LEVI FACTORS OF THE SPECIAL FIBER OF A PARAHORIC GROUP SCHEME AND TAME RAMIFICATION

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ABSTRACT. Let $\mathscr{A}$ be a Henselian discrete valuation ring with fractions K and with *perfect* residue field k of characteristic $p > 0$. Let G be a connected and reductive algebraic group over K, and let $\mathcal{P}$ be a parahoric group scheme over $\mathscr{A}$ with generic fiber $\mathcal{P}_{/K} = G$. The special fiber $\mathcal{P}_{/k}$ is a linear algebraic group over k.

If G splits over an unramified extension of K, we proved in some previous work that the special fiber $\mathcal{P}_{/k}$ has a Levi factor, and that any two Levi factors of $\mathcal{P}_{/k}$ are geometrically conjugate. In the present paper, we extend a portion of this result. Following a suggestion of Gopal Prasad, we prove that if G splits over a *tamely ramified* extension of K, then the *geometric* special fiber $\mathcal{P}_{/k_{\mathrm{alg}}}$ has a Levi factor, where k_{alg} is an algebraic closure of k.

CONTENTS

1. INTRODUCTION

1.1. **Background.** Let$G$ be a connected, linear algebraic group over a field k; thus G is a smooth group scheme over k of finite type. If $\ell \supset k$ is a field extension, we write $G_{/\ell}$ for the linear algebraic group over ℓ obtained by extension of scalars. Throughout this paper, we are going to impose the following assumption on G:

(R) there is a unipotent subgroup $R \subset G$ such that $R_{/k_{\mathrm{alg}}}$ is the unipotent radical of $G_{/k_{\mathrm{alg}}}$.

We say that R is the unipotent radical of G. When k is perfect, condition **(R)** always holds, but it can fail for imperfect k; see e.g. [CGP 10, Example 1.1.3].

Write $\pi : G \to G/R$ for the quotient mapping. By a *Levi factor* of G we mean a closed subgroup M such that the mapping $\pi_{|M} : M \to G/R$ determined by restricting π to M is an isomorphism; thus M is a complement in G to the unipotent radical. If the characteristic of k is 0, any linear group has a Levi factor; see [Mc 10, §3.1]. However, for any field k of characteristic > 0, there are linear algebraic groups over k having no Levi factor; see e.g. the example in *loc. cit.*, §3.2. If **(R)** holds and M is a Levi factor of G, then G is isomorphic to the semidirect product of M and R – i.e. G has a Levi decomposition. When **(R)** fails to hold, it can happen that G has a subgroup M for which $G_{/k_{\mathrm{alg}}}$ is isomorphic to the semidirect product of $M_{/k_{\mathrm{alg}}}$ and $R_u G_{/k_{\mathrm{alg}}}$, so that $G_{/k_{\mathrm{alg}}}$ has a Levi decomposition (over k_{alg}) while G has no Levi decomposition (over k); for an important example, see [CGP 10, Theorem 3.4.6].

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In [Mc 10], we investigated the existence and conjugacy of Levi factors; one of the goals of our previous work was to investigate Levi factors for the special fiber of a so-called parahoric group scheme. To explain this statement, we must first establish some notation and fix assumptions.

Fix a Henselian discrete valuation ring (for short: DVR) $\mathscr{A}$ with fractions K and residues k; recall that if $\mathscr{A}$ is complete, it is Henselian.

We always suppose the residue field k of $\mathscr{A}$ to be *perfect*.

Fix an algebraic closure k_{alg} of the residue field k.

Now let G be a connected and reductive algebraic group over the field K. Bruhat and Tits have associated to G certain smooth affine $\mathscr{A}$-group schemes $\mathcal{P}$ with generic fiber $\mathcal{P}_{/K} = G$ known as *parahoric* group schemes. We are interested in the linear algebraic k-group $\mathcal{P}_{/k}$ obtained as the *special fiber* of $\mathcal{P}$. In general the algebraic group $\mathcal{P}_{/k}$ is not reductive, and we will be concerned here with Levi decompositions of $\mathcal{P}_{/k}$. As we already pointed out, the question of existence of a Levi factor of $\mathcal{P}_{/k}$ is only interesting when the characteristic of k is $p > 0$, which we suppose from now on.

One of the main results of our earlier work [Mc 10] is the following:

Theorem A ([Mc 10])**.** *Let $\mathcal{P}$ be a parahoric group scheme over $\mathscr{A}$ with generic fiber $G = G_{/K}$.*

(a) If G is split over K and if S is a maximal split torus of $\mathcal{P}_{/k}$, then $\mathcal{P}_{/k}$ has a unique Levi factor containing S. In particular, any two Levi factors of $\mathcal{P}_{/k}$ are $\mathcal{P}(k)$-conjugate.

(b) If $G_{/L}$ is split for an unramified extension $K \subset L$, then $\mathcal{P}_{/k}$ has a Levi factor, and any two Levi factors of $\mathcal{P}_{/k}$ are geometrically conjugate.

If G does not split over any unramified extension and if $\mathcal{P}$ is a parahoric group scheme with generic fiber G, then in general two Levi factors of $\mathcal{P}_{/k}$ need not be geometrically conjugate; see the explicit example in §7.2 of [Mc 10]. However, the question of the *existence* of a Levi factor of $\mathcal{P}_{/k}$ does not seem to be settled in general.

Recall that the parahoric group schemes are described by points in the Bruhat-Tits building $\mathscr{I}$ of G; see [BrTi 84, §5].

There is a smooth $\mathscr{A}$-group scheme $\widehat{\mathcal{P}}$ for which $\widehat{\mathcal{P}}(\mathscr{A})$ is precisely the subgroup of those elements in $G(K)$ stabilizing x for the action of $G(K)$ on $\mathscr{I}$; see [BrTi 84, 4.6.28]. If G splits over an unramified extension, this $\mathscr{A}$-group scheme $\mathcal{P} = \widehat{\mathcal{P}}$ has connected fibers. For general G, the special fiber of $\widehat{\mathcal{P}}$ need not be connected; for an example, see [Ti 77, §3.12]. We write $\mathcal{P} \subset \widehat{\mathcal{P}}$ for the smooth $\mathscr{A}$-subgroup scheme having generic fiber G and having connected special fiber; see Proposition 2.1. Thus the subgroup $\mathcal{P}(\mathscr{A}) \subset G(K)$ is the "connected stabilizer" of x as in [BrTi 84, 4.6.28].

1.2. **The main result.** To a finite extension field L of K, one associates two integers e and f. If $\mathscr{B}$ is the integral closure of $\mathscr{A}$ in L and ℓ the residue field of the discrete valuation ring $\mathscr{B}$, then $f = [\ell : k]$ and $e = e(\mathscr{B}/\mathscr{A})$ is the ramification index of the extension. Since k is perfect, $[L : K] = e \cdot f$. The extension L of K is said to be *tamely ramified* provided that the residual characteristic p does not divide the ramification index e.

Suppose that G is a reductive group over K. Using a method suggested by G. Prasad, we are going to prove in this paper the following result:

Theorem B. *Let $\mathcal{P}$ be a parahoric group scheme over $\mathscr{A}$ with generic fiber $\mathcal{P}_{/K} = G$. If $G_{/L}$ is split for some tamely ramified extension $K \subset L$, then the geometric special fiber $\mathcal{P}_{/k_{\mathrm{alg}}}$ has a Levi factor.*

1.3. **Descent of Levi factors.** Note that Theorem B does not guarantee that the linear algebraic group $\mathcal{P}_{/k}$ has a Levi factor over k. For a connected linear algebraic group G over k for which **(R)** holds, it does not seem to be known whether the group $G_{/k_{\mathrm{sep}}}$ can have a Levi factor when G fails to have a Levi factor. The author has considered this question in a recent manuscript [Mc 13] and has obtained the following partial results.

Let G be a linear algebraic group over the field k and suppose that the unipotent radical R is defined and split over k.

Theorem C ([Mc 13, Theorem A]). *Let Γ be a finite group acting by automorphisms on G, and suppose that the order of Γ is invertible in k. If G has a Levi decomposition, there is a Levi factor $M \subset G$ invariant under the action of Γ. In particular, M^Γ is a Levi factor of G^Γ.*

Theorem D ([Mc 13, Theorem B]). *Let L/k be a Galois extension, suppose that $[L:k]$ is relatively prime to p, and that $G_{/L}$ has a Levi decomposition. Then G has a Levi decomposition.*

Theorem E ([Mc 13, Theorem C]). *Suppose that there is a G-equivariant isomorphism of linear algebraic groups $R \simeq \mathrm{Lie}(R)$ – i.e. the unipotent radical R is a vector group and the action of G/R on R is linear. If $G_{/k_{\mathrm{sep}}}$ has a Levi decomposition then G has a Levi decomposition.*

Finally, in [Mc 13, §4] one finds an example of a disconnected abelian group G (over a perfect field k) for which $G_{/k_{\mathrm{sep}}}$ has a Levi decomposition but G has no Levi decomposition.

1.4. **Overview of the proof of Theorem B.** The proof of the main result – Theorem B – will be given in §5. For this proof, we may identify k_{alg} with the residue field of a strict Henselization $\mathscr{A}_{\mathrm{un}}$ of $\mathscr{A}$; in view of *étale descent* (Theorem 5.3), in the proof of Theorem B we may and will replace K by the field of fractions K_{un} of $\mathscr{A}_{\mathrm{un}}$ and hence suppose that $k = k_{\mathrm{alg}}$.

After these reductions, one knows G to split over a tamely and totally ramified extension L of K. We use a Theorem of Rousseau – Theorem 5.2 – to find a suitable parahoric group scheme $\mathcal{Q}$ over the integral closure $\mathscr{B}$ of $\mathscr{A}$ in L and a natural action of the galois group $\Gamma = \mathrm{Gal}(L/K)$ on $R_{\mathscr{B}/\mathscr{A}}\mathcal{Q}$ by $\mathscr{A}$-automorphisms; here $R_{\mathscr{B}/\mathscr{A}}(?)$ denotes the functor of "restriction of scalars" from $\mathscr{B}$-schemes to $\mathscr{A}$-schemes.

Since $G_{/L}$ is split, it follows from Theorem A that $\mathcal{Q}_{/k}$ has a Levi factor. Since $\mathscr{B}$ is a totally ramified extension of $\mathscr{A}$, we argue in Proposition 4.2 that $R_{\mathscr{B}/\mathscr{A}}\mathcal{Q}$ has a Levi decomposition. Since the order of Γ is relatively prime to p, Theorem C implies that also $(R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma_{/k}$ has a Levi decomposition.

Finally, we use Theorem 4.1 to show that the $\mathscr{A}$-group schemes $\mathcal{P}$ and $(R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma)^0$ are isomorphic. In particular, $\mathcal{P}_{/k}$ is isomorphic to $((R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma_{/k})^0$ and thus has a Levi decomposition.

1.5. **Terminology.** By a linear algebraic group G over a field k we mean a smooth, affine group scheme of finite type over k. When we speak of a closed subgroup of an algebraic group G, we mean a closed subgroup scheme over k; thus the subgroup is required to be "defined over k" in the language of [Sp 98] or [Bo 91]. Similar remarks apply to homomorphisms between linear algebraic groups. We occasionally use the terminology "k-subgroup" or "k-homomorphism" for emphasis.

2. AFFINE SCHEMES AND GROUP SCHEMES

Let A be a noetherian commutative ring. In this section, we formulate some generalities about affine schemes over A; any such X is determined by its affine algebra $A[X]$.

First, we consider an affine group scheme $\mathcal{G}$ in case A is an integral domain. One says that $\mathcal{G}$ is *connected* if $\mathcal{G}_{/k(x)}$ is connected for each $x \in \mathrm{Spec}(A)$, where $k(x)$ denotes the residue field of x; thus $k(x)$ is the field of fractions of $A/\mathfrak{p}_x$ where the prime ideal $\mathfrak{p}_x \subset A$ "is" the point x.

Proposition 2.1 ([BrTi 84, 1.2.12]). *If A is an integral domain and if $\mathcal{G}$ is a smooth affine group scheme over A, there is an affine, open subgroup scheme $\mathcal{G}^0$ which is smooth over A and connected.*

We now recall the functor of "restriction of scalars":

Proposition 2.2 ([CGP 10, Prop. A.5.2]). *Let $f : A \to B$ be a finite, flat homomorphism between commutative noetherian rings A and B. Let X be a smooth, affine B-scheme of finite type. Then the functor on A-algebras $\Lambda \mapsto X(\Lambda \otimes_A B)$ is represented by a smooth, affine scheme $R_{B/A}(X)$ of finite type over A. If X is a group scheme over B, then $R_{B/A}(X)$ is a group scheme over A.*

We also require the scheme of fixed points under the action of a finite group:

Proposition 2.3 ([Ed 92, 3.4]). *Let X be a smooth affine scheme of finite type over A and suppose that the finite group Σ acts on X by automorphisms over A. Then the functor on A-algebras $\Lambda \mapsto X(\Lambda)^\Sigma$ is represented by an affine scheme X^Σ of finite type over A. If $|\Sigma|$ is invertible in A, then X^Σ is smooth over A.*

A proof of the following result was written down in [Mc 13, (3.4.2)].

Proposition 2.4. *Let $K \subset L$ be a finite galois extension of fields with galois group $\Gamma = \mathrm{Gal}(L/K)$. Let G be a linear algebraic group over K. There is a natural action of Γ on $R_{L/K}(G_{/L})$ by K-automorphisms, and the natural mapping*

$$\phi : G \to R_{L/K}(G_{/L})^{\Gamma}$$

is an isomorphism of algebraic groups over K.

For the remainder of this section, we are going to suppose that A is a discrete valuation ring with field of fractions F and residue field $\mathfrak{f}$. We now record some results which are essentially found in J-K. Yu's manuscript [Yu 03].

Proposition 2.5. *Let X and Y be smooth and affine schemes of finite type over A, let $f : X \to Y$ be a morphism of A-schemes such that*

(i) $f_{/F} : X_{/F} \to Y_{/F}$ is an isomorphism, and
(ii) $f_{/\mathfrak{f}} : X_{/\mathfrak{f}} \to Y_{/\mathfrak{f}}$ is a dominant morphism.

Then f is an isomorphism of A-schemes.

Proof. Write $A[X]$ and $A[Y]$ for the affine algebras of X and Y, and write $\phi : A[Y] \to A[X]$ for the comorphism $\phi = f^*$ of f. Since X and Y are smooth over $\mathscr{A}$, $A[X]$ and $A[Y]$ are *free* $\mathscr{A}$-modules. Moreover, f is an isomorphism if and only if ϕ is an isomorphism. Finally, (i) shows that $\phi \otimes 1_F$ is an isomorphism, and (ii) shows that $\phi \otimes 1_{\mathfrak{f}}$ is injective. Thus the present Proposition follows from the Proposition which follows. □

Proposition 2.6 ([Yu 03, Lemma 7.6 and its proof.])**.** *Let M and N be free A-modules, and let $\phi : M \to N$ be an A-module homomorphism. Suppose that*

(i) $\phi \otimes 1_F$ is an isomorphism, and
(ii) $\phi \otimes 1_{\mathfrak{f}}$ is injective.

Then ϕ is an isomorphism.

Proof. This fact is proved in [Yu 03]; see the proof of Lemma 7.6. Since the argument is short, for the reader's convenience we give Yu's proof. Since M and N are free, evidentally M embeds in $M \otimes_A F$ and N embeds in $N \otimes_A F$. Since $\phi \otimes 1_F$ is injective by (i), it follows that ϕ is injective. Now identify M with a submodule of N. We must argue that $N/M = 0$. Since $\phi \otimes 1_F$ is onto by (i), N/M is a torsion A-module. Since N is free, one knows that $\mathrm{Tor}^1_A(N, \mathfrak{f}) = 0$. The long exact sequence of Tor shows that

$$0 \to \mathrm{Tor}^1_A(N/M, \mathfrak{f}) \xrightarrow{\partial} M \otimes_A \mathfrak{f} \xrightarrow{\phi \otimes 1_{\mathfrak{f}}} N \otimes_A \mathfrak{f}$$

is exact. Since $\phi \otimes 1_{\mathfrak{f}}$ is injective by (ii), conclude that $\mathrm{Tor}^1_A(N/M, \mathfrak{f}) = 0$. Since A is a discrete valuation ring, $\mathrm{Tor}^1_A(N/M, \mathfrak{f})$ identifies with the π-torsion submodule of N/M, where π is a uniformizing element for A. It follows that $N/M = 0$ and hence that ϕ is surjective; this completes the proof. □

3. Local fields and tamely ramified extensions

Let $\mathscr{A}$ be a Henselian discrete valuation ring (DVR) with maximal ideal $\mathfrak{m} = \pi_{\mathscr{A}} \mathscr{A}$. Recall that $\mathscr{A}$ is Henselian provided that the conclusion of Hensel's Lemma holds for $\mathscr{A}$; for example, the DVR $\mathscr{A}$ is Henselian if it is complete in its $\mathfrak{m}$-adic topology. We write K for the field of fractions of $\mathscr{A}$ and k for the residue field of $\mathscr{A}$.

We assume throughout §3, §4 and §5 that the residue field k of $\mathscr{A}$ is perfect.

We refer to a generator $\pi = \pi_{\mathscr{A}}$ for the unique maximal ideal of $\mathscr{A}$ as a *uniformizer*, or as a *prime element*. One sometimes refers to K as a "local field".

Fix a separable closure K_{sep} of K, and let $L \subset K_{\mathrm{sep}}$ be a finite separable extension of K of degree n. Write $\mathscr{B}$ for the integral closure of $\mathscr{A}$ in L; it is a Henselian DVR with fractions L. Since k is perfect, the residue field ℓ of $\mathscr{B}$ is a separable extension of k, and $n = [L : K] = ef$ where $f = [\ell : k]$ and $e = e(L/K)$ is the *ramification index* of the extension L/K. The extension L/K is said to be *unramified* if $e = e(L/K) = 1$, *totally ramified* if $e = [L : K]$, and *tamely ramified* if the integer e is invertible in the residue field $k = \mathscr{A}/\pi\mathscr{A}$.

Proposition 3.1. *If L is a totally ramified extension of K of degree n, then $L = K(\pi_1)$ and $\mathscr{B} = \mathscr{A}[\pi_1]$ where $\pi_1 \in \mathscr{B}$ is a prime element. The minimal polynomial $f(T) \in \mathscr{A}[T]$ of π_1 over K is an* Eisenstein *polynomial, and $\mathscr{B} \simeq \mathscr{A}[T]/\langle f \rangle$. In particular, there is an isomorphism*

$$\mathscr{B} \otimes_{\mathscr{A}} k \simeq k[T]/\langle T^n \rangle.$$

under which $\pi_1 \otimes 1 \in \mathscr{B} \otimes_{\mathscr{A}} k$ corresponds to the class of T.

Proof. The assertions follow from [Se 79, §I.6, Prop. 18]. □

Proposition 3.2. *Let L/K be a tamely and totally ramified galois extension of degree n, and write $\Gamma = \mathrm{Gal}(L/K)$ for the galois group.*

(a) The group Γ is cyclic, say $\Gamma = \langle \sigma \rangle$, and if $\mathfrak{m}$ denotes the unique maximal ideal of $\mathscr{B}$, there is a primitive n-th root of unity $\zeta \in K^\times$ such that σ acts on $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ by multiplication with ζ^i for $i \geq 1$.

(b) The action of Γ on $\mathscr{B}$ induces an action of Γ on $\mathscr{B} \otimes_{\mathscr{A}} k$ by k-algebra automorphisms. The space of Γ-invariants $(\mathscr{B} \otimes_{\mathscr{A}} k)^\Gamma = (k[T]/\langle T^n \rangle)^\Gamma$ is 1-dimensional over k and is equal to the coefficient field k.

Proof. Assertion (a) follows [Se 79, §IV.2 Cor. 1]. Using (a) and Proposition 3.1 together with the complete reducibility of $k\Gamma$-representations, (b) follows since a generator σ of Γ acts non-trivially on $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ for $1 \leq i \leq n-1$. □

4. Restriction of scalars of group schemes

We preserve the notations $\mathscr{A}$, K and k of the preceding section. Moreover, we suppose now that K is a *strictly Henselian* local field. Thus K coincides with its maximal unramified extension K_{un}, and in particular the residue field $k = k_{\mathrm{alg}}$ of $\mathscr{A} = \mathscr{A}_{\mathrm{un}}$ is algebraically closed.

Let $K \subset L$ be a finite, galois extension of K, write Γ for the galois group $\mathrm{Gal}(L/K)$, and write $\mathscr{B}$ for the integral closure $\mathscr{B}$ of $\mathscr{A}$ in L. Then $\mathscr{B}$ is also strictly Henselian, and the extension L/K is totally ramified.

We suppose that L is tamely ramified over K; thus by Proposition 3.2(a), the group $\Gamma = \mathrm{Gal}(L/K)$ is cyclic of order relatively prime to p.

Let $\mathcal{P}$, respectively $\mathcal{Q}$, be smooth group schemes of finite type over $\mathscr{A}$, respectively $\mathscr{B}$. Write $G = \mathcal{P}_{/K}$ for the generic fiber of $\mathcal{P}$, and suppose that $\mathcal{Q}_{/L} \simeq G_{/L}$.

We suppose $\mathcal{P}$ is connected; see the discussion preceding Proposition 2.1. Recall that this means that the linear algebraic groups $G = \mathcal{P}_{/K}$ and $\mathcal{P}_{/k}$ are connected.

Theorem 4.1. *With the above notations, assume that*

(A1) $\mathcal{P}(\mathscr{A}) \subset \mathcal{Q}(\mathscr{B})$ (viewed as subgroups of $G(L)$), and

(A2) For each $\gamma \in \Gamma$, we have $\gamma(\mathcal{Q}(\mathscr{B})) = \mathcal{Q}(\mathscr{B})$.

Then the action of Γ on $R_{L/K}G$ by automorphisms over K prolongs to an action of Γ on $R_{\mathscr{B}/\mathscr{A}}\mathcal{Q}$ by automorphisms over $\mathscr{A}$, and there is a unique morphism of $\mathscr{A}$-schemes $\psi : \mathcal{P} \to ((R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma)^0$ such that $\psi_{/K}$ is the isomorphism of Proposition 2.4. If in addition

(A3) the index of $\psi(\mathcal{P}(\mathscr{A}))$ in $((R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma)^0(\mathscr{A})$ is finite

then ψ is an isomorphism of group schemes $\psi : \mathcal{P} \xrightarrow{\sim} ((R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma)^0$.

Proof. First recall that – in the terminology of [BrTi 84, §I.7] – a scheme $\mathscr{X}$ over $\mathscr{A}$ is *étoffe* if whenever $\mathscr{Y}$ is an $\mathscr{A}$-scheme and $\phi : \mathscr{X}_{/K} \to \mathscr{Y}_{/K}$ is a morphism over K such that $\phi(\mathscr{X}(\mathscr{A})) \subset \mathscr{Y}(\mathscr{A})$, there is a (necessarily unique) morphism $\psi : \mathscr{X} \to \mathscr{Y}$ with $\phi = \psi_{/K}$. Since $\mathscr{A}$ is strictly Henselian, [BrTi 84, I.7.3] shows that any smooth scheme $\mathscr{X}$ over $\mathscr{A}$ is étoffe.

By Proposition 2.2, the $\mathscr{A}$-scheme $R_{\mathscr{B}/\mathscr{A}}\mathcal{Q}$ is smooth and hence étoffe. Thanks to (A2), the action of Γ on $R_{L/K}G_{/L}$ indeed induces an action of Γ on $R_{\mathscr{B}/\mathscr{A}}\mathcal{Q}$. In particular, we may speak of the $\mathscr{A}$-group scheme $(R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma$. Since Γ has order invertible in $\mathscr{A}$, Proposition 2.3 shows that $(R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma$ is smooth over $\mathscr{A}$.

If $\phi : G \to R_{L/K}G_{/L}$ is the isomorphism of Proposition 2.4, condition (A1) implies that $\phi(\mathcal{P}(\mathscr{A}))$ is contained in $\mathcal{Q}(\mathscr{B})^\Gamma = (R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma(\mathscr{A})$; since $\mathcal{P}$ is étoffe, it follows that there is a unique morphism of $\mathscr{A}$-group schemes $\psi : \mathcal{P} \to (R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma$ for which $\psi_{/K} = \phi$. Since $\mathcal{P}$ is connected, in fact ψ determines a morphism $\psi : \mathcal{P} \to ((R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma)^0$.

Since $\mathscr{A}$ is Henselian and since $\mathcal{P}$ and $((R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma)^0$ are smooth group schemes over $\mathscr{A}$, the natural mappings

$$\mathcal{P}(\mathscr{A}) \to \mathcal{P}(k) \quad \text{and} \quad ((R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma)^0(\mathscr{A}) \to ((R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma)^0(k)$$

are surjective [Li 02, Cor. 2.13]. Thus (A3) implies that the index of $\psi(\mathcal{P}(k))$ in $((R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma)^0(k)$ is finite.

Since $\mathcal{P}_{/K}$ and $(R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma_{/K} \simeq (R_{L/K}\mathcal{Q}_{/K})^\Gamma$ are isomorphic by Proposition 2.4, and since $\mathcal{P}$ and $(R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma$ are smooth over $\mathscr{A}$, the algebraic k-groups $\mathcal{P}_{/k}$ and $(R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma_{/k}$ have the same dimension. Since the image of ψ on k-points has finite index, and since k is algebraically closed, $\psi_{/k}$ determines a dominant mapping $\mathcal{P}_{/k} \to ((R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma_{/k})^o$

It now follows from Proposition 2.5 that ψ is an isomorphism $\psi : \mathcal{P} \xrightarrow{\sim} ((R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma)^0$ as required. □

Proposition 4.2. *Suppose that $K \subset L$ is a totally ramified extension, let $\mathscr{B}$ be the integral closure of $\mathscr{A}$ in L and let k be the residue field (of $\mathscr{A}$ and of $\mathscr{B}$). If $\mathcal{Q}$ is a smooth affine group scheme over $\mathscr{B}$ and if $\mathcal{Q}_{/k}$ has a Levi factor, then $(R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})_{/k}$ has a Levi factor.*

Proof. Write $B = \mathscr{B} \otimes_{\mathscr{A}} k$; then by Proposition 3.1, $B \simeq k[T]/\langle T^n \rangle$ where $n = [L : K]$. Evidently $(R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})_{/k}$ identifies naturally with $R_{B/k}(\mathcal{Q}_{/B})$.

Write $i : k \to B$ and $j : B \to k$ for the *unique* k-algebra maps. Then i and j induce morphisms

$$j : R_{B/k}\mathcal{Q}_{/B} \to \mathcal{Q}_{/k} \quad \text{and} \quad i : \mathcal{Q}_{/k} \to R_{B/k}\mathcal{Q}_{/B}$$

which by some abuse of notation we'll also denote by i and j. Write U for the kernel of j and M for the image of i. It follows from [CGP 10, Prop. A.5.11](2) that U is connected and unipotent. Since $j \circ i$ is the identity mapping, $R_{B/k}\mathcal{Q}_{/B}$ is the semidirect product of U and M.

Since $M \simeq \mathcal{Q}_{/k}$ and since $\mathcal{Q}_{/k}$ has a Levi factor by hypothesis, the result now follows. □

5. Reductive groups over a local field

We keep the assumptions and notations of 3; in particular, K is the field of fractions of a Henselian DVR $\mathscr{A}$ with residue field k. Let G be a connected and reductive group over K.

Proposition 5.1. *If $G_{/L}$ is split over a tamely ramified extension $L \supset K$, then $G_{/L_{\text{un}}}$ is split for a tamely ramified, finite, galois extension $L_{\text{un}} \supset K_{\text{un}}$, where K_{un} is the maximal unramified extension of K in the fixed separable closure K_{sep}.*

Proof. According to a theorem of Lang [Se 97, II.3.3], K_{un} is a C_1 field. It then follows from an important result of Steinberg (in case K is perfect) and Borel-Springer [BS 68] that $G_{/K_{\text{un}}}$ is *quasi-split*; i.e. $G_{/K_{\text{un}}}$ has a Borel subgroup defined over K_{un}.

Since $G_{/K_{\text{un}}}$ is quaisplit, it follows from [BrTi 84, 4.1.2] that $G_{/K_{\text{un}}}$ has a minimal splitting field $L_{\text{un}} \supset K_{\text{un}}$ which is precisely the field of invariants for the kernel of the representation of $\text{Gal}(K_{\text{sep}}/K_{\text{un}})$ on $X^*(T)$ where the torus T is the centralizer of a maximal K_{un}-split torus of G. The minimality of L_{un} implies that L_{un} is contained in the compositum $L_1 = L \cdot K_{\text{un}}$, since L_1 is evidentally a splitting field for G. Since L_1 is a tamely ramfied extension of K_{un}, it follows that L_{un} is tamely ramified over K_{un} as well. □

For a field extension L of K, let $\mathscr{I}_L$ be the affine building of $G_{/L}$ defined by Bruhat and Tits; see e.g. [BrTi 84, §5]. Write $\mathscr{I} = \mathscr{I}_K$. If L is galois over K, there is a natural action of Γ on $\mathscr{I}_L$. The following theorem was proved by Rousseau [Ro 77, §5], with a simplified proof given later by Prasad [Pr 01]:

Theorem 5.2 (Rousseau's Theorem)**.** *Let $K \subset L$ be a finite, galois, tamely ramified extension with galois group $\Gamma = \mathrm{Gal}(L/K)$. The natural map $j : \mathscr{I} \to (\mathscr{I}_L)^\Gamma$ is bijective.*

For a separable extension $L \supset K$, recall that we write $\mathscr{B}$ of the integral closure of $\mathscr{A}$ in L, and recall from the introduction 1.1 that a point $y \in \mathscr{I}_L$ determines a parahoric $\mathscr{B}$-group scheme $\mathcal{Q}$ with generic fiber $G_{/L}$.

Theorem 5.3 (Étale descent)**.** *Let $K \subset L$ be an unramified galois extension. For $x \in \mathscr{I}$, write $y = j(x) \in \mathscr{I}_L$. Let $\mathcal{P}$ be the parahoric $\mathscr{A}$-group scheme determined by x, and let $\mathcal{Q}$ be the parahoric $\mathscr{B}$-group scheme determined by y. Then the identification of generic fibers $\mathcal{P}_{/L} \xrightarrow{\sim} G_{/L} \xleftarrow{\sim} \mathcal{Q}_{/L}$ prolongs to an isomorphism*

$$\alpha : \mathcal{P}_{/\mathscr{B}} \xrightarrow{\sim} \mathcal{Q}$$

of group schemes over $\mathscr{B}$. If ℓ denotes the residue field of $\mathscr{B}$, we have in particular an isomorphism

$$\alpha_{/\ell} : \mathcal{P}_{/\ell} \xrightarrow{\sim} \mathcal{Q}_{/\ell}.$$

Sketch. When $L = L_{\mathrm{un}}$ is strictly Henselian, G is quasisplit and [BrTi 84, §4] provides a definition of the parahoric group scheme attached to y. It follows from [BrTi 84, 4.6.30] that the action of Γ on $L[G_{/L}] = K[G] \otimes_K L$ leaves invariant the subalgebra $\mathscr{B}[\mathcal{Q}]$, the coordinate algebra of $\mathcal{Q}$. Thus [BrTi 84, 5.1.8] shows that $\mathcal{Q}$ arises by base-change $\mathscr{A} \to \mathscr{B}$ from a canonical smooth $\mathscr{A}$-group scheme $\mathcal{P}$, and $\mathcal{P}$ *is by definition* the parahoric group scheme attached to x.

In general – i.e. when L is not necessarily strictly Henselian – the assertion follows since the the preceding construction is canonical; see [BrTi 84, §5]. □

We are now ready to prove:

Theorem 5.4. *Let $\mathcal{P}$ be a parahoric group scheme over $\mathscr{A}$ with generic fiber $G = G_{/K}$. If $G_{/\Lambda}$ is split for some tamely ramified extension $K \subset \Lambda$, then the geometric special fiber $\mathcal{P}_{/k_{\mathrm{alg}}}$ has a Levi factor.*

This is Theorem B from the introduction.

Proof. Since G splits over a tamely ramified extension of K, it follows from Proposition 5.1 that G splits over a finite, galois, tamely ramified extension $L_{\mathrm{un}} \supset K_{\mathrm{un}}$ where K_{un} is the maximal unramfied extension of K.

Since the result only describes the geometric special fiber, in view of 5.3, we may and will replace K by K_{un}. Thus, we suppose that $\mathscr{A} = \mathscr{A}_{\mathrm{un}}$ is strictly Henselian, that k is algebraically closed, and that G splits over a tamely ramfied galois extension L of K. As usual, we write $\mathscr{B}$ for the integral closure of $\mathscr{A}$ in L and $\Gamma = \mathrm{Gal}(L/K)$ for the galois group. Since the extension $K \subset L$ is tamely ramified, the order of Γ is relatively prime to the characteristic p of the residue field k.

Now, the parahoric group scheme $\mathcal{P}$ is determined by a point x in the building $\mathscr{I}$ of G; more precisely, $\mathcal{P}$ is the group scheme for which $\mathcal{P}(\mathscr{A})$ is the "connected stabilizer" of x – cf. [BrTi 84, 4.6.28 and 5.2.6] and the discussion in §1.1. With notation as in Rousseau's Theorem 5.2, let $y = j(x) \in (\mathscr{I}_L)^\Gamma$. Thus y determines a parahoric group scheme $\mathcal{Q}$ over $\mathscr{B}$ with generic fiber $\mathcal{Q}_{/L} = G_{/L}$ for which $\mathcal{Q}(\mathscr{B})$ is the connected stabilizer of y.

Since $\mathcal{P}$ has connected fibers, since $\mathcal{P}(\mathscr{A})$ stabilizes y, and since $\mathcal{Q}(\mathscr{B})$ is the connected stabilizer of y, we have $\mathcal{P}(\mathscr{A}) \subset \mathcal{Q}(\mathscr{B})$ as subgroups of $G(L)$; thus condition (A1) of Theorem 4.1 holds. Since x is Γ-stable, evidentally the connected stabilizer $\mathcal{Q}(\mathscr{B}) \subset G(L)$ is Γ-stable, so that condition (A2) of Theorem 4.1 holds as well.

Thus according to Theorem 4.1 there is a unique homomorphism of $\mathscr{A}$-group schemes

$$\psi : \mathcal{P} \to ((R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma)^0$$

such that $\psi_{/K} : G \to (R_{L/K}G_{/L})^\Gamma$ is the isomorphism of Proposition 2.4.

We have evident containments:

$$\mathcal{P}(\mathscr{A}) \subset \mathcal{Q}(\mathscr{B}) \cap G(K) \subset \mathrm{Stab}_{G(K)}(x) \subset \mathrm{Stab}_{G(L)}(y).$$

Moreover, $(R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma(\mathscr{A}) = \mathcal{Q}(\mathscr{B}) \cap G(K)$. Since $\mathcal{P}(\mathscr{A})$ has finite index in $\mathrm{Stab}_{G(K)}(x)$ by [BrTi 84, 4.6.28] it follows that the image of $\mathcal{P}(\mathscr{A})$ has finite index in $R_{\mathscr{B}/\mathscr{A}}(\mathcal{Q})^\Gamma$, so that condition (A3) of Theorem 4.1 holds. Accoring to that Theorem, ψ determines an isomorphism

$$(\sharp) \quad \psi : \mathcal{P} \xrightarrow{\sim} ((R_{\mathscr{B}/\mathscr{B}}\mathcal{Q})^\Gamma)^0$$

of $\mathscr{A}$-group schemes.

The group $G_{/L}$ is split and $\mathcal{Q}$ is a parahoric group scheme over $\mathscr{B}$ with generic fiber $G_{/L}$. Thus by Theorem A of the introduction, the special fiber $\mathcal{Q}_{/k}$ has a Levi factor. Now Proposition 4.2 shows that the special fiber $(R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})_{/k}$ has a Levi factor. Since Γ has order relatively prime to p, Theorem C shows that $((R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})_{/k})^\Gamma = (R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma_{/k}$ has a Levi factor. Finally, $(\sharp)$ shows that $\psi_{/k}$ is an isomorphism of group schemes $\mathcal{P}_{/k} \to ((R_{\mathscr{B}/\mathscr{A}}\mathcal{Q})^\Gamma)^0_{/k}$, so indeed $\mathcal{P}_{/k}$ has a Levi factor and the proof is complete. □

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