ON THE DESCENT OF LEVI FACTORS

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ABSTRACT. Let G be a linear algebraic group over a field k of characteristic $p > 0$, and suppose that the unipotent radical R of G is defined and split over k. By a Levi factor of G, one means a closed subgroup M which is a complement to R in G. In this paper, we give two results related to the descent of Levi factors.

First, suppose ℓ is a finite Galois extension of k for which the extension degree $[\ell : k]$ is relatively prime to p. If $G_{/\ell}$ has a Levi decomposition, we show that G has a Levi decomposition. Second, suppose that there is a G-equivariant isomorphism of algebraic groups $R \simeq \mathrm{Lie}(R)$ – i.e. R is a vector group with a linear action of the reductive quotient G/R. If $G_{/k_{\mathrm{sep}}}$ has a Levi decomposition for a separable closure k_{sep} of k, then G has a Levi decomposition.

Finally, we give an example of a disconnected, abelian, linear algebraic group G for which $G_{/k_{\mathrm{sep}}}$ has a Levi decomposition, but G itself has no Levi decomposition.

CONTENTS

1. INTRODUCTION

Let k be a field and let G be a linear algebraic group over k; thus G is a smooth group scheme over k of finite type. For any field extension $k \subset \ell$ we write $G_{/\ell}$ for the linear algebraic group over ℓ obtained from G by base-change. In this note, we are interested in *Levi decompositions* of G. In our investigations, we always impose the following condition:

(RS) there is a split unipotent subgroup $R \subset G$ such that $R_{/k_{\mathrm{alg}}}$ is the unipotent radical of $G_{/k_{\mathrm{alg}}}$.

Recall that a connected unipotent group U is *split* provided that there is a filtration

$$U = U^0 \supset U^1 \supset \cdots \supset U^r = 1$$

by closed normal subgroups for which each subquotient U^i/U^{i+1} is a *vector group*. When k is perfect, this condition always holds, but it can fail for imperfect k; see e.g. [CGP 10, Example 1.1.3]. We note that if **(RS)** holds, the quotient G/R is a reductive algebraic group over k.

1.1. **Levi factors.** A **Levi factor** of G is a closed k-subgroup M of G such that the product mapping

$$(x, y) \mapsto xy : M \ltimes R \to G$$

is a k-isomorphism of algebraic groups, where $M \ltimes R$ denotes the *semidirect product* (for the action of M on R by conjugation); groups possessing a Levi factor are said to have a *Levi decomposition*. Of course, to give a Levi factor of G is the same as to give a k-homomorphism $G/R \to G$ of algebraic groups which is a *section* to

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the natural quotient mapping $G \to G/R$. For more on these matters in the setting of linear algebraic groups see [Mc 10, §4.3].

Remark (1.1.1). If the unipotent radical of $G_{/k_{\text{alg}}}$ fails to arise by base-change from a subgroup of G (so in particular, **(RS)** fails to hold) then of course G has no Levi decomposition. But in this case, G may well have a Levi factor in the sense that G may possess a closed subgroup $M \subset G$ for which $G_{/k_{\text{alg}}}$ is the semidirect product of $M_{/k_{\text{alg}}}$ and R, where k_{alg} is an algebraic closure of k and R is the unipotent radical of $G_{/k_{\text{alg}}}$; for examples, see e.g. [CGP 10, Theorem 3.4.6]. Since we require **(RS)** to hold, we don't distinguish between Levi factors and Levi decompositions in this paper.

If the characteristic of k is 0, any linear algebraic group has a Levi factor; see [Mc 10, §3.1]. However, for any field k of characteristic > 0, there are linear algebraic groups over k having no Levi factor; see e.g. the example in *loc. cit.*, §3.2. We assume for the remainder of the paper that the field k has characteristic $p > 0$.

1.2. **Descent of Levi factors.** Suppose that G is a linear algebraic group over k and that **(RS)** holds; write $M = G/R$ for the reductive quotient of G.

Let k_{sep} be a separable closure of k, and suppose that the group $G_{/k_{\text{sep}}}$ has a Levi decomposition. The question we face is this: among all Levi factors of $G_{/k_{\text{sep}}}$, are any stable under the action of the absolute Galois group $\text{Gal}(k_{\text{sep}}/k)$? Otherwise said, does the group G have a Levi factor? In this paper, we give some partial answers to this question.

We first study the action of a finite group Γ on G. In this context, we find the following result; see Theorem (3.3.1).

Theorem A. *Suppose that G satisfies* ***(RS)*** *and that the finite group Γ acts on G. Assume that the order of Γ is invertible in k. If G has a Levi decomposition, there is a Levi factor $M \subset G$ invariant under the action of Γ. In particular, M^Γ is a Levi factor of G^Γ.*

If now $k \subset \ell$ is a finite, galois extension and if $G_{/\ell}$ has a Levi factor, we observe in §3.4 that the k-group $R_{\ell/k}G_{/\ell}$ obtained from $G_{/\ell}$ by *restriction of scalars* has a Levi factor. Moreover, there is a natural action of $\Gamma = \text{Gal}(\ell/k)$ by k-automorphisms on $R_{\ell/k}G_{/\ell}$, and G may be identified with the group of fixed points $(R_{\ell/k}G_{/\ell})^\Gamma$; see (3.4.2).

These observations together with Theorem A now allow us to obtain our first main result concerning the descent of Levi factors; in Theorem (3.4.3) we prove the following:

Theorem B. *Suppose that G satisfies* ***(RS)****. Let $k \subset \ell$ be a finite Galois extension for which $[\ell : k]$ is not divisible p. If $G_{/\ell}$ has a Levi decomposition, then G has a Levi decomposition.*

Finally, in §4 we consider linear algebraic groups G which are extensions of a linear algebraic group H by a vector group V. If H acts on V by automorphisms of algebraic groups, then $\text{Lie}(V)$ determines an H-module, i.e. a linear representation of H. We say that the action of H on V is *linear* if there is an H-equivariant isomorphism of algebraic groups $V \simeq \text{Lie}(V)$. There are vector groups V with G-action for which there is no G-equivariant isomorphism $\text{Lie}(V) \simeq V$ – see e.g. the examples in [Mc 12]. When H is reductive and the action of H on V is linear, we find unconditional "descent" results for Levi factors. More precisely, we have the following result, proved in Corollary (4.5.2).

Theorem C. *Suppose that G satisfies* ***(RS)*** *and suppose that there is a G-equivariant isomorphism of linear algebraic groups $R \simeq \text{Lie}(R)$ – i.e. the unipotent radical R is a vector group and the action of G/R on R is linear. If $G_{/k_{\text{sep}}}$ has a Levi decomposition then G has a Levi decomposition.*

In a final section §5, we give an example of a disconnected group G for which $G_{/k_{\text{sep}}}$ has a Levi decomposition while G has no Levi decomposition.

2. Cohomology and automorphisms of extensions

2.1. **Some generalities.** Let k be a field, choose a separable closure k_{sep} of k and write $\Gamma = \text{Gal}(k_{\text{sep}}/k)$ for the absolute Galois group of k.

As in [Se 97, §I.1], we consider functors defined on the category of $\mathscr{E}$ separable algebraic field extensions of k. Let $T : \mathscr{E} \to \text{Sets}$ be such a functor, and impose the following conditions:

(G1) $T(K) = \varinjlim T(K_i)$ for K_i running over the set of sub-extensions of K having finite type over k.

(G2) If $K \to K'$ is an injection of fields, the corresponding morphism $T(K) \to T(K')$ is injective.

(G3) If K'/K is a Galois extension, then $T(K) = T(K')^{\mathrm{Gal}(K'/K)}$.

These conditions guarantee for any choice of separable closure k_{sep} of k that the action of the profinite absolute Galois group $\Gamma = \mathrm{Gal}(k)$ on $T(k_{\mathrm{sep}})$ is continuous when $T(k_{\mathrm{sep}})$ is given the discrete topology.

By a group-functor, we mean a functor $A : \mathscr{E} \to$ Groups with values in the category of groups such that the composite $\mathscr{E} \xrightarrow{A}$ Groups $\xrightarrow{\mathcal{F}}$ Sets satisfies conditions (G1)–(G3), where $\mathcal{F}$ is the *forgetful functor*; cf. [Se 97, §II.1.1]. Conditions (G1)–(G3) imply that the Galois cohomology set

$$H^1(k, A) := H^1(\mathrm{Gal}(k_{\mathrm{sep}}/k), A(k_{\mathrm{sep}}))$$

is independent of the choice of separable closure of k.

We say that a functor $T : \mathscr{E} \to$ Sets satisfying (G1)–(G3) is a *torsor* for the group-functor A, or simply an A-torsor, if

(i) for each separable algebraic field extension K of k, the group $A(K)$ acts on $T(K)$,
(ii) the actions in (i) are compatible (in an obvious sense) with extensions $K \to K'$, and
(iii) $T(k_{\mathrm{sep}})$ is a principal homogeneous space for $A(k_{\mathrm{sep}})$ for any separable closure k_{sep} of k.

(2.1.1) ([Se 97, §I.5.2]). *(a) There is a bijection between the elements of the set $H^1(k, A)$ and isomorphism classes of A-torsors under which the trivial class $1 \in H^1(k, A)$ corresponds to the trivial A-torsor A.*
(b) Assume $H^1(k, A) = 1$. Then every A-torsor is trivial. Put another way, if T is an A-torsor, then $T(k)$ is non-empty and is a principal homogeneous space for the group $A(k)$.

Remark (2.1.2). Of course, if $\ell \supset k$ is a finite Galois extension, the bijection of (2.1.1) yields a bijection between the cohomology set $H^1(\mathrm{Gal}(\ell/k), A(\ell))$ the isomorphism classes of those A-torsors T for which $T(\ell)$ is a principal homogeneous space for $A(\ell)$.

2.2. **Galois twists of extensions of linear algebraic groups.** Recall that a sequence

$$(*) \quad 1 \xrightarrow{i} N \to G \xrightarrow{\pi} M \to 1$$

of linear algebraic groups is *strictly exact* if the sequence of groups

$$1 \to N(k_{\mathrm{sep}}) \xrightarrow{i} G(k_{\mathrm{sep}}) \xrightarrow{\pi} M(k_{\mathrm{sep}}) \to 1$$

is exact and $d\pi : \mathrm{Lie}(G) \to \mathrm{Lie}(M)$ is surjective. Then i induces an isomorphism of N onto a closed, normal subgroup of G, and π induces an isomorphism $G/i(N) \simeq M$.

By an *extension* of M by N we mean a strictly exact sequence of the form $(*)$. Given a second extension

$$(**) \quad 1 \to N \xrightarrow{i'} G' \xrightarrow{\pi'} M \to 1$$

of M by N, by an *isomorphism of extensions* between $(*)$ and $(**)$ we mean a commuting diagram

$$\begin{array}{ccccccccc} (*) & 0 & \longrightarrow & N & \xrightarrow{i} & G & \xrightarrow{\pi} & M & \longrightarrow & 1 \\ & & & \| & & \downarrow \phi & & \| & & \\ (**) & 0 & \longrightarrow & N & \xrightarrow{i'} & G' & \xrightarrow{\pi'} & M & \longrightarrow & 1 \end{array}$$

where $\phi : G \to G'$ is an isomorphism of linear algebraic groups over k.

Fix an extension of M by N given by a strictly exact sequence of linear algebraic groups $(*)$. For each field extension K of k, let $\mathscr{A}(K)$ be the group of automorphisms of the extension $(*)_{/K}$ of $M_{/K}$ by $N_{/K}$ obtained from $(*)$ by extension of scalars; thus the group $\mathscr{A}(K)$ consists in those automorphisms ϕ of $G_{/K}$ such that $\phi \circ i = i$ and $\pi = \pi \circ \phi$.

(2.2.1). *$\mathscr{A}$ is a group-functor (in the sense of* §2.1*).*

Proof. First note of course that if $K \subset K'$ is a field extension, an automorphism of the extension $(*)_{/K}$ determines by base-change an automorphism of the extension $(*)_{/K'}$; with the resulting morphisms $\mathscr{A}(K) \to \mathscr{A}(K')$, $\mathscr{A}(-)$ is indeed a functor $\mathscr{A} : \mathscr{E} \to$ Groups. Conditions (G1)–(G3) are standard and left to the reader. ☐

We say that the extension $(**)$ of M by N is a *k-form* of the extension $(*)$ if there is separable algebraic field extension $K \supset k$ and a isomorphism of extensions defined over K between $(*)_{/K}$ and $(**)_{/K}$.

(2.2.2). *Fix an extension $(*)$.*

(a) There is a bijection between $H^1(\Gamma, \mathscr{A})$ and the set of isomorphisms classes of k-forms of $(*)$; under this bijection, the class of the k-form $(*)$ corresponds to the neutral element of the pointed set $H^1(\Gamma, \mathscr{A})$.
(b) Let $\ell \supset k$ be a finite Galois extension of fields. The bijection of (a) induces a bijection between the cohomology set $H^1(\mathrm{Gal}(\ell/k), \mathscr{A}(\ell))$ and the set of isomorphism classes of k-forms $(**)$ of $(*)$ for which the extensions $(*)_{/\ell}$ and $(**)_{/\ell}$ are isomorphic.

Proof. Assertion (b) follows at once from (a); see (2.1.2)

Given a k-form $(**)$ of $(*)$, let $T : \mathscr{E} \to \mathrm{Sets}$ be the functor whose value at an object K of $\mathscr{E}$ is the set $T(K) = \mathrm{Isom}((*)_{/K}, (**)_{/K})$ of all isomorphisms of extensions $(*)_{/K} \to (**)_{/K}$. Fix a separable closure k_{sep} of k. Since $(**)$ is a k-form of $(*)$, there is an isomorphism $\phi : (*)_{/k_{\mathrm{sep}}} \xrightarrow{\sim} (**)_{/k_{\mathrm{sep}}}$; it follows that $T(k_{\mathrm{sep}}) = \phi.\mathscr{A}(k_{\mathrm{sep}})$ is a principal homogeneous space for the group $\mathscr{A}(k_{\mathrm{sep}})$ so that T is an $\mathscr{A}$-torsor.

On the other hand, given an $\mathscr{A}$-torsor T, choose a separable closure k_{sep} of k. A choice of an element of $T(k_{\mathrm{sep}})$ leads to a 1-cocycle $\sigma \in Z^1(\Gamma, \mathscr{A}(k_{\mathrm{sep}})$. Using this co-cycle, one constructs a twisted k-form ${}_{\sigma}G$ of the linear algebraic group G. The action of the group Γ on the group of points ${}_{\sigma}G(k_{\mathrm{sep}})$ is obtained by "twisting" the given action of Γ on $G(k_{\mathrm{sep}})$ by the co-cycle σ. Since each $\gamma \in \Gamma$ determines an automorphism of the extension $(*)_{/k_{\mathrm{sep}}}$ one finds that the homomorphisms

$$i : N_{/k_{\mathrm{sep}}} \to G_{/k_{\mathrm{sep}}} = {}_{\sigma}G_{/k_{\mathrm{sep}}} \quad \text{and} \quad \pi : G_{/k_{\mathrm{sep}}} = {}_{\sigma}G_{/k_{\mathrm{sep}}} \to M_{/k_{\mathrm{sep}}}$$

commute with the action of Γ on $N(k_{\mathrm{sep}})$, $M(k_{\mathrm{sep}})$ and ${}_{\sigma}G(k_{\mathrm{sep}})$ and hence descend to yield a strictly exact sequence of

$$(\flat) \quad 1 \to N \xrightarrow{i} {}_{\sigma}G \xrightarrow{\pi} M \to 1;$$

it is immediate that $(\flat)$ is a k-form of the extension $(*)$.

The reader is left to verify that the indicated assignments determine the required bijection. □

2.3. **A cohomological lemma.** Let A be a ("concrete") group. Recall that the lower central series $Z_i(A)$ of A is defined as follows: $Z_1(A) = Z(A)$ is the center of A, and

$$Z_i(A) = \pi^{-1}(Z(A/Z_{i-1}(A))) \quad \text{for} \quad i > 1$$

where $\pi : A \to A/Z_{i-1}(A)$ denotes the quotient mapping (for each i). Recall that A is *nilpotent* if its lower central series terminates with $Z_j(A) = A$ for some $j \geq 1$. If A is nilpotent, the *nilpotence class* of A is the minimal $j \geq 1$ for which $Z_j(A) = A$.

(2.3.1). *Let Γ be a finite group whose order is not divisible by the prime number p, and let A be a nilpotent group on which Γ acts by group automorphisms such that for some $N = N_A \geq 1$, each element of A has order dividing p^N. Then $H^1(\Gamma, A) = 1$.*

Proof. Write m for the number of elements in Γ.

We give the proof by induction on the nilpotence class j of A. If $j = 1$, the group A is abelian; in that case A is a Γ-module, and we apply the standard result from the cohomology of finite groups which shows that multiplication by m annihilates each $H^i(\Gamma, A)$ for all $i \geq 1$ [Se 79, VIII §3 Cor. 1]. Now, the group A is annihilated by p^N, so also each $H^i(\Gamma, A)$ is annihilated by p^N. Since $\gcd(m, p^N) = 1$, it follows that $H^i(\Gamma, A) = 0$ for $i \geq 1$. In particular, $H^1(\Gamma, A)$ is trivial.

Now suppose that the nilpotence class of A satisfies $j > 1$, and that the result is know for all nilpotent groups of class $s < j$. Write $Z = Z(A)$ for the center of A, and observe that the nilpotence class of the group A/Z is $j - 1$.

Since Z is a normal subgroup of A, one finds an exact sequence of pointed sets

$$1 \to Z^{\Gamma} \to A^{\Gamma} \to (A/Z)^{\Gamma} \to H^1(\Gamma, Z) \to H^1(\Gamma, A) \to H^1(\Gamma, A/Z)$$

cf. [Se 97, I.§5]. By induction one knows that $H^1(\Gamma, Z)$ and $H^1(\Gamma, A/Z)$ are trivial, and it follows at once that $H^1(\Gamma, A)$ is trivial. Thus the required result indeed follows by induction on j. □

Remark (2.3.2). Rather than invoking group cohomology in the case of abelian A, the preceding result may perhaps be seen more directly using a "barycenter" argument. Indeed, if Γ is a finite group whose order m is prime to p and if Γ acts on an Abelian group A having p^N-torsion, then every A-torsor A_0 on which Γ acts (compatibly with its action on A) is trivial. Indeed, given such an A-torsor A_0, choose any element $x \in A_0$. For $g \in \Gamma$, $gx = a_g + x$ for $a_g \in A$. Now form the element $b = j \sum_{g \in \Gamma} a_g \in A$ where j is an integer for which $mj \equiv 1 \pmod{p^N}$, and consider $b + x$. For $h \in \Gamma$,

$$h(b+x) = j \sum_{g \in \Gamma} ha_g + hx = j \sum_{g \in \Gamma} (ha_g + a_h) + x = j \sum_{g \in \Gamma} a_{hg} + x = b + x$$

since the assignment $g \mapsto a_g$ is a 1-cocycle. Thus $b + x$ is fixed by Γ so that A_0 is indeed trivial.

Remark (2.3.3). If Γ is a finite group whose order is prime to p, if k is a field of characteristic p, and if U is a unipotent linear algebraic group over k and Γ acts on U, then (2.3.1) shows that $H^1(\Gamma, U(k_{\mathrm{sep}})) = 1$ and $H^1(\Gamma, \mathrm{Lie}(U)) = 0$.

3. PRIME-TO-p DESCENT

Throughout § 3, k denotes a field of characteristic $p > 0$, and Γ denotes a finite group of order relatively prime to p.

3.1. Fixed point results.

Theorem (3.1.1). *Let U be a connected k-split unipotent group, and suppose that Γ acts on U by automorphisms of algebraic groups. Then U^Γ is connected.*

Proof. Suppose that there is a proper, connected k-split Γ-invariant, normal subgroup $U_1 \subset U$ for which it is known that U_1^Γ and $(U/U_1)^\Gamma$ are connected. By Remark (2.3.3), $H^1(\Gamma, U_1(k_{\mathrm{alg}})) = 1$, and it follows that the sequence

$$(\clubsuit) \quad 1 \to U_1^\Gamma \to U^\Gamma \to (U/U_1)^\Gamma \to 1$$

is strictly exact. We now deduce that U^Γ is connected, as desired.

Using ($\clubsuit$), we first reduce the proof of the Theorem to the case in which U is a vector group. For this, we argue as in the proof of [Mc 12, Theorem 4.2.1]. First, the derived subgroup $U_1 = (U, U)$ is Γ-invariant. Moreover, (U, U) and the quotient $U/(U, U)$ are connected, k-split unipotent groups over k [Sp 98, Exerc. 14.3.12 (2) and (3)]. Since U is nilpotent, the strict exactness of ($\clubsuit$) and induction on $\dim U$ show that U^Γ is connected provided the Theorem is known when U is abelian. Now let U be abelian and let $U_1 = U^{(p)}$ be the subgroup generated by p-th powers; then $U^{(p)}$ is Γ-invariant. According to *loc. cit.*, $U^{(p)}$ and $U/U^{(p)}$ are again abelian split connected unipotent groups. By induction on the exponent, the strict exactness of ($\clubsuit$) shows that U^Γ is connected provided the Theorem is known for abelian U of exponent p – i.e. for vector groups U [CGP 10, Theorem B.2.5].

Now suppose the Theorem is known for those vector groups W on which Γ acts by group automorphisms and for which $\mathrm{Lie}(W)$ is a simple $k\Gamma$ module. We claim the Theorem then follows for any vector group U on which Γ acts by group automorphisms. To prove the claim, we suppose that $\mathrm{Lie}(U)$ is not a simple $k\Gamma$ module. Apply the semisimplicity of $k\Gamma$-modules together with [Mc 12, (3.2.2)] to find a Γ-equivariant separable and surjective homomorphisms of algebraic groups $\phi : U \to V$ for a simple Γ-module V [1]. Since $\mathrm{Lie}(U)$ is not simple, $\ker \phi$ is positive dimensional, and by induction on the dimension of U, the Γ-invariant subgroup $(\ker \phi)^0$ contains a Γ-invariant connected subgroup U_1 for which $\mathrm{Lie}(U_1)$ is a simple Γ-module.

Since the composition length of $\mathrm{Lie}(U/U_1)$ is less than that of $\mathrm{Lie}(U)$, we may suppose that $(U/U_1)^\Gamma$ is connected. Moreover, since $\mathrm{Lie}(U_1)$ is a simple $k\Gamma$-module, $W^\Gamma = U_1^\Gamma$ is connected by our assumption. Now the strict exactness of ($\clubsuit$) shows that U^Γ is connected.

Thus to prove the Theorem, we may assume that U is a vector group for which $\mathrm{Lie}(U)$ is a simple Γ-module. Consider the separable surjective Γ-invariant homomorphism $\phi : U \to V$ for a simple $k\Gamma$-module V constructed above; the tangent mapping $d\phi$ is an isomorphism of $k\Gamma$-modules $\mathrm{Lie}(U) \xrightarrow{\sim} V = \mathrm{Lie}(V)$. If $\mathrm{Lie}(U) \simeq V$ is the trivial 1 dimensional Γ-module, it is easy to see that Γ acts trivially on U (e.g. apply [Mc 12, Prop. (3.2.3)]) so that $U^\Gamma = U$ is indeed connected.

[1] In the notation of *loc. cit.*, choose a simple Γ-submodule $W \subset \mathscr{A}(U)$ with $W \cap \mathscr{A}^1(U) = \{0\}$ and set $V = W^\vee$, the dual Γ-module

Finally, if V is a non-trivial simple Γ-module, then $V^\Gamma = 0$ and we must argue that $U^\Gamma = 0$. Write $F = \ker\phi$ so that F is a finite linear algebraic group (a finite, smooth group scheme over k). It suffices to argue that $F^\Gamma = 0$.

Write $\mathscr{A}(U)$ for the collection of homomorphisms of algebraic groups $U \to \mathbf{G}_a$; we may view $\mathscr{A}(U)$ as a Γ-submodule of the algebra $k[U]$ of regular functions on U as in [Mc 12, §3.1]. Since Γ has order prime to p, $\mathscr{A}(U)$ is a semisimple Γ-module, and it follows from [Mc 12, Prop. (3.2.1)] that $\mathscr{A}(U)^\Gamma = 0$.

By [Mc 12, Lemma (3.1.4)] $\mathscr{A}(U)$ contains a system of k-algebra generators for $k[U]$; thus to prove $F^\Gamma = 0$, it is enough to argue that $f_{|F^\Gamma} = 0$ for all $f \in \mathscr{A}(U)$. Set $B = \{f \in \mathscr{A}(U) \mid f_{|F^\Gamma} = 0\}$. Then there is a injective Γ-equivariant group homomorphism from $\mathscr{A}(U)/B$ to the group C of all homomorphisms $F^\Gamma(k_{\text{alg}}) \to k_{\text{alg}}$. But Γ acts trivially on C, so that Γ acts trivially on $\mathscr{A}(U)/B$. Since $\mathscr{A}(U)$ is a semisimple Γ module and $\mathscr{A}(U)^\Gamma = 0$, deduce that $B = \mathscr{A}(U)$ so that $F^\Gamma = 0$ as required. This completes the proof. □

(3.1.2). *If Γ acts by group automorphisms on a reductive group H, then the identity component of H^Γ is reductive.*

Proof. Since Γ is a linearly reductive group, this follows from a result of Richardson [Ri 82, Prop. 10.1.5]. □

3.2. **Automorphisms of an extension.** Let G be a linear algebraic group over k for which **(RS)** holds, let R be the unipotent radical of G, and write $M = G/R$ for the reductive quotient of G. We view G as an extension

$$(\sharp) \quad 1 \to R \to G \xrightarrow{\pi} M \to 1$$

As in §2.2, let $\mathscr{A}$ be the group-functor given for separable extensions K of k by taking $\mathscr{A}(K)$ to be the group of all automorphisms of the extension $(\sharp)_{/K}$ obtained from $(\sharp)$ by extending scalars.

(3.2.1). *If the unipotent radical R is* abelian, *then $\mathscr{A}(\ell)$ is abelian for each separable field extension ℓ, and there is a constant N for which each element of $\mathscr{A}(\ell)$ has order dividing p^N.*

Proof. The result describes $\mathscr{A}(\ell)$; thus we may and will suppose that $k = \ell$. By a result of Rosenlicht – see e.g. [Mc 10, (2.2.3)] –we may choose a regular mapping $\sigma : M \to G$ which is a section to π.

Using σ we may describe the variety G as follows. Define regular maps $\underline{m}, \underline{x} : G \to G$ by setting $\underline{m} = \sigma \circ \pi$ and taking for $\underline{x}$ the map defined by the rule $\underline{x}(g) = \underline{m}(g)^{-1} \cdot g$. Writing $(\underline{m}, \underline{x}) : G \to G \times G$ for the map determined by $\underline{m}$ and $\underline{x}$, we find that $\mu \circ (\underline{m}, \underline{x}) = 1_G$. The image $\widetilde{M}$ of $\underline{m}$ (which is also the image of σ) coincides with the fiber $\underline{x}^{-1}(1)$, so $\widetilde{M}$ is closed and $\pi : \widetilde{M} \to M$ is an isomorphism. Moreover, the image of $\underline{x}$ is R and the mapping $G \xrightarrow{(\underline{m},\underline{x})} \widetilde{M} \times R$ is an isomorphism. In what follows, we identify *G as a variety* – though not as a group – with $M \times R$.

If $\phi : G \to G$ is a k-automorphism which determines an automorphism of the extension $(\sharp)$, there is a regular function $f_\phi : M \to R$ for which the value of ϕ at k_{alg}-points of $G = M \times R$ is given by:

$$\phi(m, x) = (m, f_\phi(m) \cdot x) \quad \text{for } m \text{ in } M(k_{\text{alg}}) \text{ and } x \text{ in } R(k_{\text{alg}})$$

Thus if $\phi, \psi \in \mathscr{A}(k)$, then

$$(\psi \cdot \phi)(m, x) = \psi(m, f_\phi(m) \cdot x) = (m, f_\psi(m) \cdot f_\phi(m) \cdot x);$$

since R is assumed to be abelian, this rule is symmetric in ϕ and ψ so indeed $\mathscr{A}(k)$ is abelian.

Since R is an abelian unipotent group, there is a constant $N \geq 1$ such that each element of $R(k_{\text{sep}})$ has order dividing p^N. Let $\phi \in \mathscr{A}(k)$. For $(m, x) \in (M \ltimes R)(k_{\text{alg}})$, we have

$$\phi^{p^N}(m, x) = (m, f_\phi(m)^{p^N} x) = (m, x)$$

so that indeed ϕ has order dividing p^N. □

(3.2.2). *Let $\ell \supset k$ be a separable extension of fields, and let $A = \mathscr{A}(\ell)$. Suppose that $G_{/\ell}$ has a Levi decomposition. Then:*

(a) A is a nilpotent group, and
(b) there is a constant N for which each element of A has order dividing p^N.

Proof. Since the result concerns $\mathscr{A}(\ell)$, we may and will suppose that $\ell = k$.

Since $G = G_{/\ell}$ is assumed to have a Levi decomposition, there is an action of the group M on R for which the extension $(\sharp)$ is isomorphic (over $\ell = k$) to the extension

$$1 \to R \to M \ltimes R \to M \to 1$$

defined by the semi-direct product.

Since R is k-split, we may find a sequence

$$1 = U_m \subset U_{m-1} \subset \cdots \subset U_1 \subset U_0 = R$$

where each U_i is a closed normal k-subgroup of G and each quotient U_i/U_{i+1} is a vector group – i.e. U_i/U_{i+1} is isomorphic to a product of groups $\mathbf{G}_{a/k}$ –; see [Mc 10, (2.2.3)]. It now follows that each element x of $R(k_{\text{sep}})$ has the property $x^{p^m} = 1$.

For $m \geq i \geq 1$, the group U_i is normal in G and thus the action of M on R by conjugation leaves U_i invariant. One may therefore consider the extension

$$(\sharp_i) \quad 1 \to U_i \to M \ltimes U_i \to M \to 1.$$

Write $\mathscr{A}_i$ for the group functor of automorphisms of $(\sharp_i)$. There is a homomorphism of group functors $\mathscr{A}_i \to \mathscr{A}$ and in particular a homomorphism of groups $\tau = \tau_i : \mathscr{A}_i(k) \to \mathscr{A}(k) = A$. Indeed, a k-automorphism Φ of $(\sharp_i)$ is given as in §2.2 by a suitable automorphism of $M \ltimes U_i$. The diagram of §2.2 shows that there is a regular function $\phi : M \to U_i$ for which this automorphism of $M \ltimes U_i$ is given by the rule $(m, x) \mapsto (m, \phi(m)x)$. The regular function $\iota \circ \phi : M \to U_i \to R$ determines an automorphism $(m, x) \mapsto (m, \phi(m)x)$ of $M \ltimes R$, where $\iota : U_i \to R$ denotes the inclusion. Finally, this automorphism of $M \ltimes R$ determines an automorphism $\tau(\Phi)$ of the extension $(**)$.

Write $A_i \subset A$ for the image of this mapping $\tau = \tau_i : \mathscr{A}_i(k) \to \mathscr{A}(k) = A$. Then we have

$$1 = A_m \subset A_{m-1} \subset \cdots \subset A_1 \subset A$$

One readily observes that each A_i is *normal* in A, and that A_i/A_{i+1} is isomorphic to a subgroup of the group of all k-automorphisms of the extension

$$1 \to U_i/U_{i+1} \to M \ltimes (U_i/U_{i+1}) \to M \to 1.$$

Thus (3.2.1) shows that A_i/A_{i+1} is an abelian group; it follows that A is nilpotent and (a) holds.

For (b), choose M large enough that each element of each of the groups A_i/A_{i+1} has order dividing p^M. Then each element of A has order dividing p^N where $N = mM$. □

3.3. **Levi factors stable under the action of a finite group.** Recall that the finite group Γ has order relatively prime to p, the characteristic of k.

Theorem (3.3.1). *If G has a Levi decomposition, there is a Levi factor $M \subset G$ invariant under the action of Γ. In particular, M^Γ is a Levi factor of G^Γ.*

Proof. As in 3.2, write $\mathscr{A}$ for the group of automorphisms of the extension

$$(\sharp) : \quad 1 \to R \to G \to G/R \to 1.$$

Set $A = \mathscr{A}(k)$. According to (3.2.2), A is a nilpotent group and for some $N \geq 1$, each element of A has order dividing p^N. Moreover, Γ acts on A by the rule $\gamma \star a = \gamma \circ a \circ \gamma^{-1}$ for $a \in A$ and $\gamma \in \Gamma$.

Fix a Levi factor $L \subset G$. For $\gamma \in \Gamma$, the subgroup $\gamma L \subset G$ is another Levi factor. By definition the group $A = \mathscr{A}(k)$ acts transitively on the the collection of all Levi factors of G; thus for $\gamma \in \Gamma$ we may find $a_\gamma \in A$ for which $a_\gamma \cdot \gamma L = L$.

We now observe that the assignment $a \mapsto a_\gamma$ is a 1-cocycle. Indeed, for $\sigma, \tau \in \Gamma$,

$$\sigma\tau L = \sigma a_\tau^{-1} L = (\sigma \star a_\tau^{-1})\sigma L = (\sigma \star a_\tau^{-1}) a_\sigma^{-1} L.$$

Since an automorphism a of $(\flat)$ is trivial if and only if $aL = L$, it follows that $a_{\sigma\tau}^{-1} = (\sigma \star a_\tau^{-1}) a_\sigma^{-1}$ so that indeed $a_{\sigma\tau} = a_\sigma(\sigma \star a_\tau)$; i.e. the (non-Abelian) 1-cocycle identity holds.

Since Γ has order relatively prime to p, $H^1(\Gamma, A) = 1$ by (2.3.1). Thus, there is $b \in A$ such that

$$1 = b \cdot a_\sigma \cdot (\sigma \star b^{-1}) \quad \text{for each } \sigma \in \Gamma.$$

Put $M = bL$. For $\sigma \in \Gamma$, we find that

$$\sigma M = \sigma bL = (\sigma \star b)\sigma L = (\sigma \star b)a_\sigma^{-1} L = bL = M$$

so indeed M is a Levi factor of G stable under the action of Γ.

Now, the unipotent radical R of G is Γ-stable, and R^Γ is a connected unipotent group by (3.1.1). Write $\pi : G \to G/R$ for the quotient mapping. Since M is a Levi factor, π induces a Γ-equivariant isomorphism $M \to G/R$ and hence an isomorphism $M^\Gamma \to (G/R)^\Gamma$.

Since $H^1(\Gamma, R^\Gamma(k_{\mathrm{alg}})) = 1$ and $H^1(\Gamma, \mathrm{Lie}(R^\Gamma)) = 0$ by Remark (2.3.3), the sequence

$$1 \to R^\Gamma \to G^\Gamma \to (G/R)^\Gamma \to 1$$

is strictly exact. It now follows from (3.1.1) and (3.1.2) that M^Γ is a Levi factor of G^Γ. □

3.4. **Descent of Levi factors for prime-to-p Galois extensions.** Let L/k be a finite Galois extension for which $[L : k]$ is relatively prime to p. For an algebraic group H over L, we write $R_{L/k}(H)$ for the algebraic group over k obtained from H by *restriction of scalars*; cf. [CGP 10, A.5].

(3.4.1) ([DG 70, II.1 Theorem 3.6] or [Ja 03, I.2.6(10)]). *Let X be an affine scheme of finite type over k and suppose that the finite group Σ acts on X by automorphisms over k. Then the functor on k-algebras $\Lambda \mapsto X(\Lambda)^\Sigma$ is represented by an affine scheme X^Σ of finite type over k.*

Write $\Gamma = \mathrm{Gal}(L/k)$.

(3.4.2). *Let G be a linear algebraic group over k. There is a natural action of Γ on $R_{L/k}(G_{/L})$ by automorphisms over k, and the natural mapping*

$$\phi : G \to R_{L/k}(G_{/L})^\Gamma$$

is an isomorphism of algebraic groups over k.

Proof. If Λ is a commutative k-algebra then Γ acts naturally on $\Lambda \otimes_k L$: an element $\gamma \in \Gamma$ acts as $1 \otimes \gamma$. Since L/k is Galois, $(\Lambda \otimes_k L)^\Gamma = \Lambda$. The action of Γ on $\Lambda \otimes_k L$ yields a natural action of Γ on $(R_{L/k}G_{/L})(\Lambda) = G(\Lambda \otimes_k L)$, and evidently $(R_{L/k}G_{/L})(\Lambda)^\Gamma = G((\Lambda \otimes_k L)^\Gamma) = G(\Lambda)$.

The natural maps $\Lambda \to \Lambda \otimes_k L$ given by $x \mapsto x \otimes 1$ determine a functorial group homomorphism $\phi_\Lambda : G(\Lambda) \to G((\Lambda \otimes_k L)^\Gamma) = (R_{L/k}(G_{/L})(\Lambda)^\Gamma$ and hence a homomorphism $\phi : G \to R_{L/k}(G_{/L})^\Gamma$; since ϕ_Λ is an isomorphism for each Λ, ϕ is an isomorphism. □

Theorem (3.4.3). *Let G be a linear algebraic group over k, let L/k be a Galois extension, and suppose that $[L : k]$ is relatively prime to p. If $G_{/L}$ has a Levi decomposition, then G has a Levi decomposition.*

Proof. Let H be a linear algebraic group over L for which **(RS)** holds. Suppose that H has a Levi factor M, there is an L-isomorphism $H \simeq R \rtimes M$ where R is the unipotent radical of H. Then $R_{L/k}H \simeq R_{L/k}R \rtimes R_{L/k}M$. Since $R_{L/k}R$ is unipotent and $(R_{L/k}M)^0$ is reductive [Oe 84, A.3.4], this description shows that the k-group $R_{L/k}H$ has a Levi decomposition. Thus, by hypothesis the k-group $R_{L/k}G_{/L}$ has a Levi decomposition. Since the order of Γ is relatively prime to p, it now follows from (3.4.2) together with Theorem (3.3.1) that G has a Levi decomposition. □

4. Extensions of a reductive group by a linear representation

4.1. **Galois cohomology of a vector space.** Let V be a vector space over k which is not necessarily of finite dimension. Then V defines a group-functor A_V as in §2.1 by the rule $A_V(\ell) = V \otimes_k \ell$. Note that when $\dim V$ is not finite, the functor A_V is *not representable* by a k-scheme of finite type.

It is straightforward to see the following:

(4.1.1). *Conditions (G1)–(G3) hold for the functor A_V.*

Using the additive version of Hilbert's Theorem 90, one finds:

(4.1.2). *$H^i(k, A_V) = 0$ for $i > 0$. In particular, every A_V-torsor is trivial.*

Proof. Let $\Gamma = \mathrm{Gal}(k_{\mathrm{sep}}/k)$. The cohomology groups are defined [Se 97, §I.2.2] using the complex

$$(C^\bullet, \partial) = (C^\bullet(\Gamma, V \otimes_k k_{\mathrm{sep}}), \partial)$$

where C^i consists of continuous maps $\prod^i \Gamma \to V \otimes_k k_{\mathrm{sep}}$.

Let $i > 0$ and $f \in Z^i = \ker(\partial : C^i \to C^{i+1})$. Since f is continuous, it is constant on the cosets of some subgroup $U \subset \Gamma$ having finite index. In particular, the image of f is finite and is thus contained in some finite dimensional vector subspace $W \subset V \otimes_k k_{\mathrm{sep}}$. We may evidently find a finite dimensional subspace $W_0 \subset V$ such that $W \subset k_{\mathrm{sep}} W_0$.

Then the triviality of the class $[f] \in H^i(k, A_V)$ will follow from the triviality of $[f]$ in $H^i(k, A_{W_0})$; thus we may and will suppose that V is finite dimensional.

In the finite dimensional case, the assertion may be easily proved using induction on the dimension of V together with the additive version of Hilbert's Theorem 90 [Se 97, §II.1.2]; details are left to the reader. □

4.2. **Cohomology of G-modules.** If V is G-module, the (Hochschild) cohomology groups $H^\bullet(G, V)$ are defined as the derived functors of the functor $W \mapsto W^G$ of G-fixed points on the category of G-representations over k [Ja 03, §I.4].

On the other hand, one can consider the Hochschild complex $C^\bullet(G, V)$ of V; for finite dimensional G-modules V, the m-th degree term of this complex $C^m = C^m(G, V)$ is the set of all regular functions of varieties (over k)

$$G \times \cdots \times G = \prod_{i=1}^{m} G \to V,$$

with the "usual" boundary mappings $\partial^m : C^m \to C^{m+1}$; see e.g. [Ja 03, I.4.14]. In particular, for $f \in C^1$, $\partial^1(f) : G \times G \to V$ is given by the rule $(g, g') \mapsto f(g) + gf(g')$.

(4.2.1) ([Ja 03, I,4,16]). *The derived functor cohomology $H^\bullet(G, V)$ may be identified naturally with the cohomology $H^\bullet(C^\bullet(G, V))$ of the Hochschild complex $(C^\bullet(G, V), \partial^\bullet)$*

Remark (4.2.2). When G is reductive and V is finite dimensional, the groups $H^\bullet(G, V)$ are finite dimensional as k-vector spaces; see [Ja 03, II.4.10 and II.4.7(c)]. On the other hand, $H^1(\mathbf{G}_a, k)$ is of countably infinite dimension as a k-vector space; [Ja 03, I.4.21(b)].

4.3. **Extensions.** Let V be a finite dimensional linear representation of the linear algebraic group H. We may view V as a linear algebraic group over k. Recall from §2.2 that an extension of H by V is a strictly exact sequence

$$(*) \quad 0 \to V \xrightarrow{i} E \xrightarrow{\pi} H \to 1$$

of linear algebraic groups; when considering such extensions *we always insist* that the action of $E/V = H$ on V is given by the action of H on the linear representation V. We sometimes write E for this extension when no ambiguity seems likely.

Consider a second extension

$$(**) \quad 0 \to V \xrightarrow{i} E' \xrightarrow{\pi} H \to 1$$

In §2.2 we considered isomorphisms of extensions, but we may also consider morphisms of extensions; a morphism between extensions E and E' is a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{i} & E & \xrightarrow{\pi} & H & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\phi} & & \| & & \\ 0 & \longrightarrow & V & \xrightarrow{i'} & E' & \xrightarrow{\pi'} & H & \longrightarrow & 1 \end{array}$$

where $\phi : E \to E'$ is an morphism of linear algebraic groups $E \to E'$; we write ϕ for this morphism.

In fact, this notion is not more general, as the following result shows:

(4.3.1). *Any morphism ϕ of extensions E and E' is already an isomorphism of extensions.*

Proof. Since $\pi' \circ \phi = \pi$ and $\phi \circ i = i'$, one readily argues that ϕ is injective and surjective on k_{alg}-points, where k_{alg} is an algebraic closure of k. One then argues that $d\phi$ is a linear isomorphism, and the result follows. □

Given an extension $(*)$, it follows from the result of Rosenlicht already mentioned in the proof of (3.2.1) – see [Mc 10, (2.2.3)] – that we may choose a regular mapping $\sigma : H \to E$ which is a section to π.

(4.3.2). *Consider the regular function*

$$f = f_{E,\sigma} : H \times H \to V \quad \textit{via} \quad (g, g') \mapsto \sigma(g)\sigma(g')\sigma(gg')^{-1}.$$

We have:

(1) $f \in Z^2(H, V)$.
(2) The extension $(*)$ *is trivial if and only if* $0 = [f] \in H^2(H, V)$.
(3) More generally, if $f' = f_{E',\sigma'} : H \times H \to V$ *arises from a second extension* E' *of* H *by* V *using a section* $\sigma' : H \to E'$*, the extensions are isomorphic* $(**)$ *if and only if* $[f] = [f']$ *in* $H^2(H, V)$.

Proof. [DG 70, II §3.2.3] □

Let E_1 and E_2 be two extensions of H by the linear representation V. We form a new extension $(E_1 + E_2)$ as follows. First, consider the linear algebraic group $\widetilde{F} = \widetilde{F}_{E_1,E_2}$ for which the following diagram is Cartesian:

$$\begin{array}{ccc} \widetilde{F} & \xrightarrow{f_1} & E_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ E_2 & \xrightarrow{\pi_2} & H \end{array}$$

The universal property of $\widetilde{F}$ shows there to be unique homomorphisms $j_1, j_2 : V \to \widetilde{F}$ such that $f_1 \circ j_2 = f_2 \circ j_1 = 0$, and such that $f_i \circ j_i$ is the inclusion $V \to E_i$ for $i = 1, 2$.

The above conditions show that $j_1 \oplus j_2 : V \oplus V \to \widetilde{F}$ is injective, so that $\widetilde{F}$ is an extension of H by $V \oplus V$. The image W of $(j_1, -j_2) : V \to V \oplus V$ is normal in $\widetilde{F}$, and the quotient $F = \widetilde{F}/W$ is an extension of H by V; we set $(E_1 + E_2) = F$.

(4.3.3). *(a) If* $\alpha_i \in H^2(H, V)$ *correspond to the extensions* E_i *for* $i = 1, 2$*, then* $\alpha_1 + \alpha_2 \in H^2(H, V)$ *corresponds to the extension* $(E_1 + E_2)$.
(b) If $\phi_i : E_i \to E_i'$ *are isomorphisms of extensions for* $i = 1, 2$*, there is an induced isomorphism* $(\phi_1 + \phi_2) : (E_1 + E_2) \to (E_1' + E_2')$ *of extensions.*

Proof. Assertion (a) follows from a simple computation using 2-cocyles $f_i \in Z^2(H, V)$ representing the classes α_i; details are left to the reader.

As to (b), note that ϕ_1 and ϕ_2 induce a mapping of linear algebraic groups $\widetilde{\phi} : F_{E_1,E_2} \to F_{E_1',E_2'}$ which is a morphism of extensions of H by $V \oplus V$. It is straightforward to see that $\widetilde{\phi}$ induces a morphism – and hence an isomorphism – $\phi : (E_1 + E_2) \to (E_1' + E_2')$ as required. □

4.4. **Automorphisms of the trivial extension.** Let H be a linear algebraic group over k, and let V be a finite dimensional linear representation of H. We may view V as a vector group and form the semidirect product $V \rtimes H$; it is also a linear algebraic group over k. We view $V \rtimes H$ as the trivial extension of H by the vector group V.

As in (2.2.1), let $\mathscr{A}$ be the automorphism group-functor of the trivial extension

$$0 \to V \to (H \ltimes V) \to H \to 1$$

of H by V.

We are going to prove the following:

Theorem (4.4.1). *(a) With notation as in §4.1,* $\mathscr{A} = A_Z$ *for a certain* k*-vector space* Z.
(b) $H^1(k, \mathscr{A}) = 1$.

Proof. Note that (b) is an immediate consequence of (a) together with (4.1.2).

For (a), we show that $\mathscr{A} = A_Z$ where $Z = Z^1(H, V)$ is the vector space of 1-cocycles $H \to V$. For any separable field extension $K \supset k$ an element $f \in A_Z(K) = Z^1(H_{/K}, V_{/K})$ determines an map $\Phi_f : (V \rtimes H)_{/K}$ by the rule

$$\Phi_f : (v, h) \mapsto (v + f(h), h).$$

Using the cocycle condition $\partial^1 f = 0$, one checks that Φ_f is a morphism of extensions, and it follows from (4.3.1) that Φ_f is an automorphism of $(V \rtimes H)_{/K}$. We get in this way a homomorphism

$$\Phi : A_Z \to A$$

of group functors.

On the other hand, let ϕ be an automorphism of the extension

$$0 \to V \xrightarrow{i} (V \rtimes H) \xrightarrow{\pi} H \to 1.$$

Since $\phi \circ i = i$ and $\pi \circ \phi = \phi$, and since ϕ is a group homomorphism, one sees that the regular map ϕ has the form

$$(v, h) \mapsto (v + \Psi_\phi(h), h)$$

for some morphism $\Psi_\phi : H \to V$ of varieties. Using the fact that ϕ is a group homomorphism, one argues that $\partial^1 \Psi_\phi = 0$ so that $\Psi_\phi \in Z^1 = A_Z(k)$. Repeating this construction after replacing k by any extension K of k, we have describe an assignment $\phi \mapsto \Psi_\phi$ which determines a homomorphism $\Psi : A \to A_Z$ of group functors.

Now Φ and Ψ are inverse to one another. □

4.5. **Descent of isomorphisms between extensions by linear representations.** Let H be a linear algebraic group and let V be a finite dimensional linear representation of H.

Theorem (4.5.1). *Fix two extensions*

$$(*) \quad 0 \to V \xrightarrow{i} E \xrightarrow{\pi} H \to 1 \quad \textit{and} \quad (**) \quad 0 \to V \xrightarrow{i'} E' \xrightarrow{\pi'} H \to 1$$

of H by V over k, and suppose that $\psi : (*)_{/k_{\mathrm{sep}}} \to (**)_{/k_{\mathrm{sep}}}$ is an isomorphism of extensions over k_{sep}. Then there is a k-isomorphism between the extensions $(*)$ and $(**)$.

Proof. Since the extensions $(E)_{/k_{\mathrm{sep}}}$ and $(E')_{/k_{\mathrm{sep}}}$ are isomorphic, (4.3.3)(b) shows that there is an isomorphism of extensions between $(E - E')_{/k_{\mathrm{sep}}}$ and the trivial extension $(0)_{/k_{\mathrm{sep}}} = (V \rtimes H)_{/k_{\mathrm{sep}}}$. Moreover, if we prove that $(E - E')$ is isomorphic to (0), then another application of (4.3.3)(b) will show that (E) is isomorphic to (E').

Thus in proving the Theorem, we may and will suppose $(**)$ to be the trivial extension $E' = H \ltimes V$ given by the semi-direct product.

Denote by $\mathscr{A}$ the group-functor of automorphisms of the trivial extension $(**)$ as in § 4.4. Then Theorem (4.4.1) shows that $H^1(k, \mathscr{A})$ is trivial, thus it follows from (2.2.2) that $(*)$ and $(**)$ are already isomorphic over k. □

As a consequence, we now find a proof of Theorem B from the introduction:

Corollary (4.5.2). *Suppose that H is a reductive group over k, and let E be an extension of H by the finite dimensional linear representation V viewed as a vector group. Suppose that the group $E_{/k_{\mathrm{sep}}}$ has a Levi factor. Then E has a Levi factor.*

Proof. The group E has a Levi factor if and only if the extension $0 \to V \to E \to H \to 1$ is k-isomorphic to the trivial extension given by the semi-direct product of H and V. Thus the Corollary follows immediately from the preceding Theorem. □

5. Example of failure of descent for Levi decomposition

Let k be a perfect field of characteristic $p > 0$, and let $\ell \supset k$ be a Galois extension with $[\ell : k] = p$. Consider the linear algebraic group $W = W_2$ of *length two Witt vectors* over k: thus W is isomorphic to $\mathbf{A}^2$ as a k-variety, and the group operation in W is given by the rule

$$(a, b) + (a', b') = (a + a', h(a, a') + b + b')$$

for a certain polynomial $h(Y, Z)$ with integral coefficients.

For the following, see e.g. [Se 88, §VII.2].

(5.0.3). *There is a strictly exact sequence of linear algebraic groups*

$$0 \to \mathbf{G}_a \to W \to \mathbf{G}_a \to 0$$

where the inclusion $\mathbf{G}_a \to W$ *is given by* $b \mapsto (0,b)$ *and the surjection* $W \to \mathbf{G}_a$ *is given by* $(a,b) \mapsto a$. *If* $w = (a,b)$ *and* $w' = (a',b')$ *are elements of* $W(k_{\text{alg}})$, *then* $pw = pw'$ *if and only if* $a = a'$. *In particular, the order of the element* w *is* p^2 *if and only if* $a \neq 0$.

Write $F = \mathbf{Z}/p\mathbf{Z}$, and consider the embedding $F \subset \mathbf{G}_a \xrightarrow{i} W$ determined by the embedding $\mathbf{Z}/p\mathbf{Z} \subset k$ of fields. Now form the quotient group $U = W/F$. We may find $\tilde{e} \in W(k)$ for which $\pi(\tilde{e}) \neq 0$ and $p\tilde{e} \in F$; then the image $0 \neq e \in U(k)$ of $\tilde{e}$ satisfies $pe = 0$.

View the linear algebraic group $H = U \times \mathbf{Z}/p\mathbf{Z}$ as an extension

$$(\flat) \quad 0 \to U \to H \to \mathbf{Z}/p\mathbf{Z} \to 0.$$

Since e has order p, there is a k-automorphism τ of the extension $(\flat)$ determined by the rule

$$\tau(u, n + p\mathbf{Z}) = (u + ne, n + p\mathbf{Z}) \quad \text{for} \quad u \in U(k_{\text{alg}}), n \in \mathbf{Z}.$$

Fix a generator σ for the Galois group $\text{Gal}(\ell/k) = \langle \sigma \rangle$ and write $A = \mathscr{A}(\ell)$ for the group of automorphisms of the extension $(\flat)_{/\ell}$ obtained from $(\flat)$ be scalar extension as in §2.2.

The Galois group $\text{Gal}(\ell/k)$ acts on A, and the rule $\sigma^i \mapsto s_{\sigma^i} = \tau^i$ defines a group homomorphism $\text{Gal}(\ell/k) \to A$. Since τ is a k-automorphism, we have ${}^{\sigma}\tau = \tau$ so that s is in fact a 1-cocycle.

Using the 1-cocycle s, form the "twisted" extension ${}_s(\flat)$

$${}_s(\flat) \quad 0 \to U \to {}_sH \xrightarrow{\pi} \mathbf{Z}/p\mathbf{Z} \to 0$$

as in (2.2.2); thus the group $G = {}_sH$ is obtained from H by twisting with s.

(5.0.4). *(a) The extension* $(\flat)$ *is not k-isomorphic to the extension* ${}_s(\flat)$
(b) ${}_s(\flat)$ *becomes isomorphic to* $(\flat)$ *after extending scalars to* ℓ.

Proof. (b) holds by construction. To prove (a), it will suffice to argue that $G = {}_sH$ has no k-rational element $x \in G(k)$ of order p for which $\pi(x) = 1 + p\mathbf{Z}$.

Well, if $\pi(x) = 1 + p\mathbf{Z}$, then x has the form $x = (u, 1 + p\mathbf{Z}) \in {}_sH(\ell) = H(\ell)$. Now $x \in {}_sH(k)$ if and only if $x = \sigma \star x$ for the "twisted" $\text{Gal}(\ell/k)$ action $\star$ on $H(\ell)$. By the definition of this twisted action, we have $x \in {}_sH(k)$ if and only if

$$u = {}^{\sigma}u + e.$$

Now, $u \in U(\ell)$ is represented by some element $w = (a,b) \in W(k_{\text{alg}})$ where $a, b \in k_{\text{alg}}$. Denoting by $\tilde{\sigma}$ an element of $\text{Gal}(k_{\text{alg}}/k)$ whose restriction to ℓ coincides with σ, we find that

$$(a,b) - ({}^{\tilde{\sigma}}a, {}^{\tilde{\sigma}}b) \equiv \tilde{e} \pmod{F}$$

in the group $W(k_{\text{alg}})$. Since $\tilde{e}$ has order p^2, it follows that $a \neq {}^{\sigma}a$ so that $a \notin k$. In particular, it follows from (5.0.3) that $p(a,b) \notin F$, so the image u of (a,b) in $U(k)$ has order p^2. Thus also $x \in H(k)$ has order p^2 and the proof is complete. ∎

Thus $G = {}_sH$ provides an example of a (disconnected) linear algebraic group such that $G_{/k_{\text{sep}}} = {}_sH_{/k_{\text{sep}}} = H_{/k_{\text{sep}}}$ has a Levi decomposition, but $G = {}_sH$ has no Levi decomposition.

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