ADJOINT JORDAN BLOCKS

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ABSTRACT. Let G be a quasisimple algebraic group over an algebraically closed field of characteristic $p > 0$. We suppose that p is *very good* for G; since p is good, there is a bijection between the nilpotent orbits in the Lie algebra and the unipotent classes in G. If the nilpotent $X \in \mathrm{Lie}(G)$ and the unipotent $u \in G$ correspond under this bijection, and if u has order p, we show that the partitions of $\mathrm{ad}(X)$ and $\mathrm{Ad}(u)$ are the same. When G is classical or of type G_2, we prove this result with no assumption on the order of u.

In the cases where u has order p, the result is achieved through an application of results of Seitz concerning good A_1 subgroups of G. For classical groups, the techniques are more elementary, and they lead also to a new proof of the following result of Fossum: the structure constants of the representation ring of a 1-dimensional formal group law $\mathscr{F}$ are independent of $\mathscr{F}$.

1. INTRODUCTION

Let G be a quasisimple algebraic group over an algebraically closed field k of characteristic $p > 0$. In case p is *good* for G, there is a bijection between the nilpotent orbits in the Lie algebra, and the unipotent classes in G.

Suppose the class of the unipotent u and the nilpotent X correspond under this bijection. Assume further that u has order p (or equivalently, that X is p-nilpotent, i.e. $X^{[p]} = 0$). We prove in Theorem 10 that if p is *very good*, the partition of $\mathrm{Ad}(u)$ is the same as the partition of $\mathrm{ad}(X)$ (where Ad and ad denote the respective adjoint actions of G and $\mathfrak{g} = \mathrm{Lie}(G)$ on $\mathfrak{g}$). This is carried out in §3.1 using results of Seitz [S00] on good A_1-type subgroups of G.

When G is a classical group, or has type G_2, we show in Theorems 24 and 30 that the partitions of $\mathrm{Ad}(u)$ and $\mathrm{ad}(X)$ coincide with no assumption on the order of u. The techniques for classical groups involve comparison of "group like" and "Lie algebra like" tensor products; see §4. One may consider more generally the tensor product determined by any 1 dimensional formal group law. In this context, our techniques provide a new proof of a theorem of Fossum: the representation ring of a formal group law $\mathscr{F}$ is independent of $\mathscr{F}$; see Corollary 18.

Lawther [Law95] has found (with the aid of a computer) the adjoint partitions of the unipotent classes in exceptional simple algebraic groups. Combined with his work, Theorem 10 computes the adjoint partitions for those nilpotent classes of exceptional groups in good characteristic which are p-nilpotent.

In a final section §7, we consider adjoint partitions in characteristic 0. Let G be a semisimple group of rank r in characteristic 0. We show that for certain nilpotent elements X, the eigenvalues (on a Cartan subalgebra) of a corresponding Weyl group element account for r parts of the partition of $\mathrm{ad}(X)$.

At the risk of being pedantic, we recall: The partition of a nilpotent linear endomorphism X of a vector space V over a field is the ordered sequence of Jordan block sizes of X. The partition of a unipotent linear automorphism u of V is the partition of the nilpotent map $u - 1$.

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2. SIMPLE GROUPS

Let G be a quasisimple algebraic group over k with Lie algebra $\mathfrak{g}$; thus the root system R of the semisimple group G is irreducible.

2.1. **The bad, the good, and the very good.** The characteristic p of k is said to be bad for R in the following circumstances: $p = 2$ is bad whenever $R \neq A_r$, $p = 3$ is bad if $R = G_2, F_4, E_r$, and $p = 5$ is bad if $R = E_8$. Otherwise, p is good.

Moreover, p is very good if $r \not\equiv -1 \pmod{p}$ in case $R = A_r$.

2.2. **Classical groups.** Let V be a finite dimensional vectorspace over k, suppose that φ is a bilinear form on V, let Ω be the full group of isometries of V with respect to φ, and let $\mathfrak{o} = \mathrm{Lie}(\Omega)$. We will say that Ω is a *classical group* in the following situations:

CG1. $\varphi = 0$, so that $\Omega = \mathrm{GL}(V)$ and $\mathfrak{o} = \mathfrak{gl}(V)$.

CG2. φ is nondegenerate and alternating, so that $\Omega = \mathrm{Sp}(V, \varphi)$ is a symplectic group, and $\mathfrak{o} = \mathfrak{sp}(V, \varphi)$.

CG3. φ is nondegenerate and symmetric, so that $\Omega = O(V, \varphi)$ is an orthogonal group, and $\mathfrak{o} = \mathfrak{so}(V, \varphi)$.

In each case, Ω is a reductive group (though it is not connected in case CG3, and it is not semisimple in case CG1). The prime 2 is bad in cases CG2 and CG3; all other primes are good for classical groups. For convenience, we write (ν, V) for the natural representation of Ω.

We record the following well-known characterization of the adjoint representation of a classical group Ω.

Lemma 1. *There is an isomorphism of Ω-representations*

$$(\mathrm{Ad}, \mathfrak{o}) \simeq \begin{cases} (\nu \otimes \nu^\vee, V \otimes V^\vee) & \text{in case CG1,} \\ (\mathrm{Sym}^2 \nu, \mathrm{Sym}^2 V) & \text{in case CG2,} \\ (\bigwedge^2 \nu, \bigwedge^2 V) & \text{in case CG3.} \end{cases}$$

Here, $V^\vee$ denotes the dual vector space (and $\nu^\vee$ the contragredient representation). We also record:

Lemma 2. *Let W be a k-vector space. If the characteristic of k is not 2, there is an isomorphism of $\mathrm{GL}(W)$-representations*

$$W \otimes W \simeq \mathrm{Sym}^2 W \oplus \bigwedge^2 W.$$

2.3. **Springer's isomorphism.** Let G be connected, reductive with an irreducible root system, let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent variety , and let $\mathcal{U} \subset G$ be the unipotent variety of G. We recall the following result (which depends on Bardsley and Richardson's construction of a "Springer isomorphism").

Proposition 3. *If the characteristic of k is very good for G, there is a G-equivariant isomorphism $\theta : \mathcal{N} \to \mathcal{U}$ with the property $\theta(X^{[p]}) = \theta(X)^p$ for each $X \in \mathcal{N}$.*

Proof. [M, Lemma 27, Theorem 35]. □

2.4. **Distinguished nilpotents and associated co-characters.** In this paragraph, G may be an arbitrary reductive group in good characteristic. A nilpotent $X \in \mathfrak{g}$ is *distinguished* if the connected center of G is a maximal torus of $C_G(X)$. The Bala-Carter theorem, proved by Pommerening for reductive groups in good characteristic, implies that any nilpotent $X \in \mathfrak{g}$ is distinguished in the Lie algebra of some Levi subgroup L of G.

Let $X \in \mathfrak{g}$ be nilpotent. A co-character $\phi : k^\times \to G$ is said to be associated to X if $\mathrm{Ad} \circ \phi(t)X = t^2 X$ for all $t \in k^\times$, and if the image of ϕ is contained in the derived group L' of a Levi subgroup L such that X is distinguished in $\mathrm{Lie}(L)$. The existence of a co-character associated to a nilpotent $X \in \mathfrak{g}$ follows from Pommerening's proof of the Bala-Carter theorem (more precisely: is a key step in his proof of that theorem). Any two co-characters associated with X are conjugate via $C_G^o(X)$; see e.g. [Jan, Lemma 5.2].

The following useful observation is a consequence of definitions:

Lemma 4. *Let $\phi : \mathrm{SL}_{2/k} \to G$ be a homomorphism and suppose that $X = d\phi(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$ is a distinguished nilpotent element of G. Then the rule $\psi(t) = \phi(diag(t, t^{-1}))$ yields a co-character associated to X.*

3. ADJOINT JORDAN BLOCKS FOR ELEMENTS OF ORDER p

3.1. **Partitions and $\mathrm{SL}_2(k)$.** Let $S = \mathrm{SL}_2(k)$, and write $S(p) = \mathrm{SL}_2(\mathbf{F}_p)$. The dominant weights of a maximal torus of S may be identified with $\mathbf{Z}_{\geq 0}$. To each $d \in \mathbf{Z}_{\geq 0}$, there correspond various S-representations: a simple module $L(d)$, a Weyl module $V(d)$ (of dimension $d + 1$), and an indecomposable tilting module $T(d)$; all three have highest weight d. If $d < p$, then $L(d) = V(d) = T(d)$. We refer to [Don93] for more details concerning tilting modules. Note especially that each tilting S-module is a direct sum $\bigoplus_{d_i} T(d_i)$ for various $d_i \geq 0$; see [Don93, Theorem 1.1].

Proposition 5. *Suppose that $p \neq 2$, and $p \leq d \leq 2p - 2$.*
  (1) *Each unipotent element $1 \neq u \in S$ acts on $T(d)$ with partition (p, p).*
  (2) *Each nilpotent element $0 \neq X \in \mathrm{Lie}(S) = \mathfrak{sl}_2(k)$ acts on $T(d)$ with partition (p, p).*

Proof. By [S00, Lemma 2.3], the $S(p)$-module $\mathrm{res}_{S(p)}^S T(d)$ is projective of dimension $2p$. If $U(p) < S(p)$ is the subgroup generated by some element of order p, it follows that $\mathrm{res}_{U(p)}^S T(d)$ is free of rank 2 over $kU(p)$. Now (1) is immediate, since u is S-conjugate to an element in $U(p)$.

The Lie algebra case is essentially the same argument. Write $\mathfrak{s} = \mathrm{Lie}(S)$. One then knows that $T(d)$ is a direct summand of $L(r + 1) \otimes L(p - 1)$, where $d = r + p$. Since the Steinberg module $L(p - 1)$ is projective as a module for the restricted enveloping algebra of $\mathfrak{s}$ [Jan87, Prop. II.10.2], the same holds for $T(d)$. Put $\mathfrak{u} = kX \subset \mathfrak{s}$; together with [Jan98, Cor. 3.4], the above shows that $T(d)$ is free of rank two over Λ, the restricted enveloping algebra of $\mathfrak{u}$. Since Λ is a truncated polynomial algebra, X must act with the indicated partition. □

Let u, X as in the previous proposition. On a simple module $L(d) = T(d)$ with $d < p$, both u and X act as a single Jordan block. We thus obtain:

Corollary 6. *Let (ρ, V) be a tilting module for S such that each weight d of a maximal torus of S on V satisfies $d \le 2p - 2$. If $1 \ne u \in S$ is unipotent and $0 \ne X \in \mathrm{Lie}(S)$ is nilpotent, then the partition of $\rho(u)$ is the same as the partition of $d\rho(X)$.*

3.2. **Some results on classical groups.** Let $\Omega = \Omega(V)$ be a classical group in good characteristic (§2.2).

Proposition 7. *Let $u \in \Omega$ be a unipotent element of order p, and let $\lambda = (\lambda_i)_{1 \le i \le t}$ be the partition of u. There is a unique Ω-conjugacy class of homomorphisms $\psi : \mathrm{SL}_2(k) \to \Omega$ with the following properties:*

(1) *The image of ψ meets the conjugacy class of u.*
(2) *The character of the $\mathrm{SL}_2(k)$-module $(\nu \circ \psi, V)$ is*

$$\chi(\lambda_1 - 1) + \chi(\lambda_2 - 1) + \cdots + \chi(\lambda_t - 1).$$

Proof. For existence of such a ψ, see [S00, Prop. 4.1]. To see that any two such homomorphisms are conjugate, see the proof of [S00, Prop. 7.1]. □

Proposition 8. *Let $\psi : \mathrm{SL}_2(k) \to \Omega$ be a homomorphism satisfying (1) and (2) of Proposition 7. Then $(\mathrm{Ad} \circ \psi, \mathfrak{o})$ is a tilting module for $\mathrm{SL}_2(k)$. Moreover, each weight d of this representation satisfies $d \le 2p - 2$.*

Proof. Since the tensor product of tilting modules is again a tilting module [Don93, Prop. 1.2], and since each weight e of $(\nu \circ \psi, V)$ satisfies $e \le p - 1$, Lemmas 1 and 2 yield the result. □

Let $X \in \mathfrak{o}$ be nilpotent, and let λ_X be the partition of X (recall that for classical groups, the nilpotent and unipotent classes are classified by their partitions; see e.g. [Hum95, §7.11] or [Jan, Theorem 1.6] for a description of those λ which may arise as λ_X). As in §2.4, let ϕ be a co-character associated with X. In fact, the existence of such a ϕ is quite easy to prove for a classical group; see [Jan, §3.5].

Proposition 9. *Let $X \in \mathfrak{o}$ be nilpotent with partition $\lambda = (\lambda_i)$. Suppose that $\lambda_i \le p$ for each i. Let $\psi : \mathrm{SL}_2(k) \to \Omega$ be a homomorphism such that X is in the image of $d\psi$, and such that the restriction of ψ to a maximal torus of $\mathrm{SL}_2(k)$ is a co-character associated to X. Then ψ satisfies (1) and (2) of Proposition 7 for the unipotent class with partition λ.*

Proof. The explicit recipe [Jan, §3.5] for a co-character associated to X implies that the character of the $\mathrm{SL}_2(k)$-representation $(\mathrm{Ad} \circ \psi, \mathfrak{o})$ is as in (2) of Proposition 7. For condition (1), it suffices to observe that $(\mathrm{Ad} \circ \psi, \mathfrak{o})$ is a restricted, semisimple representation of $\mathrm{SL}_2(k)$ (by the linkage principle), so that a unipotent element in the image of ψ indeed acts with the partition λ. □

3.3. **The main result for unipotent elements of order p.** Let G be quasisimple in very good characteristic, and let $\theta : \mathcal{N} \to \mathcal{U}$ be a G-equivariant homeomorphism as in Proposition 3. Suppose that $u = \theta(X)$, and that $u^p = 1$. According to Proposition 3, this is equivalent to $X^{[p]} = 0$.

Theorem 10. *The partition of $\mathrm{Ad}\, u$ is the same as that of $\mathrm{ad}\, X$.*

Before giving the proof, we note the following lemmas which provide a helpful reduction in case the root system of G is classical.

Lemma 11. *Let $\pi : G \to G_{ad}$ be the isogeny to the adjoint group. The partition of $\mathrm{Ad}(u)$ is the same as that of $\mathrm{Ad}(\pi(u))$, and the partition of $\mathrm{ad}(X)$ is the same as that of $\mathrm{ad}(d\pi(X))$.*

Proof. Since p is very good, $d\pi$ is a bijection and the result is clear. $\qquad\square$

Lemma 12. *Suppose that the root system R of G is classical $(A, B, C$ or $D)$, that the characteristic is very good for G, and that G is of adjoint type. There is a classical group Ω (§2.2), an $\mathrm{Ad}(\Omega)$ invariant decomposition $\mathfrak{o} = \mathfrak{o}_1 \oplus \mathfrak{o}_2$, a unipotent $u' \in \Omega$, and a nilpotent $X' \in \mathfrak{o}$ such that*

   (1) *u' and X' correspond under a Springer isomorphism (Propoosition 3) for Ω^o,*
   (2) *the partition of $\mathrm{Ad}(u)$ coincides with that of $\mathrm{Ad}(u')_{|\mathfrak{o}_1}$,*
   (3) *the partition of $\mathrm{ad}(X)$ coincides with that of $\mathrm{ad}(X')_{|\mathfrak{o}_1}$,*
   (4) *$\mathfrak{o}_2$ is either zero, or $\mathrm{Ad}(\Omega)$ acts trivially on it.*

Proof. By the previous lemma, we may replace G be the corresponding adjoint group. Since G has a "classical type" root system, it is well known that G is a quotient of a classical group Ω by a central subgroup. It suffices to find a decomposition $\mathfrak{o} = \mathfrak{o}_1 \oplus \mathfrak{o}_2$ such that the quotient map $d\pi$ induces an Ω-equivariant isomorphism $\mathfrak{o}_1 \simeq \mathfrak{g}$, and such that $\mathfrak{o}_2$ is 0 or a trivial module. Indeed, [M, Prop. 26] then shows that $d\pi$ will also induce Ω-equivariant isomorphisms between the nilpotent variety of $\mathfrak{o}$ and that of $\mathfrak{g}$, and that π will induce an isomorphism between the unipotent variety of Ω and that of G, and the lemma will follow.

When $R = B, C$ or D, the map π is an isogeny, and since $\mathfrak{o} = \mathfrak{o}_1$ and $\mathfrak{g}$ are simple representations, $d\pi$ induces an isomorphism on Lie algebras.

When $R = A_r$, we may take $\Omega = \mathrm{GL}_{r+1}(k)$ and our hypothesis shows that p does not divide $r + 1$. Set $\mathfrak{o}_1 = \mathfrak{sl}_{r+1}(k)$ and let $\mathfrak{o}_2$ be the space spanned by the identity matrix. Then $\mathfrak{o} = \mathfrak{o}_1 \oplus \mathfrak{o}_2$ is an Ω-stable decomposition, and Ω acts trivially on $\mathfrak{o}_2$. Moreover, $\mathfrak{o}_1$ and $\mathfrak{g}$ are simple modules, so $d\pi$ induces an isomorphism $\mathfrak{o}_1 \simeq \mathfrak{g}$, which completes the proof of the lemma. $\qquad\square$

Proof of Theorem 10. Fix $u \in G$ unipotent of order p. Let $\phi : \mathrm{SL}_2(k) \to G$ be a homomorphism whose image contains u and is a good A_1 subgroup, in the sense of [S00]; this means that the weights of the $\mathrm{SL}_2(k)$-module $(\mathrm{Ad} \circ \phi, \mathfrak{g})$ are all $\leq 2p-2$ (the existence of such a ϕ is proved in *loc. cit.*). It follows from Theorem 1.1 of *loc. cit.* that $(\mathrm{Ad} \circ \phi, \mathfrak{g})$ is a tilting module for $\mathrm{SL}_2(k)$ (recall that we suppose p to be *very good* so that the exceptional situation in *loc. cit.* does not occur).

Now let $\psi : \mathrm{SL}_2(k) \to G$ be a "sub-principal" homomorphism, as constructed in [M], whose image contains u. Write $u = \theta(X)$ for $X \in \mathcal{N}$. Then [M, Theorem 12] shows that the image of $d\psi$ meets the nilpotent orbit containing X. So Theorem 10 will follow from Corollary 6 once we show that ψ is conjugate to ϕ. According to [S00, Prop. 7.2], all good A_1-subgroups meeting the class of u are conjugate in G. So we are reduced to proving that the image of ψ is a good A_1-subgroup.

By [M, Theorem 2], one knows that the weights of $(\mathrm{Ad} \circ \psi, \mathfrak{g})$ are determined by the labeled Dynkin diagram of u. When G has an exceptional-type root system (E,F, or G) it follows from the proof of [S00, Prop. 4.2] that those weights are $\leq 2p - 2$, so that the image of ψ is indeed a good A_1-subgroup.

When G has a classical-type root system (A, B, C, or D), we may suppose, according to the previous lemma, that G is a classical group. By [M, Theorem 2], the restriction of ψ to a maximal torus of $\mathrm{SL}_2(k)$ is a co-character associated to a conjugate of X. Thus Proposition 9 implies that ψ satisfies the conditions of Proposition 7 for u. Then Proposition 8 shows that the image of ψ is a good A_1 subgroup, as desired. $\qquad\square$

3.4. **Further remarks on conjugacy of A_1-subgroups.** In this section, G denotes a connected, quasisimple group. If G has a classical-type root system, we suppose that the characteristic is very good for G. If G has an exceptional-type root system R, we suppose that $p > N$ where $N = 3, 3, 5, 7, 7$ when $R = G_2, F_4, E_6, E_7, E_8$ respectively.

Theorem 13. *Let $X \in \mathfrak{g}$ be a distinguished nilpotent element satisfying $X^{[p]} = 0$. There is a unique G-conjugacy class of homomorphisms $\phi : \mathrm{SL}_2(k) \to G$ such that the image of $d\phi$ meets the orbit of X. For any such ϕ, the image of ϕ meets the unipotent class containing $\theta(X)$, where θ is an equivariant isomorphism as in Proposition 3.*

Proof. That there is at least one such ϕ follows from [M, Theorem 12]; moreover it follows from *loc. cit.* that the image of ϕ meets the class of $\theta(X)$, so the last assertion of the theorem will follow from conjugacy. If ϕ is any such homomorphism, we may suppose that $X = d\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Since X is distinguished, it follows from Lemma 4 that $\nu(t) = \phi(\mathrm{diag}(t, 1/t)) : k^\times \to G$ is a co-character associated to X.

Suppose first that G has a classical root system. Since any central extension of $\mathrm{SL}_2(k)$ is split, we may replace G by the corresponding adjoint group, as in Lemma 11. As in the proof of Lemma 12, we may therefore suppose that $G = \Omega = \Omega(V)$ is a classical group. In this case, the existence of at least one ϕ follows also from Proposition 7 (this is more elementary than [M]). The remaining assertions follow from Proposition 9.

Now supose that G has an exceptional root system, and that $\phi' : \mathrm{SL}_2(k) \to G$ is a second homomorphism with $X = d\phi' \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\nu'(t) = \phi'(\mathrm{diag}(t, 1/t)$ is a second co-character associated with X. According to 2.4, ν and ν' are conjugate by $C_G^o(X)$; thus we may suppose that $\nu = \nu'$. Under our assumptions on p, a result of Liebeck–Seitz and Lawther–Testerman now shows that ϕ and ϕ' are G-conjugate; see [LT99, §2 and Theorem 4]. This completes the proof.

$\square$

Remark 14. There are of course homomorphisms $\phi : \mathrm{SL}_2(k) \to G$ whose image meets a distinguished unipotent class, but for which $d\phi$ fails to meet the corresponding nilpotent class. To give an example, recall for $0 \le m < p$ that the $\mathrm{SL}_2(k)$-module $L(m)$ carries an invariant non-degenerate bilinear form β_n which is alternating when m is odd, and symmetric when m is even. Now, suppose that $p > 2n > 0$, and consider the $\mathrm{SL}_2(k)$-module $W = L(2n) \otimes L(1)^{[1]}$ (where the exponent in the second factor denotes a Frobenius twist). The tensor product $\beta = \beta_{2n} \otimes \beta_1$ is an invariant alternating form on W. If $\rho : \mathrm{SL}_2(k) \to \mathrm{Sp}(W, \beta)$ is the defining homomorphism, then the image of ρ meets the (distinguished) unipotent class with partition $(2n + 2, 2n)$ on W, but $X = d\rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has partition $(2n + 1, 2n + 1)$. Note that X is *not* distinguished.

4. FORMAL GROUPS AND NILPOTENT ENDOMORPHISMS

Fix in this section an arbitrary field k.

4.1. **Similar nilpotent endomorphisms.** Denote by $\mathscr{C}$ the category whose objects are pairs (V, ϕ) where V is a (finite dimensional) k-vector space and ϕ is a nilpotent

k-endomorphism; a morphism $f : (V, \phi) \to (W, \psi)$ in $\mathscr{C}$ is a linear map which intertwines ϕ and ψ.

We begin with the simple observation

Lemma 15. *Let $f \in tk[\![t]\!] \setminus t^2 k[\![t]\!]$. For each (V, ϕ) in $\mathscr{C}$, there is an isomorphism $(V, \phi) \simeq (V, f(\phi))$.*

Proof. Since any subspace invariant by ϕ is also invariant by $f(\phi)$, we may suppose that ϕ acts as a single Jordan block. We identify (V, ϕ) with $(k[t]/(t^n), \mu_t)$ where μ_t is multiplication by (the image of) t, so we must find a linear automorphism Λ of $k[t]/(t^n)$ with $\Lambda \circ \mu_t = \mu_{f(t)} \circ \Lambda$.

It is well known that f has a "compositional inverse"; i.e. a series $g \in tk[\![t]\!]$ with $f(g(t)) = t$ and $g(f(t)) = t$. It follows that the algebra homomorphism $\Lambda : k[t]/(t^n) \to k[t]/(t^n)$ satisfying $t \mapsto f(t)$ is an automorphism. Since Λ clearly intertwines μ_t and $\mu_{f(t)}$, the proof is complete. $\qquad\square$

Now suppose that $Y_1, \ldots, Y_m$ are indeterminants, and put $A = k[\![Y_1, \ldots, Y_m]\!]$. For an m-tuple of positive integers $\vec{r} = (r_1, \ldots, r_m)$, put

$$A_{\vec{r}} = A/(Y_1^{r_1}, \ldots, Y_m^{r_m}).$$

If $\xi_1, \ldots, \xi_r \in k^\times$ and $f_1, \ldots, f_r$ are formal series without constant term, there is a uniquely determined continuous algebra homomorphism $\Lambda : A \to A$ satisfying $\Lambda(Y_i) = Y_i(\xi_i + f_i)$. Evidently Λ induces an algebra homomorphism $\Lambda_{\vec{r}} : A_{\vec{r}} \to A_{\vec{r}}$.

Proposition 16. Λ *and* $\Lambda_{\vec{r}}$ *are algebra automorphisms.*

Proof. Giving A the $(Y_1, \ldots, Y_m)$-adic filtration, the associated graded ring $\operatorname{gr} A$ identifies with A, and $\operatorname{gr} \Lambda$ is given by $y_i \mapsto \xi_i y_i$, hence is an automorphism. Since A is complete and Hausdorff in the $(Y_1, \ldots, Y_m)$-adic topology, [Ser00, II.A.4 Prop 6] shows that Λ is surjective. It follows that $\Lambda_{\vec{r}}$ is surjective; since $A_{\vec{r}}$ is finite dimensional, $\Lambda_{\vec{r}}$ is an automorphism. If $f \in \ker \Lambda$, the image of f in $A_{\vec{r}}$ is 0 for each $\vec{r}$; thus $f = 0$ and the proposition follows. $\qquad\square$

Theorem 17. *Let $F(u, v) \in k[\![u, v]\!]$ satisfy $F(u, v) \equiv \xi_1 u + \xi_2 v \pmod{(u, v)^2}$ with $\xi_1, \xi_2 \in k^\times$, and suppose that (V, ϕ) and (W, ψ) are objects in $\mathscr{C}$. Then there is an isomorphism in $\mathscr{C}$*

$$(V \otimes W, \phi \otimes 1 + 1 \otimes \psi) \simeq (V \otimes W, F(\phi \otimes 1, 1 \otimes \psi)).$$

Proof. Write $X = \phi \otimes 1 + 1 \otimes \psi$ and $X' = F(\phi \otimes 1, 1 \otimes \psi)$; we must show that X and X' are conjugate by the adjoint action of $\operatorname{GL}(V \otimes W)$. If $V = V_1 \oplus V_2$ with each V_i invariant by ϕ, then $V_i \otimes W$ is invariant by both X and X' for $i = 1, 2$. So we may suppose that ϕ (and also ψ) acts as a single Jordan block. We identify (V, ϕ) with $(k[Y]/(Y^n), \mu_y)$ and (W, ψ) with $(k[Z]/(Z^m), \mu_z)$, where we write y and z for the images of Y and Z in the quotient algebras.

We must find a linear automorphism Λ of

$$B = k[Y]/(Y^n) \otimes k[Z]/(Z^m) = k[Y, Z]/(Y^n, Z^m)$$

satisfying $\Lambda \circ (\mu_{y+z}) = \mu_{F(y,z)} \circ \Lambda$.

Choose formal series $H_1(u, v), H_2(u, v) \in uk[\![u, v]\!] + vk[\![u, v]\!]$ such that $F(u, v) = \xi_1 u + \xi_2 v + H_1(u, v)u + H_2(u, v)v$ [in general, H_1 and H_2 are not uniquely determined by this condition]. Then according to Proposition 16, there is an algebra automorphism $\Lambda : B \to B$ satisfying $\Lambda(y) = f_1(y, z)$ and $\Lambda(z) = f_2(y, z)$ where

$$f_1(Y, Z) = Y(\xi_1 + H_1(Y, Z)) \quad \text{and} \quad f_2(Y, Z) = Z(\xi_2 + H_2(Y, Z)).$$

Since

$$\Lambda(y+z) = f_1(y,z) + f_2(y,z) = \xi_1 y + \xi_2 z + y H_1(y,z) + z H_2(y,z) = F(y,z),$$

Λ intertwines μ_{y+z} and $\mu_{F(y,z)}$ as desired. $\qquad\square$

4.2. **Formal groups.** A *formal group law* over k is a series $\mathscr{F}(u,v) \in k[\![u,v]\!]$ satisfying $\mathscr{F}(u,v) \equiv u + v \pmod{(u,v)^2}$ and certain other axioms that are written down in a number of places, se e.g. [Haz78]. Note that $\mathscr{F}(u,v)$ is automatically commutative; see [Haz78, Theorem I,1.6.7].

The representation ring $R_{\mathscr{F}}$ of $\mathscr{F}$ introduced in [Fos89] is first of all the Abelian group generated by symbols $[M]$ for each object M in $\mathscr{C}$ subject to relations as follows: $[M]$ and $[N]$ are identified if $M \simeq N$. The addition is given by the direct sum in $\mathscr{C}$:

$$[(V,\phi)] + [(W,\psi)] = [(V \oplus W, \phi \oplus \psi)].$$

The tensor product (with respect to $\mathscr{F}$) of two objects in $\mathscr{C}$ is defined by

(4.1) $$(V,\phi) \otimes_{\mathscr{F}} (W,\psi) = (V \otimes W, \mathscr{F}(\phi \otimes 1, 1 \otimes \psi)).$$

The product in $R_{\mathscr{F}}$ is defined by $[(V,\phi)] \cdot [(W,\psi)] = [(V,\phi) \otimes_{\mathscr{F}} (W,\psi)]$; the axioms of a formal group imply that $R_{\mathscr{F}}$ is indeed a ring.

Let J_n denote the class in $R_{\mathscr{F}}$ obtained from the object (k^n, ϕ_n) where ϕ_n is a single $n \times n$ Jordan block. The J_n form a $\mathbf{Z}$-basis for $R_{\mathscr{F}}$, and the multiplication in $R_{\mathscr{F}}$ is completely determined by the structure constants $a^i_{n,m}$ for the products $[J_n] \cdot [J_m] = \sum_i a^i_{n,m} [J_i]$ for each $n, m \geq 0$.

Now suppose that k has characteristic $p > 0$. For each $n \geq 1$, the full subcategory of $\mathscr{C}$ consisting of those objects (V,ϕ) such that $\phi^{p^n} = 0$ (p^n-nilpotent objects) is closed under the tensor product (4.1) for any formal group $\mathscr{F}$.

Let C be a cyclic group of order p^n with generator g. The category of kC modules is then equivalent to the p^n-nilpotent subcategory; the C-module V corresponds to the object $(V, g-1)$ of $\mathscr{C}$. The usual tensor product for group representations corresponds to the tensor product (4.1) with respect to the *multiplicative* formal group law $\mathscr{F}_m = u + v + uv$.

Let $\mathfrak{w}_n$ be the Abelian p-Lie algebra $\sum_{i=0}^{n-1} k X^{[p^i]}$ (with the indicated p-power map). The category of restricted $\mathfrak{w}_n$-representations is equivalent to the p^n-nilpotent subcategory. The usual tensor product of Lie algebra representations corresponds to the tensor product (4.1) with respect to the additive formal group law $\mathscr{F}_a = u + v$.

Corollary 18 (Fossum). *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be any two formal group laws over k. The structure constants for the representation rings $R_{\mathscr{F}_1}$ and $R_{\mathscr{F}_2}$ are the same.*

Proof. This is an immediate consequence of Theorem 17. $\qquad\square$

In characteristic 0, all 1-dimensional formal group laws are isomorphic (see [Haz78, Theorem 1.6.2]) so one has in that case a simple proof of the corollary. The structure constants in characteristic 0 are the "Clebsch-Gordan" coefficients; see [Sri64, (2.3)]. In characteristic $p > 0$, there are non-isomorphic formal group laws; for instance, $\mathscr{F}_a \not\simeq \mathscr{F}_m$.

The representation ring of the multiplicative law $\mathscr{F}_m$ has been studied by J. A. Green in [Gre62] and B. Srinivasan in [Sri64]; see also the references in [Fos89]. Green computed enough of the structure constants to show that $R_{\mathscr{F}_m} \otimes_{\mathbf{Z}} \mathbf{C}$ is a

semisimple algebra. Srinivasan determined the structure constants of $R_{\mathscr{F}_m}$ explicitly.

The above corollary was proved by Fossum in [Fos89, III]. He showed that the eight formulas obtained by Green in [Gre62] hold for any formal group law; the corollary follows since those formulas determine the multiplication in $R_{\mathscr{F}}$ uniquely. This proof is in the spirit of "modular representation theory"; it exploits the fact that (k^q, ϕ_q) is a projective object in a suitable subcategory of $\mathscr{C}$ when $q = p^m$.

4.3. **A result on symmetric series.** Let Σ_m be the symmetric group on m letters. Fix $Y_1, \ldots, Y_m$ indeterminants (each given degree 1), and consider the graded algebras $A = k[Y_1, \ldots, Y_m]$ and $\hat{A} = k[\![Y_1, \ldots, Y_m]\!]$.

The rule $\sigma(Y_i) = Y_{\sigma^{-1}(i)}$ for $\sigma \in \Sigma_m$ defines representations of Σ_m by graded algebra automorphisms on A and by continuous graded algebra automorphisms on $\hat{A}$. We use the notations A^{Σ_m} and $\hat{A}^{\Sigma_m}$ for the subalgebras of invariants. A_d and $\hat{A}_d$ denote the respective d-th homogeneous components, $A_{\geq d}$ is $\bigoplus_{e \geq d} A_e$, and $\hat{A}_{\geq d}$ is the ideal of $\hat{A}$ generated by $A_{\geq d}$ (the d-th power of the unique maximal ideal of $\hat{A}$).

Lemma 19. *Suppose that $m!$ is invertible in k. Let $H \in A_d^{\Sigma_m}$ be a homogeneous invariant. Then there are elements $H_i \in A_d$ for $1 \leq i \leq m$ such that $H = \sum_i H_i$, Y_i divides H_i, and $\sigma H_i = H_{\sigma^{-1}(i)}$ for each $\sigma \in \Sigma_m$.*

Proof. Observe that if $H', H'' \in A_d^{\Sigma_m}$ and the conclusion of the lemma holds for H' and H'', then it holds for $H' + H''$. Similarly, if $H' \in A_d^{\Sigma_m}$ and the conclusion of the lemma holds for H', then it holds for $H' \cdot H''$ for any $H'' \in A_e^{\Sigma_m}$.

Since A^{Σ_m} is a polynomial algebra in the elementary symmetric polynomials $s_j \in A_j$, $1 \leq j \leq m$, we may therefore suppose $H = s_j$. Thus we fix $1 \leq j \leq m$.

For $I \subseteq \{1, \cdots, m\}$, let $Y_I = \prod_{k \in I} Y_k$. For each $1 \leq i \leq m$, we define

$$H_i = (1/j) \sum_{|I|=j,\ i \in I} Y_I;$$

note that j is invertible in k by assumption. Evidently Y_i divides H_i. It is a simple matter to verify that $\sum_{i=1}^m H_i = s_j$. Let $\sigma \in \Sigma_m$. Since $\sigma Y_I = \prod_{k \in I} Y_{\sigma^{-1}(k)}$, it follows that $\sigma H_i = H_{\sigma^{-1}(i)}$. $\square$

Proposition 20. *Suppose that $m!$ is invertible in k and that $f \in \hat{A}^{\Sigma_m}$ satisfies $f \equiv Y_1 + \cdots + Y_m \pmod{\hat{A}_{\geq 2}}$ with $\xi \in k^\times$. Then we may find series $f_i \in \hat{A}$ for $1 \leq i \leq m$ such that*

(1) $f_i \equiv Y_i \pmod{\hat{A}_{\geq 2}}$,
(2) $Y_i \mid f_i$,
(3) $\sigma f_i = f_{\sigma^{-1}(i)}$ for all $\sigma \in \Sigma_m$.
(4) $f = f_1 + \cdots + f_m$.

Proof. Since Σ_m acts by graded algebra automorphisms, f is invariant if and only if its homogeneous components are. So the proposition follows from the previous lemma. $\square$

4.4. **Exterior and symmetric powers in $\mathscr{C}_{\mathscr{F}}$.** Let $\mathscr{F}(u,v)$ be a formal group law as before. Let $Y_1, Y_2, \ldots$ be indeterminants, and put

$$\textstyle\bigotimes_{\mathscr{F}}^1(Y_1) = Y_1, \quad \bigotimes_{\mathscr{F}}^m(Y_1, \ldots, Y_m) = \mathscr{F}(\bigotimes_{\mathscr{F}}^{m-1}(Y_1, \ldots, Y_{m-1}), Y_m), \quad m \geq 2.$$

Thus $\mathscr{F}(Y_1, Y_2) = \bigotimes_{\mathscr{F}}^2(Y_1, Y_2)$,, and if (V, ϕ) is an object of $\mathscr{C}$, the m-fold power of (V, ϕ) for the tensor product (4.1) is $(V^{\otimes m}, \bigotimes_{\mathscr{F}}^m(\phi))$, where $\bigotimes_{\mathscr{F}}^m(\phi)$ is obtained from $\bigotimes_{\mathscr{F}}^m(Y_1, \ldots, Y_m)$ by specializing $Y_i \mapsto 1^{\otimes(i-1)} \otimes \phi \otimes 1^{\otimes(m-i)}$.

Recall that $\mathscr{F}_a(u,v) = u + v$ denotes the additive formal group law. We have for each $m \geq 1$:

$$\textstyle\bigotimes_{\mathscr{F}_a}^m(Y_1, \ldots, Y_m) = Y_1 + \cdots + Y_m.$$

For any $\mathscr{F}(u,v)$ we have

$$\textstyle\bigotimes_{\mathscr{F}}^m(Y_1, \ldots, Y_m) \equiv Y_1 + \cdots + Y_m \pmod{k[\![Y_1, \ldots, Y_m]\!]_{\geq 2}}.$$

There is a linear representation of the symmetric group Σ_m on $V^{\otimes m}$ given by

$$(4.2) \qquad\qquad \sigma(v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$$

for $\sigma \in \Sigma_m$ and $v_i \in V$, $1 \leq i \leq m$. Any map $X \in \operatorname{End}_{\Sigma_m}(V^{\otimes m})$ induces maps $X' \in \operatorname{End}(\bigwedge^m V)$ and $X'' \in \operatorname{End}(\operatorname{Sym}^m V)$.

Since any 1-dimensional formal group law $\mathscr{F}(u,v)$ is commutative, we have

$$\textstyle\bigotimes_{\mathscr{F}}^m(Y_1, \ldots, Y_m) \in k[\![Y_1, \ldots, Y_m]\!]^{\Sigma_m}.$$

It follows that $\bigotimes_{\mathscr{F}}^m(\phi)$ is a Σ_m-homomorphism and hence induces an endomorphism $\bigwedge_{\mathscr{F}}^m(\phi)$ of the exterior power $\bigwedge^m V$ and an endomorphism $\operatorname{Sym}_{\mathscr{F}}^m(\phi)$ of the symmetric power $\operatorname{Sym}^m V$.

Theorem 21. *Let $\mathscr{F}$ be a formal group law, and (V, ϕ) an object in $\mathscr{C}$. Assume that $m \geq 1$ is an integer such that $m!$ is non-zero in k. Then there are isomorphisms in $\mathscr{C}$*

$$\textstyle(\bigwedge^m V, \bigwedge_{\mathscr{F}_a}^m(\phi)) \simeq (\bigwedge^m V, \bigwedge_{\mathscr{F}}^m(\phi))$$

and

$$(\operatorname{Sym}^m V, \operatorname{Sym}_{\mathscr{F}_a}^m(\phi)) \simeq (\operatorname{Sym}^m V, \operatorname{Sym}_{\mathscr{F}}^m(\phi)),$$

where $\mathscr{F}_a = u + v$ is the additive formal group law.

Proof. If $V = W \oplus W'$ where W and W' are proper subspaces invariant by ϕ, one has the decomposition

$$\textstyle\bigwedge^m V = \bigoplus_{i+j=m} \bigwedge^i W \otimes \bigwedge^j W'$$

into subspaces invariant by both $\bigwedge_{\mathscr{F}}^m(\phi)$ and $\bigwedge_{\mathscr{F}_a}^m(\phi)$.

Let $W_{i,j} = \bigwedge^i W \otimes \bigwedge^j W'$ with $i + j = m$. On $W_{i,j}$, $\bigwedge_{\mathscr{F}}^m(\phi)$ acts as

$$\textstyle\mathscr{F}(\bigwedge_{\mathscr{F}}^i(\phi) \otimes 1, 1 \otimes \bigwedge_{\mathscr{F}}^j(\phi)),$$

which by Theorem 17 is similar to $\bigwedge_{\mathscr{F}}^i(\phi) \otimes 1 + 1 \otimes \bigwedge_{\mathscr{F}}^j(\phi)$. If the result were known for $\bigwedge^i W$ and $\bigwedge^j W'$, one would know this last endomorphism to be similar to $\bigwedge_{\mathscr{F}_a}^i(\phi) \otimes 1 + 1 \otimes \bigwedge_{\mathscr{F}_a}^j(\phi)$, which is precisely the restriction of $\bigwedge_{\mathscr{F}_a}^m(\phi)$ to $W_{i,j}$. We deduce that if the theorem is known for W and W', then it holds for V. Similar remarks hold for the symmetric powers. Thus, we may suppose that ϕ acts as a single Jordan block on V.

Let $B = k[Y_1, \ldots, Y_m]/(Y_1^n, \ldots, Y_m^n)$ where $n = \dim V$; writing y_i for the image of Y_i in B, we identify $(V^{\otimes m}, \bigotimes_{\mathscr{F}}^m(\phi))$ with (B, μ) where μ is multiplication

by $\bigotimes_{\mathscr{F}}^m(y_1,\ldots,y_m)$. Similarly, we identify $(V^{\otimes m}, \bigotimes_{\mathscr{F}}^m(\phi))$ with (B, μ') where μ' is multiplication by $y_1 + \cdots + y_m$. These identification are isomorphisms of Σ_m representations, where Σ_m acts on $V^{\otimes m}$ as in (4.2), and on B by the action induced from that on the polynomial algebra.

The theorem will follow if we find a $k\Sigma_m$-linear automorphism Λ of B such that $\Lambda \circ \mu = \mu' \circ \Lambda$. Since $\bigotimes_{\mathscr{F}}^m(Y_1,\ldots,Y_m) \in k[\![Y_1,\ldots,Y_m]\!]$ satisfies the hypothesis of Proposition 20, we may find power series $f_1,\ldots,f_m$ satisfying conditions 1-4 of that Proposition. In view of conditions 1 and 2, Proposition 16 implies that the rule $\Lambda(y_i) = f_i(y_1,\ldots,y_m)$ defines an algebra automorphism of B. Property 3 implies that Λ intertwines the Σ_m action. Finally, property 4 implies that Λ intertwines μ and μ'. This completes the proof of the theorem. $\qquad\square$

Example 22. In [Fos89], it is proved that the classes of

$$(\textstyle\bigwedge^m V, \bigwedge_{\mathscr{F}}^m(\phi)) \quad \text{and} \quad (\mathrm{Sym}^m V, \mathrm{Sym}_{\mathscr{F}}^m(\phi))$$

in the representation ring are independent of $\mathscr{F}$ provided $\phi^p = 0$, if p is the characteristic of k. Theorem 21 shows that these classes are independent of $\mathscr{F}$ provided that $p > m$ (or $p = 0$ of course).

To see that some hypothesis on the characteristic is essential, suppose that $p = 2$. Recall that we write (V_n, ϕ_n) for the n dimensional Jordan block in $\mathscr{C}$. Then some computer calculations yield:

n	$\mathscr{F}$	$[V_n^{\otimes 2}, \bigotimes_{\mathscr{F}}^2(\phi_n)]$	$[(\bigwedge^2 V_n, \bigwedge_{\mathscr{F}}^2(\phi_n))]$	$[(\mathrm{Sym}^2 V_n, \mathrm{Sym}_{\mathscr{F}}^2(\phi_n))]$
4	$\mathscr{F}_m$	$4J_4$	$J_2 + J_4$	$2J_4 + J_2$
4	$\mathscr{F}_a$	$4J_4$	$2J_3$	$2J_4 + 2J_1$
5	$\mathscr{F}_m$	$2J_8 + 2J_4 + J_1$	$J_7 + J_3$	$J_8 + J_4 + J_3$
5	$\mathscr{F}_a$	$2J_8 + 2J_4 + J_1$	$J_7 + J_3$	$J_8 + J_4 + 3J_1$
6	$\mathscr{F}_m$	$4J_8 + 2J_2$	$J_8 + J_6 + J_1$	$2J_8 + J_4 + J_1$
6	$\mathscr{F}_a$	$4J_8 + 2J_2$	$2J_7 + J_1$	$2J_8 + J_2 + 3J_1$
7	$\mathscr{F}_m$	$6J_8 + J_1$	$2J_8 + J_5$	$3J_8 + J_4$
7	$\mathscr{F}_a$	$6J_8 + J_1$	$3J_7$	$3J_8 + 4J_1$

5. ADJOINT JORDAN BLOCKS FOR CLASSICAL GROUPS

Let $\Omega = \Omega(V)$ be a classical group in good characteristic (see §2.2).

Proposition 23. *There is a series $\varepsilon(t) \in tk[\![t]\!] \setminus t^2 k[\![t]\!]$ such that $X \mapsto 1 + \varepsilon(X)$ defines a Ω-equivariant isomorphism of varieties $\mathcal{N} \to \mathcal{U}$.*

Proof. See [Hum95, §6.20]. $\qquad\square$

Note that we identify $\mathcal{N}$ and $\mathcal{U}$ with subvarieties of $\mathrm{End}_k(V)$. In case CG1, *any* series $\varepsilon(t) \in tk[\![t]\!] \setminus t^2 k[\![t]\!]$ defines an isomorphism as in the proposition. In cases CG2, CG3, one may take the Cayley transform series

$$\varepsilon(t) = (1-t)(1+t)^{-1} - 1 = 2\sum_{i=1}^{\infty}(-1)^i t^i.$$

We may also use in these cases the Artin-Hasse exponential series [Ser88, V§16]. This follows from [M02, 7.4]; see also [Pro01].

Theorem 24. *Let $\theta : \mathcal{N} \to \mathcal{U}$ be an Ω-equivariant isomorphism of varieties. Then $\mathrm{Ad}(\theta(X))$ and $\mathrm{ad}(X)$ have the same partition.*

Proof. We may suppose that θ is determined by a formal series ε as in the previous proposition. Using the identifications of the adjoint modules in Lemma 1, the action of $\mathrm{ad}(X)$ is determined by the additive formal group law $\mathscr{F}_a$. We have:

$$\mathrm{ad}(X) = \begin{cases} X \otimes 1 + 1 \otimes X^{\vee} & \text{in case CG1} \\ \bigwedge^2_{\mathscr{F}_a}(X) & \text{in case CG2} \\ \mathrm{Sym}^2_{\mathscr{F}_a}(X) & \text{in case CG3.} \end{cases}$$

Similarly, we have

$$\mathrm{Ad}(\theta(X)) = \begin{cases} (1 + \varepsilon(X)) \otimes (1 + \varepsilon(X^{\vee})) & \text{in case CG1} \\ 1 + \bigwedge^2_{\mathscr{F}_m}(\varepsilon(X)) & \text{in case CG2} \\ 1 + \mathrm{Sym}^2_{\mathscr{F}_m}(\varepsilon(X)) & \text{in case CG3.} \end{cases}$$

By Lemma 15, $\mathrm{Ad}(\theta(X))$ is similar to

$$\begin{aligned} (1 + X) \otimes (1 + X^{\vee}) & \quad \text{in case CG1,} \\ 1 + \bigwedge^2_{\mathscr{F}_m}(X) & \quad \text{in case CG2, and} \\ 1 + \mathrm{Sym}^2_{\mathscr{F}_m}(X) & \quad \text{in case CG3.} \end{aligned}$$

The result now follows from Theorem 17 and Theorem 21. □

Remark 25. In bad characteristic, the adjoint partition of the regular nilpotent class can already differ from that of the regular unipotent class. Let $p = 2$ and let Ω be the symplectic group $\mathrm{Sp}_{2n}(k)$. If $u \in G$ is a regular unipotent element, and $X \in \mathfrak{o}$ is a regular nilpotent element, them from example 22 one obtains the following partitions:

n	partition of $\mathrm{Ad}(u)$	partition of $\mathrm{ad}(X)$
2	$(4^2, 2)$	$(4^2, 1^2)$
3	$(8^3, 4)$	$(8^2, 2, 1^3)$

Note that the dimensions of $\ker(\mathrm{Ad}(u) - 1)$ and $\ker \mathrm{ad}(X)$ agree with the tables in [Hes79, p. 179].

If again $p = 2$, Ω is the orthogonal group $\mathrm{O}_7(k)$, and u, X are regular, then the same example shows that the partition of $\mathrm{Ad}(u)$ is $(8^2, 5)$ while that of $\mathrm{ad}(X)$ is (7^3).

6. ADJOINT JORDAN BLOCKS FOR G_2

In this section, we consider a group G with root system of type G_2. Denote by ϖ_1, ϖ_2 the fundamental dominant weights in the character group of a fixed maximal torus T of G with respect to a choice of simple roots α_1, α_2, where α_1 is *short*. Suppose that $p > 3$.

Since $p \neq 2$, the Weyl module $V = V(\varpi_1)$ for G is irreducible of dimension 7. Moreover, G leaves invariant a non-degenerate quadratic form on V; this defines a closed embedding $\rho : G \to H = SO(V)$. Recall that H has a root system of type B_3.

6.1. **Simple root vectors and root subgroups.** Fix non-zero elements $x_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$, $i = 1, 2$.

Lemma 26. *For suitable choices of a maximal torus of H containing T and a system γ_i ($i = 1, 2, 3$) of simple roots for H, we have*

(1) $d\rho(x_{\alpha_1}) = y_{\gamma_1} + y_{\gamma_3} \in \mathfrak{h}_{\gamma_1} + \mathfrak{h}_{\gamma_3}$
(2) $d\rho(x_{\alpha_2}) = y_{\gamma_2} \in \mathfrak{h}_{\gamma_2}$

with $y_{\gamma_i} \neq 0$ for $i = 1, 2, 3$.

Proof. The weights of the G-module V are $\pm f_1, \pm f_2, \pm f_3$ and 0, where $f_1 = \varpi_1$, $f_2 = \varpi_1 - \alpha_1$, and $f_3 = \varpi_1 - \alpha_1 - \alpha_2$.

Since V is a simple, restricted G-module, a theorem of Curtis implies that it is also simple as a $\mathfrak{g}$-module. Let $\mathfrak{n}$ be the Lie algebra of the unipotent radical of the Borel sub-group of G corresponding to our choice of positive roots. Since V is simple for $\mathfrak{g}$, we must have $V^{\mathfrak{n}} = V_{f_1}$. Since $\mathfrak{n}$ is generated as a Lie algebra by x_{α_1} and x_{α_2}, it follows that for any weight $\gamma \neq f_1$ of V, the restriction of either x_{α_1} or x_{α_2} to V_λ is non-zero.

Thus, we see that

(6.1) $x_{\alpha_1} V_{f_2} = V_{f_1}, \quad x_{\alpha_1} V_0 = V_{f_3}, \quad x_{\alpha_1} V_{-f_3} = V_0, \quad x_{\alpha_1} V_{-f_1} = V_{-f_2}$

and x_{α_1} acts trivially on all other weight spaces of V. Similarly,

(6.2) $$x_{\alpha_2} V_{f_3} = V_{f_2}, \quad x_{\alpha_2} V_{-f_2} = V_{-f_3}.$$

and x_{α_2} acts trivially on all other weight spaces of V.

Choose a non-zero weight vector $e_{\pm i} \in V_{\pm f_i}$ and $e_0 \in V_0$; this yields a basis of V. Since T-weight spaces V_λ and V_μ are orthogonal under the invariant form unless $\lambda + \mu = 0$, it follows from [Bor91, §23.4] that the subgroup S of H which acts diagonally in this basis is a maximal torus (containing T). If $\varepsilon_i \in X^*(S)$ is the S-weight of e_i, then the $\varepsilon_1, \varepsilon_2, \varepsilon_3$ form a **Z**-basis for $X^*(S)$. Moreover, $\gamma_1 = \varepsilon_1 - \varepsilon_2, \gamma_2 = \varepsilon_2 - \varepsilon_3, \gamma_3 = \varepsilon_3$ is a system of simple roots for H. The roots of H are: $R_H = \{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq 3\} \cup \{\pm e_3\}$.

Write $x_{\alpha_i} = \sum_{\alpha \in R_H} z_{i,\alpha} \in \bigoplus_{\alpha \in R_H} \mathfrak{h}_\alpha$ for $i = 1, 2$ (since x_{α_i} is nilpotent, it has no component in $\mathfrak{h}_0$). To prove (1), it suffices to show that $z_{1,\alpha} = 0$ whenver $\alpha \neq \gamma_1, \gamma_3$; this follows from (6.1). To prove (2), it suffices to show that $z_{2,\alpha} = 0$ whenver $\alpha \neq \gamma_2$; this follows from (6.2).   □

Let $U_{\alpha_i} \leq G$ ($i = 1, 2$) be the simple root subgroups. We claim that U_{α_i} is the image of the map $\mathscr{X}_{\alpha_i} : \mathbf{G}_a \to G$ satisfying $\rho(\mathscr{X}_{\alpha_i}(s)) = \exp(s d\rho(x_{\alpha_i}))$. One can see the claim in several ways. One is to realize G as a Chevalley group (see [Ste68]). Alternatively, one may realize G as the automorphisms of an 8 dimensional Cayley algebra $\mathbf{O}$; the module V is the space orthogonal to the identity element of $\mathbf{O}$. Since the x_{α_i} act as derivations of $\mathbf{O}$, their exponentials (which are defined thanks to our assumption on p) yield automorphisms of $\mathbf{O}$. The images of these exponentials yield 1 dimensional unipotent subgroups of G normalized by T which are then necessarily the desired root subgroups.

Since $p \neq 2$, it is well-known that the simple root subgroups $V_{\gamma_i} \leq H$ are the images of the maps $\mathscr{Y}_{\gamma_i} : \mathbf{G}_a \to H$ given by $\mathscr{Y}_{\gamma_i}(s) = \exp(s y_{\gamma_i})$.

Lemma 27. *We have $\rho(\mathscr{X}_{\alpha_1}(s)) = \mathscr{Y}_{\gamma_1}(s)\mathscr{Y}_{\gamma_3}(s)$, and $\rho(\mathscr{X}_{\alpha_2}(s)) = \mathscr{Y}_{\gamma_2}(s)$.*

Proof. Since $[y_{\gamma_1}, y_{\gamma_3}] = 0$, we have $\exp(s d\rho(x_{\alpha_1})) = \exp(s y_{\gamma_1}) \exp(s y_{\gamma_3})$ and the formula for α_1 follows. The formula for α_2 is simpler.   □

Proposition 28. *Let $X \in \mathfrak{g}$ be regular nilpotent, and $u \in G$ be regular unipotent. Then X and u both act as a single Jordan block on $V = V(\varpi_1)$.*

Proof. Recall that the regular nilpotent and regular unipotent classes for H act as a single Jordan block on the natural module of H. Also recall that for any reductive group L, if $z_\alpha \in \mathfrak{g}_\alpha$ are non-zero root vectors for a system of simple roots, then $\sum_\alpha z_\alpha$ is regular nilpotent. If the images of $\mathscr{X}_\alpha : \mathbf{G}_a \to L$ are the simple root subgroups and $a_\alpha \in k^\times$, then $\prod_\alpha \mathscr{X}_\alpha(a_\alpha)$ is regular unipotent (one may take the product in any fixed order).

In particular, $x_\alpha + x_\beta$ is regular nilpotent in $\mathfrak{g}$, and by Lemma 26 we see that $d\rho(x_\alpha + x_\beta) = y_{\gamma_1} + y_{\gamma_2} + y_{\gamma_3}$ is regular nilpotent in $\mathfrak{h}$. Thus X acts as a single Jordan block as claimed.

Similarly, $v = \mathscr{X}_{\alpha_1}(1)\mathscr{X}_{\alpha_2}(1)$ is regular unipotent in G, and by Lemma 27,

$$\rho(v) = \mathscr{Y}_{\gamma_1}(1)\mathscr{Y}_{\gamma_3}(1)\mathscr{Y}_{\gamma_2}(1)$$

is regular unipotent in H. Thus u also acts as a single Jordan block. $\square$

6.2. **The adjoint representation.** Since $p \neq 3$, the Weyl module $V(\varpi_2)$ is irreducible; it is isomorphic with the adjoint representation of G.

If W is a G-module, write $\operatorname{ch} W = \sum_\mu \dim W_\mu \cdot e^\mu$ for its character. For a dominant weight μ, let $\chi(\mu) = \operatorname{ch} V(\mu)$, where $V(\mu)$ is the Weyl module with high weight μ. Brauer's formula [Hum80, ex. 24.4(9)] yields

$$\operatorname{ch} V(\varpi_1) \otimes V(\varpi_1) = \chi(0) + \chi(\varpi_1) + \chi(\varpi_2) + \chi(2\varpi_1).$$

Since $\dim V(2\varpi_1) = 27$ (by Weyl's formula) the character of $\operatorname{Sym}^2 V(\varpi_1)$ must be $\chi(0) + \chi(2\varpi_1)$, and Lemma 2 yields $\operatorname{ch} \bigwedge^2 V(\varpi_1) = \chi(\varpi_1) + \chi(\varpi_2)$ (despite the application of Lemma 2, these character formulas are valid in all characteristics). Since $p > 3$, the Weyl modules $V(\varpi_1)$, $V(\varpi_2)$ are simple and we deduce from [Jan87, Prop. II.2.14] that $0 = \operatorname{Ext}^1_G(V(\varpi_1), V(\varpi_2)) = \operatorname{Ext}^1_G(V(\varpi_2), V(\varpi_1))$. Thus:

Proposition 29. $\bigwedge^2 V(\varpi_1) \simeq V(\varpi_1) \oplus V(\varpi_2)$ *is semisimple.*

Theorem 30. *Let G be a group with root system G_2 and suppose $p > 3$. Let $\theta : \mathcal{N} \to \mathcal{U}$ be a G-equivariant isomorphism. For each $X \in \mathcal{N}$, $\operatorname{ad} X$ and $\operatorname{Ad} \theta X$ have the same partition.*

Proof. Since $p > 3$, the only nilpotent class which is not p-nilpotent is the regular class when $p = 5$; see for example [M02, Example §6.2]. Let $X \in \mathcal{N}$ be regular nilpotent; hence also $u = \theta(X)$ is regular unipotent. Let $\rho : G \to H$ be as before. Proposition 28 implies that $d\rho(X)$ and $\rho(u)$ are respectively regular nilpotent and regular unipotent in H. Thus Theorem 24 and Lemma 1 together imply that the partition of X acting on $\bigwedge^2 V(\varpi_1)$ is the same as that of u.

Since both $\operatorname{ad} X$ and $\operatorname{Ad} u$ stabilize the direct sum decomposition in Proposition 29, and since Proposition 28 shows that both act as a single Jordan block on the direct summand $V(\varpi_1)$, the theorem follows. $\square$

6.3. **Adjoint partitions.** We now use the techniques just developed to find the adjoint partition for each nilpotent class. We begin with some lemmas.

Let L be a reductive group with derived group $L' \simeq \operatorname{SL}_2(k)$ and suppose that $\dim L = 4$. Let Z be a 1 dimensional central torus in L, and let W be a 4 dimensional simple L module. Fix a maximal torus S of L'; the S weights on W must be

$\pm 3, \pm 1$. Since W is simple, Z acts with a single weight ζ on W; suppose that the integer ζ is non-0 $\pmod{p}$.

Lemma 31. *If $0 \neq v_1 \in W_1$ and $0 \neq v_{-1} \in W_{-1}$, then the L orbit of $v = v_1 + v_{-1}$ has dimension 4.*

Proof. Let $e_{\pm} \in \mathrm{Lie}(L)_{\pm 2}$ be non-0 weight vectors for S. Since W is a restricted simple module for L', a theorem of Curtis implies that W is simple for $\mathrm{Lie}(L')$; it follows that the vectors $e_+ v_1 \in W_3$ and $e_- v_{-1} \in W_{-3}$ are non-zero.

Choose $0 \neq h \in \mathrm{Lie}(S)$ and $0 \neq z \in \mathrm{Lie}(Z)$. Then (z, h, e_+, e_-) is a basis for $\mathrm{Lie}(L)$. Let $x = az + bh + ce_+ + de_-$ with $a, b, c, d \in k$, and suppose that $x.v = 0$.

Since $ce_+ v_1$ is the component of $x.v$ in W_3, we deduce that $c = 0$. Similarly, $d = 0$. Thus $x = az + bh$, and $x.v = (a\zeta + b)v_1 + (a\zeta - b)v_{-1}$. It follows that $0 = a\zeta + b = a\zeta - b$; since the characteristic is not 2 and ζ is non-0 in k, one has $a = b = 0$.

To finish the proof, write $H = \mathrm{Stab}_L(v)$. Then $\mathrm{Lie}(H)$ is the stabilizer in $\mathrm{Lie}(L)$ of v, hence is 0 by the above remarks. Thus $\dim H = 0$, so that $\dim L/H = 4$ as desired. $\square$

Remark 32. The proof of the lemma shows that the orbit map $\pi : L \to Lv$ is separable.

Lemma 33. *Let $\mathfrak{p} = \mathfrak{p}_{\alpha_1} = \mathfrak{g}_{-\alpha_1} + \mathfrak{b}$, where $\mathfrak{b}$ is the Borel subalgebra of $\mathfrak{g}$ corresponding to the choice of positive roots determined by α_1, α_2. Let $\mathfrak{u}$ be the Lie algebra of the unipotent radical U of the parabolic subgroup P with $\mathrm{Lie}(P) = \mathfrak{p}$. Let*

$$x_{\alpha_1 + \alpha_2} = [x_{\alpha_1}, x_{\alpha_2}] \in \mathfrak{u}_{\alpha_1 + \alpha_2} \quad and \quad x_{2\alpha_1 + \alpha_2} = [x_{\alpha_1}, x_{\alpha_1 + \alpha_2}] \in \mathfrak{u}_{2\alpha_1 + \alpha_2}.$$

Then

(1) *$x = x_{\alpha_1 + \alpha_2} + x_{2\alpha_1 + \alpha_2}$ is a representative for the dense (Richardson) orbit of P on $\mathfrak{u}$.*

(2) *Put $y_{\gamma_1 + \gamma_2} = [y_{\gamma_1}, y_{\gamma_2}], y_{\gamma_2 + \gamma_3} = [y_{\gamma_2}, y_{\gamma_3}]$, and $y_{\gamma_2 + 2\gamma_3} = [y_{\gamma_3}, y_{\gamma_2 + \gamma_3}] \in \mathfrak{h}$. Then $d\rho(x) = y_{\gamma_1 + \gamma_2} + y_{\gamma_2 + \gamma_3} + y_{\gamma_2 + 2\gamma_3}$.*

Proof. Let $L = P/U$; thus L is reductive and its derived group L' is simple of type A_1. The image of the co-character $\alpha_1^{\vee}$ is a maximal torus of L'. Now, L acts on $\overline{\mathfrak{u}} = \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$ and in fact has a dense orbit on this space. Moreover, if $\overline{y} \in \overline{\mathfrak{u}}$ is a representative for the dense L orbit, then any $y \in \mathfrak{u}$ with image $\overline{y}$ lies in the dense (Richardson) P-orbit on $\mathfrak{u}$.

We have $\mathfrak{u} = \mathfrak{g}_{\alpha_2} + \mathfrak{g}_{\alpha_1 + \alpha_2} + \mathfrak{g}_{2\alpha_1 + \alpha_2} + \mathfrak{g}_{3\alpha_1 + \alpha_2} + \mathfrak{g}_{3\alpha_1 + 2\alpha_2}$, and $[\mathfrak{u}, \mathfrak{u}] = \mathfrak{g}_{3\alpha_1 + 2\alpha_2}$. Since $p > 3$, the latter assertion follows from the commutator formulas in [Ste68, Theorem 1]; these commutator formulas also show that $x_{\alpha_1 + \alpha_2}$ and $x_{2\alpha_1 + \alpha_2}$ are non-0.

The weights of $\alpha_1^{\vee}$ on $\overline{\mathfrak{u}}$ are thus $\pm 3, \pm 1$; it follows that $\overline{\mathfrak{u}}$ affords a restricted simple 4-dimensional representation for L'. Consider the co-character $\gamma = \alpha_1^{\vee} + 2\alpha_2^{\vee}$. Since $\langle \alpha_1, \gamma \rangle = 0$, the image Z of γ is a central torus in L. Since $\langle \alpha_2, \gamma \rangle = 1$, γ acts with weight 1 on $\overline{\mathfrak{u}}$. Let $\overline{x}$ be the image of x in $\overline{\mathfrak{u}}$. Applying Lemma 31, we see that the L orbit through $\overline{x}$ has dimension 4, and (1) follows.

For (2) note first that $d\rho(x_{\alpha_1 + \alpha_2}) = y_{\gamma_1 + \gamma_2} + y_{\gamma_2 + \gamma_3}$. Since

$$d\rho(x_{2\alpha_1 + \alpha_2}) = [y_{\gamma_1}, y_{\gamma_2 + \gamma_3}] + [y_{\gamma_3}, y_{\gamma_1 + \gamma_2}] + [y_{\gamma_3}, y_{\gamma_2 + \gamma_3}]$$

an application of the Jacobi identity shows that $d\rho(x_{2\alpha_1 + \alpha_2}) = [y_{\gamma_3}, y_{\gamma_2 + \gamma_3}]$ and (2) follows. $\square$

Lemma 34. *Let $\mathscr{F}$ be the additive formal group law, and let $R = R_{\mathscr{F}}$ be the its represen-tation ring (see §4.2). For $p > 3$, we have the following identities in R:*

(1) $\bigwedge^2(2J_3 + J_1) = J_5 + 5J_3 + J_1$.

(2) $\bigwedge^2(J_3 + 2J_2) = 2J_4 + 2J_3 + 2J_2 + J_1$.

(3) $\bigwedge^2(2J_2 + 3J_1) = J_3 + 6J_2 + 6J_1$.

(4) *If $p = 5$ or $p \geq 11$, $\bigwedge^2(J_7) = J_{11} + J_3$. If $p = 7$, $\bigwedge^2(J_7) = 2J_7$.*

Proof. The first three assertions are an immediate consequence of the following facts which the reader may easily check: $\bigwedge^2 J_2 = J_1$, $\bigwedge^2 J_3 = J_3$, $J_3 \otimes J_3 = J_5 + J_3 + J_1$, $J_3 \otimes J_2 = J_4 + J_2$, and $J_2 \otimes J_2 = J_3 + J_1$.

For $p \geq 7$, (4) follows by considering the tilting $\mathrm{SL}_2(k)$-module $\bigwedge^2 L(6)$, and applying Proposition 5. When $p = 5$ one checks (4) by hand (or by computer). $\square$

Theorem 35. *The partitions of the non-0 nilpotent classes of $\mathfrak{g}$ on $V(\varpi_1)$ and on the adjoint representation are as follows (recall that $p > 3$):*

(6.3)

nilpotent orbit	partition on $V(\varpi_1)$	adjoint partition	
A_1	$(2^2, 1^3)$	$(3, 2^4, 1^3)$	
$\widetilde{A_1}$	$(3, 2^2)$	$(4^2, 3, 1^3)$	
$G_2(a_1)$	$(3^2, 1)$	$(5, 3^3)$	
G_2	(7)	$(11, 3)$	if $p \geq 11$ or $p = 5$
		(7^2)	if $p = 7$.

Proof. The adjoint partitions may be obtained from the partitions on $V(\varpi_1)$ by applying Lemma 34 and Proposition 29.

To obtained the partitions on $V = V(\varpi_1)$ note the following. x_{α_2} is a repre-sentative for the class A_1, and x_{α_1} is a representative for the class $\widetilde{A_1}$; one now deduces the partitions from (6.1) and (6.2). The partition of the regular class G_2 is obtained from Proposition 28.

Lemma 33 gives a representative x for the class $G_2(a_1)$; moreover, it shows that $d\rho(x) = a + b + c$ with $0 \neq a \in \mathfrak{h}_{\gamma_1 + \gamma_2}$, $0 \neq b \in \mathfrak{h}_{\gamma_2 + \gamma_3}$ and $0 \neq c \in \mathfrak{h}_{\gamma_2 + 2\gamma_3}$. If a, b, c are *any* non-0 elements of the indicated weight spaces of $\mathfrak{h}$, a direct calculation shows that the partition of $a + b + c$ on V is $(3, 3, 1)$. $\square$

Remark 36. According to Theorem 30 the partitions in (6.3) are also valid for the unipotent classes. Lawther [Law95] has computed adjoint Jordan blocks for unipo-tent elements in exceptional groups, and our results agree with his in this case.

The descriptions in §6.1 imply that the partitions of the unipotent classes on $V(\varpi_1)$ are given by (6.3); this again agrees with Lawther's calculations.

7. Adjoint Jordan blocks in characteristic 0

In this section, we work over an algebraically closed field of characteristic 0; in order to emphasize the characteristic, we will call this field F rather than k. G will be a simple algebraic group over F, and $\mathfrak{g} = \mathrm{Lie}(G)$ will be its Lie algebra.

Fix a distinguished nilpotent element $X \in \mathfrak{g}$, and choose an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ containing X, and let $\phi : \mathrm{SL}_2(F) \to G$ be the homomorphism such that the image of $d\phi$ is the span of this $\mathfrak{sl}_2$-triple. We may suppose that the cocharacter $\nu(t) = \phi(\mathrm{diag}(t, t^{-1})) : F^\times \to G$ satisfies $\mathrm{Ad}\,\nu(t)X = t^2 X$ ($t \in F^\times$).

Let $\mathfrak{g}(i) = \mathfrak{g}(i; \nu)$ denote the i-th weight space for $\mathrm{Ad}\,\nu$. Since X is distinguished, [Car93, 5.7.6] shows that $\mathfrak{g}(2i + 1) = 0$ for $i \in \mathbf{Z}$.

Let $n \geq 0$ be the largest integer for which $\mathfrak{g}(2n) \neq 0$. Fix $\zeta \in F^\times$ a root of unity of order $n + 1$, and let $s = \nu(\zeta^{1/2}) \in G$ (choose either square root of ζ in F). The eigenvalues of $\mathrm{Ad}(s)$ on $\mathfrak{g}$ are integeral powers of ζ; the ζ^i eigenspace of $\mathrm{Ad}(s)$ is

$$V_i = \mathfrak{g}(2i) \oplus \mathfrak{g}(2i - 2n - 2) \quad \text{for } 0 \leq i \leq n.$$

Let $M \in \mathfrak{g}(-2n)$, and put $S = X + M \in V_1$.

Lemma 37 (Springer)**.** [Spr74, Lemmas 9.3, 9.5, and 9.6]
  (1) $\mathrm{Ad}(s)$ *stabilizes* $C_{\mathfrak{g}}(S)$ *and has no non-0 fixed points in* $C_{\mathfrak{g}}(S)$.
  (2) V_1 *contains a regular semisimple element if and only if* S *is regular semisimple for some choice of* $M \in \mathfrak{g}(-2n)$.
  (3) *If* $\dim \mathfrak{g}(4) = \dim \mathfrak{g}(2) - 1$, *then* V_1 *contains a regular semisimple element.*

Let $\mathfrak{n} = \bigoplus_{i>0} \mathfrak{g}(i)$ and $\mathfrak{n}^- = \bigoplus_{i<0} \mathfrak{g}(i)$. The following lemma may be found in [Kos59, Lemma 6.4A] for the case where X is regular; we have essentially copied Kostant's argument.

Lemma 38. *Let* $\rho : \mathfrak{g} \to \mathfrak{n}$ *be the projection with kernel* $\mathfrak{n}^- \oplus \mathfrak{g}(0)$. *Then* $\rho \circ \mathrm{Ad}(s) = \mathrm{Ad}(s) \circ \rho$, *the restriction* $\rho_{|\mathfrak{c}}$ *to* $\mathfrak{c} = \mathfrak{c}_{\mathfrak{g}}(S)$ *takes values in* $\mathfrak{c}_{\mathfrak{g}}(X)$, *and* $\rho_{|\mathfrak{c}}$ *is injective.*

Proof. The $\mathrm{Ad}(s)$-equivariance of ρ is immediate. Let $Y \in \mathfrak{c}$ and write

$$Y = U + A + V \in \mathfrak{n}^- \oplus \mathfrak{g}(0) \oplus \mathfrak{n}.$$

We claim that $V = \rho(Y) \in \mathfrak{c}_{\mathfrak{g}}(X)$. Indeed, note that

$$[M, Y] = [M, A] + [M, V] \in \mathfrak{n}^- + \mathfrak{g}(0),$$

so that $\rho([M, Y]) = 0$. Since $Y \in \mathfrak{c}$, one has $0 = \rho([S, Y]) = \rho([X, Y] + [M, Y]) = \rho([X, Y])$. Since $[X, U] \in \mathfrak{n}^- + \mathfrak{g}(0)$, we have

$$0 = \rho([X, Y]) = \rho([X, A]) + \rho([X, V]) \in \mathfrak{g}(2) \oplus \bigoplus_{i \geq 4} \mathfrak{g}(i).$$

It follows that $[X, A] = \rho([X, A]) = 0$ and $[X, V] = \rho([X, V]) = 0$; this proves that $V \in \mathfrak{c}_{\mathfrak{g}}(X)$.

Since X is distinguished, $\mathrm{ad}(X) : \mathfrak{g}(0) \to \mathfrak{g}(2)$ is bijective [Car93, 5.7.4]; it follows that $\mathfrak{c}_{\mathfrak{g}}(X) \subset \mathfrak{n}$ (view $\mathfrak{g}$ as a representation for the $\mathfrak{sl}_2(F)$-triple to see that $\mathfrak{c}_{\mathfrak{g}}(X) \subset \mathfrak{g}(0) + \mathfrak{n}$). Since $[X, A] = 0$, we have $A = 0$. Thus we see that $Y \in \mathfrak{c}$ has the form $Y = U + V \in \mathfrak{n}^- \oplus \mathfrak{n}^+$.

Now suppose that $Y \in \ker \rho \cap \mathfrak{c}$; by the previous remarks we have $Y \in \mathfrak{n}^-$. Considering homogeneous components of Y, one sees that $[M, Y] = 0$. It follows that $0 = [S, Y] = [M, Y] + [X, Y] = [X, Y]$ i.e. $Y \in \mathfrak{c}_{\mathfrak{g}}(X)$. Since $\mathfrak{c}_{\mathfrak{g}}(X) \cap \mathfrak{n}^- = 0$, we have $Y = 0$ and the lemma is proved. $\square$

Let $\mathfrak{h} = \mathrm{Lie}(T)$ be the Lie algebra of a maximal torus of G with Weyl group $W = N_G(T)/T$. Then W acts by graded algebra automorphisms on the coordinate ring $F[\mathfrak{h}]$. Since W is a reflection group, a theorem of Chevalley says that the algebra of invariants $F[\mathfrak{h}]^W$ is generated by $r = \dim \mathfrak{h}$ algebraically independent homogeneous polynomials $\phi_1, \ldots, \phi_r$ whose degrees are uniquely determined, up to order. The exponents of W are the numbers $e_i = \deg \phi_i - 1$, $1 \leq i \leq r$.

The following theorem generalizes the result proved by Kostant [Kos59] in the case where X is regular nilpotent.

Theorem 39. *Suppose that V_1 contains a regular semisimple element of $\mathfrak{g}$. Let $e_1, \ldots, e_r$ be the exponents of the Weyl group of G, and for $1 \le i \le r$, let $1 \le f_i \le n$ be the unique quantity satisfying $f_i \equiv -e_i \pmod{n}$. Then $\operatorname{ad} X$ has a Jordan block of size $2f_i + 1$ for each $1 \le i \le r$.*

Proof. Since V_1 contains a regular semisimple element, Lemma 37(2) shows that we may choose $M \in \mathfrak{g}(-2n)$ such that $S = M + X$ is regular semisimple. Then the centralizer $\mathfrak{h}' = \mathfrak{c}_{\mathfrak{g}}(S)$ is the Lie algebra of a maximal torus T' of G.

Now part (1) of Lemma 37 shows that $s = \nu(\zeta^{1/2})$ determines an element w of the Weyl group $N_G(T')/T'$. Moreover, w has a regular eigenvector in $\mathfrak{h}'$, namely S; it is thus a *regular* Weyl group element; see [Spr74, §4] for these notions. In particular, [Spr74, Theorem 4.2] shows that the eigenvalues of w (i.e. of $\operatorname{Ad}(s)$) on $\mathfrak{h}'$ are $\zeta^{-e_i} = \zeta^{f_i}$ for $1 \le i \le r$.

Let $\mathfrak{s}$ be the $\mathfrak{sl}_2(X)$-triple containing X; as $\mathfrak{s}$-module, we may write

$$\mathfrak{g} = L(2\lambda_1) \oplus L(2\lambda_2) \oplus \cdots \oplus L(2\lambda_t).$$

Here $L(\mu)$ is the simple $\mathfrak{s}$-module of dimension $\mu + 1$; since S is distinguished, any simple $\mathfrak{s}$-summand of $\mathfrak{g}$ has even highest weight.

We may chose $v_i \in \mathfrak{g}(2\lambda_i)$ for $1 \le i \le t$ such that $v_1, \ldots, v_t$ is a basis for $\mathfrak{c}_{\mathfrak{g}}(X)$. It follows that the eigenvalues of $\operatorname{Ad}(s)$ on $\mathfrak{c}_{\mathfrak{g}}(X)$ are ζ^{λ_j} for $1 \le j \le t$.

By Lemma 38 the map $\rho : \mathfrak{c}_{\mathfrak{g}}(S) \to \mathfrak{c}_{\mathfrak{g}}(X)$ is injective and $\operatorname{Ad}(s)$-equivariant. Reordering the λ_i if necessary, we may therefore suppose that $\zeta^{\lambda_j} = \zeta^{f_j}$ for $1 \le j \le r$. For each j we have $1 \le \lambda_j \le n$ and $1 \le f_j \le n$, whence $\lambda_j = f_j$ and the theorem follows. $\square$

Remark 40. The hypothesis that V_1 contains a regular semisimple element is satisfied if the condition in (3) of Lemma 37 holds. In [Spr74], Springer lists the nilpotent classes for which (3) holds (though there is class for type G_2 in that list that doesn't belong).

Springer's results yield a map from nilpotent orbits for which V_1 contains a regular semisimple element to conjugacy classes in the Weyl group. Kazhdan and Lusztig [KL88] have defined a map from all nilpotent orbits to conjugacy classes in the Weyl group; their map agrees with Springer's when his is defined.

For a nilpotent $X \in \mathfrak{g}$, let $\sigma(X)$ denote the corresponding conjugacy class in W under the Kazhdan-Lusztig map. In the case where X is distinguished and $\sigma(X)$ consists of regular elements of W in Springer's sense, it appears from empirical observation that Theorem 39 is still valid, i.e. one obtains r of the Jordan block sizes of $\operatorname{ad}(X)$ from the eigenvalues of $\sigma(X)$. I have not so far been able to give a proof which explains this phenomenon.

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