  $\mathcal{P}_{x,k}$  of  $\mathcal{P}$  is a split reductive group over  $k$  with root system  $\Phi_x$ .*

*Proof.* For (a), note first that if  $k$  is *perfect*, the unipotent radical of *any* linear algebraic  $k$ -group – and in particular, that of  $\mathcal{P}_k$  – is defined and split over  $k$ ; see e.g. [Spr98, Prop 14.4.5(v)]. Thus for the proof of (a) we may and will suppose that the characteristic of  $k$  is  $p > 0$ .

To proceed, we first suppose that the characteristic of  $K$  is also  $p > 0$ . In particular, there is an inclusion  $\mathbf{F}_p \subset \mathcal{A}$  where  $\mathbf{F}_p$  denotes the field with  $p$  elements. Let  $\omega \in \mathcal{A}$  be a uniformizer; in particular,  $\omega$  is transcendental over  $\mathbf{F}_p$ . Since  $\mathcal{A}$  is complete, the inclusion  $\mathbf{F}_p[\omega]_{(\omega)} \subset \mathcal{A}$  prolongs to an inclusion of the completion  $\mathcal{B} = \mathbf{F}_p[[\omega]]$  in  $\mathcal{A}$ . Writing  $L = \mathbf{F}_p((\omega))$  for the field of fractions of  $\mathcal{B}$ , we find an embedding  $L \subset K$ .

Now consider the split reductive group  $H$  over  $L$  with the same root datum as  $G$ ; cf. e.g. the *Existence Theorem* found in [CGP15, Theorem A.4.6]. We may and will identify the split reductive  $K$ -groups  $G$  and  $H_K$ .

With the identification of the preceding paragraph, the torus  $T$  is – up to  $G(K)$ -conjugacy – obtained by base change from a split maximal  $L$ -torus  $T'$  of  $H$ . Identifying  $V = X_*(T)$  with  $V' = X_*(T')$ , we denote by  $\mathcal{Q}$  the parahoric  $\mathcal{B}$ -group scheme with  $\mathcal{Q}_L = H$  determined by (the point in  $V'$  corresponding to)  $x$ . It is then clear that  $\mathcal{P} = \mathcal{Q}_{\mathcal{A}}$  – i.e. that  $\mathcal{P}$  arises by base change from  $\mathcal{Q}$ . Since  $\mathbf{F}_p$  is the residue field of  $\mathcal{B}$ , it follows that  $\mathcal{P}_k$  arises by extension of scalars from  $\mathcal{Q}_{\mathbf{F}_p}$ . Since  $\mathbf{F}_p$  is *perfect*, the unipotent radical of  $\mathcal{Q}_{\mathbf{F}_p}$  is defined and split over  $\mathbf{F}_p$  and the result now follows.

If instead  $K$  has characteristic zero, one must make a different choice of  $\mathcal{B}$ . In this case, it follows from [Ser79, II.§5 Theorem 4] that there is an injective homomorphism  $W(k) \rightarrow \mathcal{A}$  where  $W(k)$  denotes the ring of *Witt vectors* having residue field  $k$ . Setting  $\mathcal{B} = W(\mathbf{F}_p) = \mathbf{Z}_p$ , functoriality of the construction of Witt vectors yields a canonical mapping  $\mathcal{B} \rightarrow W(k)$ . Now one argues as before to see that  $\mathcal{P}$  arises by base change from a smooth group scheme  $\mathcal{Q}$  over  $\mathcal{B}$ ; since the residue field of  $\mathcal{B}$  is perfect, this completes the proof of (a).

For (b), combine Lemma 4.3.6 with [BT84, Cor. 4.6.12(a)]. □

*Remark 4.3.8.* In fact, the previous result remains valid when  $\mathcal{A}$  is only Henselian rather than complete. The proofs of (b) and (c) require no change; in the proof of (a), one instead takes for  $\mathcal{B}$  either the

Henselization of  $\mathbf{F}_p[\omega]_{\langle\omega\rangle}$  in the equal characteristic case, or the Henselization of  $\mathbf{Z}_{\langle p\rangle}$  in the mixed-characteristic case.

**Proposition 4.3.9.** *If  $x, x' \in V$  and  $x - x' \in X_*(T) \subset V$ , then  $\mathcal{D}_{x'}$  is obtained from  $\mathcal{D}_x$  via the inner automorphism  $\text{Int}(h)$  for some  $h \in G(\mathbf{K})$ . In particular,  $h$  determines an isomorphism of  $\mathcal{A}$ -group schemes  $\mathcal{P}_{x'} \simeq \mathcal{P}_x$*

*Proof.*  $\phi = x - x' \in X_*(T) \subset V$ . Then  $\phi : \mathbf{G}_m \rightarrow T$  is a cocharacter. For each  $\alpha \in \Phi$ , the element  $h = \phi(\pi) \in T(\mathbf{K})$  satisfies

$$\text{Int}(h)\mathcal{U}_{\alpha, x'} = \text{Int}(\phi(\pi))\mathcal{U}_{\alpha, x'} = \mathcal{U}_{\alpha, x' + \phi} = \mathcal{U}_{\alpha, x}.$$

Thus indeed  $\mathcal{D}_{x'} = \text{Int}(h)\mathcal{D}_x$ . Now the isomorphism  $\mathcal{P}_{x'} \simeq \mathcal{P}_x$  follows from the uniqueness in Theorem 4.2.2.  $\square$

**Proposition 4.3.10.** *Suppose that the split reductive  $G$  has semisimple rank 1, let  $T$  be a split maximal torus of  $G$ , and write  $\alpha, -\alpha$  for the roots. Let  $x \in X_*(T) \otimes \mathbf{Q}$ . After choosing a Chevalley system,  $x$  determines a schematic root datum  $(\mathcal{U}_{\alpha, x}, \mathcal{U}_{-\alpha, x})$  as in Theorem 4.3.3 and thus an  $\mathcal{A}$ -group scheme  $\mathcal{P} = \mathcal{P}_x$  as in Theorem 4.2.2. Suppose that*

$$[\langle \alpha, x \rangle] + [\langle -\alpha, x \rangle] = 0$$

*Then  $\mathcal{P} = \mathcal{P}_x$  is (split) reductive over  $\mathcal{A}$ .*

*Proof.* Consider the reductive  $\mathbf{K}$ -group  $H = G/Z$  where  $Z$  is the center of  $G$ . Then  $H$  is a rank 1 adjoint group. Consider the 1-dimensional split maximal torus  $S = T/Z \subset H$ . Of course, the roots  $\pm\alpha \in X^*(T)$  are in the image of the natural mapping  $X^*(S) \rightarrow X^*(T)$ . There is a cocharacter  $\phi \in X_*(S)$  for which  $\langle \alpha, \phi \rangle = 1$ ; in fact  $X_*(S) = \mathbf{Z}\phi$ .

Now, the action of  $G$  on itself by inner automorphisms determines an action of  $H$  on  $G$ . Write  $r = [\langle \alpha, x \rangle]$ . The automorphism of  $G$  determined by the element  $\phi(\pi^r) \in S(\mathbf{K}) \subset H(\mathbf{K})$  yields  $\mathcal{A}$ -isomorphisms  $\mathcal{U}_{\pm\alpha} \xrightarrow{\sim} \mathcal{U}_{\pm\alpha, x'}$ , hence this automorphism of  $G$  determines an isomorphism  $\mathcal{G} \xrightarrow{\sim} \mathcal{P}$  of  $\mathcal{A}$ -group schemes.  $\square$

**4.4. Reductive subgroup schemes of a parahoric group scheme for split  $G$ .** We keep the assumptions and notations of the Section 4.3; in particular,  $G$  is a split reductive group over  $\mathbf{K}$  with split maximal torus  $T$ , and  $x \in V = X_*(T) \otimes \mathbf{Q}$ .

Our goal in this section is to prove the validity of the conclusion of Theorem 1 for the split group  $G$ . Thus, we must exhibit a suitable reductive subgroup scheme of the parahoric group scheme  $\mathcal{P}_x$  which was described in Section 4.3.

We are going to require the following result which provides a condition for a linear algebraic group to be reductive. To state the result, consider any field  $\mathcal{F}$  and let  $H$  be a linear algebraic group over  $\mathcal{F}$ . Let  $T \subset H$  be a non-trivial  $\mathbf{F}$ -split torus, let  $\Delta \subset X^*(T)$  be a linearly independent subset, and for each  $\alpha \in \Delta$  suppose that there is a reductive  $\mathcal{F}$ -subgroup  $H_\alpha \subset H$  containing  $T$  as a maximal torus and having roots  $\alpha, -\alpha$ . Write  $V_{\pm\alpha}$  for the root subgroups of  $H_\alpha$ .

**Proposition 4.4.1.** *Suppose for each pair  $\alpha, \beta \in \Delta$  with  $\alpha \neq \beta$  that  $V_{-\alpha}$  and  $V_\beta$  commute, and that  $H = \langle H_\alpha \mid \alpha \in \Delta \rangle$ . Then  $H$  is reductive.*

*Proof.* Our formulation is essentially that found in [Ste99, Theorem 5.4], though Steinberg works there over an algebraically closed base field. The result in the required generality is a consequence of [CGP15, Theorem C.2.29].  $\square$

We now return to the study of the parahoric group scheme  $\mathcal{P} = \mathcal{P}_x$ . Recall that the image of  $x$  in  $X_*(T) \otimes \mathbf{Q}/\mathbf{Z} = Y \otimes \mathbf{Q}/\mathbf{Z}$  determines a  $\mu$ -homomorphism with values in  $T$  as in section 3.4. In fact, as in Remark 3.4.5, the class of  $x$  determines a  $\mu$ -homomorphism  $\phi_x : \mu_{\mathcal{A}} \rightarrow \mathcal{T}$  which is defined over  $\mathcal{A}$ , where  $\mathcal{T}$  is the split  $\mathcal{A}$ -torus with generic fiber  $T$  used in the construction of  $\mathcal{P}_x$ .

Let  $M = C_G^0(\phi_{x, \mathbf{K}})$  denote the identity component of the centralizer in  $G$  of the image of the  $\mu$ -homomorphism  $\phi_{x, \mathbf{K}}$ ; thus  $M$  is a subgroup of  $G$  of type  $C(\mu)$  described in Theorem 3.4.6. We now consider the scheme-theoretic centralizer  $\hat{C}_{\mathcal{P}_x}(\phi_x)$  of the image of  $\phi_x$ , and - as in [BT84, §1.3.5] - we

consider the identity component  $\mathcal{M} = C_{\mathcal{P}_x}^0(\phi_x)$  of this centralizer. Thus the fibers  $\mathcal{M}_K$  and  $\mathcal{M}_k$  are connected linear algebraic groups over  $K$  resp.  $k$ .

We are going to prove:

**Theorem 4.4.2.** *Let  $\mathcal{P} = \mathcal{P}_x$ .*

- (a)  $\mathcal{M}$  is a locally closed subgroup scheme of  $\mathcal{P}$  which is smooth over  $\mathcal{A}$  and has generic fiber  $\mathcal{M}_K = M$ .
- (b)  $\mathcal{M}$  is a split reductive  $\mathcal{A}$ -group scheme.
- (c) The special fiber  $\mathcal{M}_k$  is a Levi factor of  $\mathcal{P}_k$ .

*Proof.* In (a), the assertion that  $\mathcal{M}_K = M$  is immediate from definitions. Since  $\mu_{n,\mathcal{A}}$  is a diagonalizable group scheme, [SGA3<sub>I</sub>, Exp XI, Cor 5.3] shows the centralizer  $C_G(\phi_x)$  to be a closed subgroup scheme of  $\mathcal{P}_x$  which is smooth over  $\mathcal{A}$ . Now, according to [BT84, §1.3.5], the identity component  $\mathcal{M}$  is an open subgroup scheme of this centralizer, so indeed  $\mathcal{M}$  is smooth over  $\mathcal{A}$  and locally closed in  $\mathcal{P}_x$ ; (a) is now proved.

Since  $\mathcal{M}$  is smooth over  $\mathcal{A}$  with reductive generic fiber  $\mathcal{M}_K$ , (b) will follow if we argue that the special fiber  $\mathcal{M}_k$  is reductive.

Fix a pinning  $(\psi_\alpha)_{\alpha \in \Delta}$  for  $G$  and hence a Chevalley schematic root datum  $\mathcal{D}_{\text{Chev}(\Phi)} = (\mathcal{U}_\alpha)_{\alpha \in \Phi}$  as in Theorem 4.3.1. Now let  $\mathcal{D}_x = (\mathcal{U}_{\alpha,x})_{\alpha \in \Phi}$  be the schematic root datum obtained from  $\mathcal{D}_{\text{Chev}(\Phi)}$  using  $x$ , as in Theorem 4.3.3, so that  $\mathcal{P}$  is the parahoric group scheme determined by  $\mathcal{D}_x$ , as in section 4.3.

Recall that according to Proposition 4.2.3, the schematic root datum  $(\mathcal{U}_{\alpha,x}, \mathcal{U}_{-\alpha,x})$  determines a closed  $\mathcal{A}$ -subgroup scheme  $\mathcal{P}_\alpha$  of  $\mathcal{P}$ . When  $\alpha \in \Phi_x$  it follows from Lemma 4.3.6 together with Proposition 4.3.10 that the subgroup scheme  $\mathcal{P}_\alpha$  is  $\mathcal{A}$ -reductive. Moreover, it is clear from the construction that  $\mathcal{P}_\alpha$  is centralized by  $\phi_x$ , so that  $\mathcal{P}_\alpha$  is contained in  $\mathcal{M}$ .

Fix a basis  $\Delta$  for the root system  $\Phi_x$  – see e.g. Theorem 2.5.3. Now let  $H \subset \mathcal{M}_k$  be the  $k$ -subgroup generated by the subgroups  $\mathcal{P}_{\alpha,k}$  for  $\alpha \in \Delta$ . Writing  $\mathcal{U}_{\pm\alpha} \subset \mathcal{P}_\alpha$  for the subgroup schemes determined by our schematic root data, one knows that  $\mathcal{U}_\alpha$  commutes with  $\mathcal{U}_{-\beta}$  for  $\alpha, \beta \in \Delta$  with  $\alpha \neq \beta$ ; indeed, it suffices to observe that these subgroups commute on the generic fiber.

Now Proposition 4.4.1 implies that the subgroup  $H \subset \mathcal{M}_k$  is reductive. Now write  $\pi$  for the quotient mapping from the  $k$ -group  $\mathcal{P}_k$  to its reductive quotient  $\mathcal{P}_{k,\text{red}}$ . It is clear for  $\beta \in \Phi_x$  that  $\pi$  maps  $\mathcal{U}_{\beta,k}$  onto the corresponding root subgroup of  $\mathcal{P}_{k,\text{red}}$ . According to Proposition 4.3.7,  $\mathcal{P}_{k,\text{red}}$  has root system  $\Phi_x$ . It follows that  $\pi$  maps  $H$  onto  $\mathcal{P}_{k,\text{red}}$ . Since  $\ker \pi$  is unipotent, it follows that  $H$  is a Levi factor of  $\mathcal{P}_k$  and in particular  $H$  is isomorphic to  $\mathcal{P}_{k,\text{red}}$ . In particular, the dimension of  $H$  coincides with the dimension of the generic fiber  $\mathcal{M}_K$ . Since  $\mathcal{M}$  is smooth with connected fibers, and since  $H \subset \mathcal{M}_k$ , it follows that  $H = \mathcal{M}_k$ ; (b) and (c) are now proved.  $\square$

*Remark 4.4.3.* (a) In the proof of Theorem 4.4.2, one can actually avoid the use of Proposition 4.4.1 as follows.

With notation as in the Theorem, let  $M = C_G(\phi_{x,K})$ . One first argues that  $\mathcal{D}_{x,M} = (\mathcal{U}_{\alpha,x})_{\alpha \in \Phi_x}$  is a schematic root datum for  $M$ . By an argument like that in Proposition 4.3.10, one now argues that  $\mathcal{D}_{x,M}$  is conjugate by an element of  $T(K)$  to a Chevalley schematic root datum for  $M$ . Thus  $\mathcal{D}_{x,M}$  determines a reductive subgroup scheme  $\mathcal{Q}$  of  $\mathcal{P} = \mathcal{P}_x$ . Evidently  $\mathcal{Q}$  is centralized by  $\phi_x$ . It remains to argue that  $\mathcal{Q}$  coincides with the centralizer  $\mathcal{M}$  subgroup scheme; this holds essentially for dimension reasons.

- (b) I thank Gopal Prasad for communicating to me the suggestion to use the result Proposition 4.4.1 to prove that the centralizer  $\mathcal{M}$  in Theorem 4.4.2 is reductive.
- (c) As a referee pointed out to me, the preceding Theorem can be nicely stated using the *affine building* of  $G$ . Namely, the parahoric group scheme  $\mathcal{P}$  is determined by a point  $x$  in the building. Now, each *apartment*  $D$  of the building which contains  $x$  determines a split  $\mathcal{A}$ -torus  $\mathcal{T}_D$  in  $\mathcal{P}$  – see [BT84, Prop. 5.1.10]. The point  $x$  determines a  $\mu$ -homomorphism  $\phi_x : \mu \rightarrow \mathcal{T}_D$  and the connected centralizer of the image of  $\phi_x$  yields a reductive subgroup scheme  $\mathcal{M}_D \subset \mathcal{P}$  determined by  $D$ .

**4.5. Reductive subgroup schemes of a parahoric group scheme.** Now suppose that  $G$  splits over an unramified, separable field extension of  $K$ . The following result provides more precise information:

**Proposition 4.5.1.** [BT84, §5.1.12] *There is a finite, unramified, Galois extension  $L$  of  $K$ , a maximal  $K$ -split torus  $S$  of  $G$  and a maximal  $K$ -torus  $T$  of  $G$  containing  $S$  for which  $T_L$  is  $L$ -split.*

Fix an extension  $L$  as in the preceding proposition, and write  $\mathcal{B}$  for the integral closure of  $\mathcal{A}$  in  $L$ . Let us fix a maximal  $K$ -split torus  $S$ . Also write  $\Gamma = \text{Gal}(L/K)$ . For any maximal  $K$ -torus  $T$ , note that  $\Gamma$  acts on the cocharacter  $X_*(T_L)$  and hence on  $V_L = X_*(T_L) \otimes \mathbf{Q}$ .

Certain parahoric group schemes for  $G_L$  descend to  $\mathcal{A}$ -group schemes, as follows:

**Proposition 4.5.2** (Étale descent). *Let  $T$  be a maximal  $K$ -torus of  $G$  which contains  $S$  and splits over  $L$ , and let  $V_L = X_*(T_L) \otimes \mathbf{Q}$ . Let  $F$  be a  $\Gamma$ -invariant facet in  $V_L$ , and let  $\mathcal{Q} = \mathcal{Q}_F = \mathcal{Q}_x$  be the corresponding parahoric group scheme for  $G_L$  determined by any point  $x \in F$ . There is a smooth, affine  $\mathcal{A}$ -group scheme  $\mathcal{P}$  with connected fibers such that*

- (i) *generic fiber  $\mathcal{P}_K$  may be identified with  $G$ , and*
- (ii)  *$\mathcal{Q} \simeq \mathcal{P} \otimes_{\mathcal{A}} \mathcal{B}$ .*

*Proof.* Since  $K \subset L$  is unramified, this follows from [BT84, §4.6.30 and §5.1.8].  $\square$

*Definition 4.5.3.* Assume that  $G$  splits over an unramified extension  $L$  of  $K$ . The *parahoric group schemes* attached to  $G$  are the  $G(K)$ -conjugates of group schemes  $\mathcal{P}$  obtained by étale descent – see Proposition 4.5.2 – from parahoric group schemes  $\mathcal{Q} = \mathcal{Q}_F$  determined by a  $\Gamma$ -stable facet  $F \subset X_*(T_L) \otimes \mathbf{Q}$  for some maximal  $K$ -torus  $T$  of  $G$  containing  $S$  which splits over  $L$ .

*Remark 4.5.4.* Let  $\mathcal{P}$  be a parahoric group scheme for  $G$  which arise by étale descent as in Proposition 4.5.2. Thus  $\mathcal{P}_B = \mathcal{Q}_F$  for some  $\Gamma$ -stable facet  $F$  in  $V_L = X_*(T) \otimes \mathbf{Q}$ . It follows from [BT84, Cor. 5.1.21] that there is a point  $x \in F \cap X_*(S) \otimes \mathbf{Q}$  – i.e. a point in  $F$  fixed by the action of  $\Gamma$ . Thus  $\mathcal{Q}_F = \mathcal{Q}_x$ .

The analogue of Proposition 4.3.7(a) remains valid for parahoric group schemes in this more general setting, as follows:

**Proposition 4.5.5.** *Let  $\mathcal{P} = \mathcal{P}_x$  be a parahoric group scheme with generic fiber  $G = \mathcal{P}_K$ , and suppose that  $G$  splits over an unramified extension of  $K$ . Then the unipotent radical of the linear algebraic  $k$ -group  $\mathcal{P}_k$  is defined and split over  $k$ .*

*Proof.* Indeed, in view of Proposition 4.3.7(a), the Proposition follows from the following more general statement, which may be deduced from [Spr98, Prop. 14.4.5]: Suppose that  $k \subset \ell$  is a separable field extension and that  $H$  is a linear algebraic  $k$ -group, and assume moreover that the unipotent radical of  $H$  is defined and split over  $\ell$ . Then the unipotent radical of  $H$  is defined and split over  $k$ .  $\square$

We now prove Theorem 1 from the introduction. Recall the statement:

**Theorem 4.5.6.** *Assume that  $G$  splits over an unramified extension of  $K$ . There is a reductive  $\mathcal{A}$ -subgroup scheme  $\mathcal{M} \subset \mathcal{P}$  containing  $\mathcal{T}$  such that*

- (a) *the special fiber  $\mathcal{P}_k$  has a Levi decomposition with Levi factor  $\mathcal{M}_k$ , and*
- (b) *the generic fiber  $\mathcal{M}_K$  is a connected reductive subgroup of  $G$  containing  $T$ , and  $\mathcal{M}_L$  is a subgroup of  $G_L$  of type  $C(\mu)$ .*

*Proof.* In case  $G$  is already split over  $K$ , the Theorem is an immediate consequence of Theorem 4.4.2. Otherwise, using Remark 4.5.4 the parahoric group scheme  $\mathcal{P}$  arises via étale descent from the  $\mathcal{B}$ -group scheme  $\mathcal{Q}_x$  for some point  $x \in X_*(S) \otimes \mathbf{Q} \subset X_*(T) \otimes \mathbf{Q}$ ; thus  $x$  is  $\Gamma$ -stable.

Via étale descent, one finds  $\mathcal{A}$ -group schemes  $\mathcal{S} \subset \mathcal{T}$  for which  $\mathcal{S}_K = S$  and  $\mathcal{T}_K = T$ ; thus  $\mathcal{S}$  is a split  $\mathcal{A}$ -torus, and  $\mathcal{T}_B$  is a split  $\mathcal{B}$ -torus. Using Remark 3.4.5, we see that the image of  $x$  in  $x \in X_*(\mathcal{S}) \otimes \mathbf{Q}/\mathbf{Z}$  determines a  $\mu$ -homomorphism  $\phi_x : \mu \rightarrow \mathcal{S}$  over  $\mathcal{A}$ .

Now let  $\mathcal{M}_1$  denote the  $\mathcal{B}$ -subgroup scheme of  $\mathcal{Q}_x$  which is the centralizer of the image of the  $\mu$ -homomorphism  $\phi_{x,B} : \mu_B \rightarrow \mathcal{S}_B \subset \mathcal{T}_B$ . Thus  $\mathcal{M}_1$  is the subgroup scheme of  $\mathcal{Q}_x$  described by Theorem 4.4.2.

Since  $\mathcal{T}$  and  $x$  remain invariant under the action of the Galois group  $\Gamma$ , and since  $\mathcal{M}_1$  is uniquely determined by  $\mathcal{T}$  and  $x$ , it is clear that  $\mathcal{M}_1$  remains invariant under this action; more precisely, the algebra  $\mathcal{B}[\mathcal{M}_1] \subset L[\mathcal{M}_{1,L}]$  is invariant under the semilinear  $\Gamma$ -action on  $L[\mathcal{M}_{1,L}]$ .

It now follows by étale descent – see [BT84, §5.1.8] – that there is a smooth group scheme  $\mathcal{M}$  over  $\mathcal{A}$  with  $\mathcal{M}_{\mathcal{B}} = \mathcal{M}_1$ . The group scheme  $\mathcal{M}$  has the required properties  $\square$

*Remark 4.5.7.* Assume that the residue field  $k$  is *perfect*. If  $G$  is an “inner form” (by which I mean: an inner form of a split reductive group  $K$ -group), then  $G$  splits over an unramified extension of  $K$ .

Indeed, suppose that  $G$  is an inner form of a split group, and write  $L$  for the maximal unramified extension contained in a some separable closure of  $K$ . Since  $k$  is perfect, the residue field of  $L$  is an algebraic closure of  $k$ . Thus, a theorem of Lang implies that  $L$  has cohomological dimension  $\leq 1$ ; see [Ser94, Ch. II, §3.2 and §3.3]. Now a theorem of Steinberg implies that  $H^1(L, H)$  is trivial for every connected linear algebra group over  $L$ ; see [Ser94, Ch. III, Thm. 1.0 and Remark (1) in §2.3]. In particular, it follows that every reductive  $L$ -group is *quasisplit* – i.e. contains a Borel subgroup defined over  $L$ . Since by assumption the reductive group  $G_L$  is an inner form of a split group, and on the other hand, a reductive group is an inner form of a unique quasisplit group – see e.g. [Knu+98, Prop. 27.8] –, it follows that  $G$  splits over  $L$  – i.e.  $G$  splits over an unramified extension.

## 5. EXAMPLES

In this section, we give an example illustrating Theorem 4.5.6 in case  $G$  is not split but has an unramified splitting field; see section 5.1. And we show by example that in general the conclusion of Theorem 4.5.6 does not hold when  $G$  fails to split over an unramified extension; see section 5.3 and section 5.4.

**5.1. An inner form of  $SL(V)$ .** Let  $d \geq 1$ , and let  $E$  be a central  $K$ -division algebra with  $\dim_K E = d^2$ . Suppose that the *ramification index*  $e$  of  $E$  is given by  $e = d$ ; this is e.g. immediate if  $k$  is *finite* – see [Ser79, XIII.4 Exer 1]. Write  $\mathcal{E}$  for the integral closure of  $\mathcal{A}$  in  $E$ ; according to [Rei03, Theorem 12.8],  $\mathcal{E}$  is the unique maximal  $\mathcal{A}$ -order in  $E$ .

Suppose that there is a maximal subfield  $L$  of  $E$  which is an unramified extension of  $K$  (necessarily of degree  $d$ ); if  $k$  is perfect, this assumption follows from [Ser79, XII.2 Prop 2.].

Now, if  $\pi_E$  denotes a prime element of  $E$ , then  $\ell = \mathcal{E}/\pi_E \mathcal{E}$  is a commutative field and is an extension of  $k$  of degree  $d$ . Let  $\mathcal{B}$  be the integral closure of  $\mathcal{A}$  in  $L$ . Since  $L \supset K$  is unramified,  $\pi_E \mathcal{E} \cap \mathcal{B} = \pi \mathcal{B}$  where  $\pi$  is a uniformizer of  $\mathcal{A}$  and hence also of  $\mathcal{B}$ . Now deduce that  $\mathcal{B}/\pi \mathcal{B}$  embeds in  $\mathcal{E}/\pi_E \mathcal{E} = \ell$ , hence  $\mathcal{B}/\pi \mathcal{B} \simeq \ell$  since  $[\mathcal{B}/\pi \mathcal{B} : k] = [\ell : k]$ .

Let  $G$  be the “unit group scheme”  $E^\times$ ; thus  $G$  is a reductive group over  $K$  which is an inner form of the group  $GL_d$ ; we have  $G_L \simeq GL_{d,L}$ . Let  $T \simeq R_{L/K} \mathbf{G}_m$  denote the maximal  $K$ -torus of  $G$  determined by the maximal subfield  $L \subset E$ .

Since  $G$  is anisotropic modulo its center, there is a unique parahoric group scheme over  $\mathcal{A}$  associated to  $G$ ; it is the unit group scheme  $\mathcal{P} = \mathcal{E}^\times$ .

Since  $\mathcal{B}$  is integral over  $\mathcal{A}$ , we have  $\mathcal{B} \subset \mathcal{E}$ . Thus  $\mathcal{T} = R_{\mathcal{B}/\mathcal{A}} \mathbf{G}_m$  is a smooth  $\mathcal{A}$ -subgroup scheme of  $\mathcal{P} = \mathcal{E}^\times$  with special fiber  $\mathcal{T}_k \simeq R_{\ell/k} \mathbf{G}_m$ . Since  $\mathcal{E}/\pi \mathcal{E} \simeq \ell$ , we conclude that the reductive quotient of the special fiber  $\mathcal{P}_k$  is isomorphic to the  $k$ -torus  $\mathcal{T}_k$ .

In fact, the parahoric group scheme  $\mathcal{P}$  arises by étale descent from an Iwahori group scheme  $\mathcal{Q}$  over  $\mathcal{B}$  with generic fiber  $GL_{d,L}$ . The reductive quotient of the special fiber  $\mathcal{Q}_\ell$  is a split torus of dimension  $d$ . Evidently, the conclusion of Theorem 4.5.6 holds by taking the reductive subgroup scheme  $\mathcal{M} = \mathcal{T}$ , which arises by étale descent from a maximal split  $\mathcal{B}$ -torus of  $\mathcal{Q}$ .

**5.2. A Lemma.** Suppose that  $\mathcal{P}$  is a smooth group scheme over  $\mathcal{A}$  whose generic fiber  $G = \mathcal{P}_K$  is reductive, and suppose that  $\mathcal{M}$  is a reductive subgroup scheme of  $\mathcal{P}$  whose generic fiber  $\mathcal{M}_K$  is a subgroup of  $\mathcal{P}_K$  which is geometrically of type  $C(\mu)$ . We write  $\bar{k}$  and  $\bar{K}$  for algebraic closures of  $k$  and  $K$ .

**Lemma 5.2.1.** (a) *The root system of  $\mathcal{M}_{\bar{k}}$  identifies with the root system of  $\mathcal{M}_{\bar{K}}$ .*

(b) Let  $\Phi$  denote the root system of  $G_{\bar{K}}$ . Then the root system of  $\mathcal{M}_{\bar{K}}$  is a subsystem of  $\Phi$  of the form  $\Phi_x$  as in Theorem 2.5.3.

*Proof.* (a) is proved in [SGA3<sub>III</sub>, Exp. XXII Prop. 2.8]. Since by assumption  $\mathcal{M}_K$  is geometrically of type  $C(\mu)$ , the assertion of (b) follows from (a) together with Theorem 3.4.6.  $\square$

**5.3. A unitary group.** Let  $G = \mathrm{SU}(V, h)$  be a quasisplit unitary group which splits over the separable quadratic extension  $K \subset L$  with  $\dim_L V = n$ , so that the reductive  $K$ -group  $G$  is a form of  $\mathrm{SL}_{2n}$ ; in fact,  $G_L \simeq \mathrm{SL}_{2n, L}$ .

Invoking some observations in our earlier manuscript [McN10, §7.2], we note the following:

**Proposition 5.3.1.** *Assume that  $K \subset L$  is a ramified extension.*

- (a) *There is a parahoric group scheme  $\mathcal{P}$  attached to  $G$  for which the reductive quotient of  $\mathcal{P}_k$  is isomorphic to  $\mathrm{Sp}_{2n, k}$ , the split symplectic group of rank  $n$  over  $k$ .*
- (b) *The conclusion of Theorem 1 is invalid for the parahoric group scheme  $\mathcal{P}$ .*

*Proof.* (a) follows from the description in [McN10, §7.2].

For (b), suppose by way of contradiction that  $\mathcal{H} \subset \mathcal{P}$  is a reductive subgroup scheme satisfying the conditions of Theorem 1. Then Lemma 5.2.1(a) shows that the root system  $\Psi$  of  $\mathcal{H}_{\bar{K}}$  identifies with that of  $\mathcal{H}_{\bar{K}}$ .

According to the preceding Proposition, the special fiber  $\mathcal{P}_k$  is a simple  $k$ -group whose root system is of type  $C_n$ . Since  $\mathcal{P}_k \simeq \mathcal{H}_k$ , it follows that the root system  $\Psi$  is of type  $C_n$ .

By assumption  $\mathcal{H}_K$  is a reductive subgroup of  $G$  containing a maximal torus, and in particular  $\mathcal{H}_L$  is a reductive subgroup of  $G_L = \mathrm{SL}_{2n, L}$  containing a maximal torus. Lemma 5.2.1(b) now shows that  $\Psi$  has the form  $\Phi_x$  where  $\Phi$  is the root system of  $G_L$  – i.e.  $\Phi$  is of type  $A_{2n-1}$ .

It is clear that any subsystem  $\Phi_x$  of  $\Phi$  is simply laced. Since  $\Psi$  is not simply laced, we have arrived at a contradiction; this completes the proof of (b).  $\square$

**5.4. Triality  $D_4$ .** Let  $G$  be a simply connected, quasisplit group of type  ${}^3D_4$  with splitting field  $L$ , a cubic Galois extension of  $K$  – see e.g. [Spr98, §17.9]. Let us suppose that  $L$  is a *ramified* (and hence *totally ramified*) extension of  $K$ ; write  $\mathcal{B}$  for the integral closure of  $\mathcal{A}$  in  $L$ .

According to [BT84, Thm. 4.2.3], the choice of a Chevalley system for  $G_L$  determines a “Chevalley-Steinberg valuation” for  $G$  and in particular – see [BT84, §4.3] – this choice determines group schemes  $\mathcal{U}_\alpha$  for each  $\alpha$  in the set  $\Phi$  of  $K$ -roots of  $G$ .

Since  $G$  is quasisplit and simply connected, the torus  $T$  is “induced”; in fact,  $T \simeq R_{L/K}(\mathbf{G}_m) \times \mathbf{G}_m$ . Thus we may take  $\mathcal{T} = R_{\mathcal{B}/\mathcal{A}}(\mathbf{G}_m) \times \mathbf{G}_{m/\mathcal{A}}$  and then  $\mathcal{D} = (\mathcal{T}, \mathcal{U}_\alpha)_{\alpha \in \Phi}$  is a schematic root datum; see [BT84, Thm. 4.5.4]. Let  $\mathcal{P}$  be the parahoric group scheme determined by the schematic root datum  $\mathcal{D}$  as in Theorem 4.2.2. We first observe that since  $L \supset K$  is totally ramified, we have

$$\mathcal{B} \otimes_{\mathcal{A}} k \simeq k[\tau]/\langle \tau^3 \rangle;$$

it follows that a maximal torus of the special fiber of the  $\mathcal{A}$ -group scheme  $R_{\mathcal{B}/\mathcal{A}}(\mathbf{G}_m)$  is  $k$ -split and 1 dimensional. In particular, the special fiber of the group scheme  $\mathcal{T}$  is split and of dimension 2.

**Proposition 5.4.1.** (a) *The reductive quotient of the special fiber  $\mathcal{P}_k$  is a split simple  $k$ -group of type  $G_2$ .*  
 (b) *The conclusion of Theorem 1 is invalid for the parahoric group scheme  $\mathcal{P}$ .*

*Proof.* We’ve remarked already that  $\mathcal{P}_k$  has a torus which is split and of dimension 2. Since the relative root system of  $G$  is of type  $G_2$ , [BT84, Cor. 4.6.12] shows that the reductive quotient of  $\mathcal{P}_k$  is a split simple  $k$ -group of type  $G_2$ .

As to (b), first note that the absolute root system  $\Phi$  of  $G$  is of type  $D_4$ . Suppose there is a subgroup scheme  $\mathcal{M}$  of the parahoric satisfying the conclusion of Theorem 1. According to Lemma 5.2.1, the root system  $\Psi$  has the form  $\Phi_x$  where  $\Phi$  is the root system of  $G$ . Since  $\Phi$  is a root system of type  $D_4$ , any root system of the form  $\Phi_x$  is simply laced.

But the root system  $\Psi$  of  $\mathcal{M}_k$  identifies with that of the reductive quotient of  $\mathcal{P}_k$ ; since a root system of type  $G_2$  isn’t simply laced, we have arrived at a contradiction. Assertion (b) now follows.  $\square$

*Remark 5.4.2.* Suppose  $p > 2$ . Let  $U$  denote the unipotent radical of  $\mathcal{P}_k$ . One can argue that as a module for  $\mathcal{P}_k$ , the Lie algebra  $\text{Lie}(U)$  has as composition factors two copies of the 7 dimensional irreducible representation  $V_7$  for the reductive quotient group of type  $G_2$ . Since the representation  $V_7$  is a *standard highest weight module* for  $G_2$  – i.e. in the notation of [Jan03, §II.2],  $V_7$  is isomorphic to the highest weight module  $H^0(\lambda)$  for some dominant weight  $\lambda$  – it follows from [Jan03, Prop. II.4.13] that  $H^i(G_2, V_7) = 0$  for  $i \geq 0$ . It now follows from [McN10, Theorems 5.1 and 5.2] that  $\mathcal{P}_k$  indeed has a Levi decomposition that is uniquely determined up to  $U(k)$ -conjugation. The conclusion of Proposition 5.4.1 simply means that this Levi factor can't arise as the special fiber of a reductive subgroup scheme satisfying the stipulations found in Theorem 1.

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