SEMISIMPLICITY OF EXTERIOR POWERS OF SEMISIMPLE REPRESENTATIONS OF GROUPS

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ABSTRACT. This paper answers a question posed by Jean-Pierre Serre; namely, a proof is given that if V is a semisimple finite dimensional representation of a group G over a field K of characteristic $p > 0$, and $m(\dim_K V - m) < p$, then $\bigwedge^m V$ is again a semisimple representation of G.

1. INTRODUCTION

An important feature of the representation theory of a group G over a field K is the following: given representations (modules) V and W of the group algebra KG, the tensor product $V \otimes_K W$ is again a representation of KG. In this paper, all representations will be assumed finite dimensional over K. When the field K has characteristic zero, the notion of semisimplicity is stable under the tensor product; namely, if V and W are semisimple KG modules then $V \otimes_K W$ is again semisimple ([Che54], p. 88). In particular, when K has characteristic 0 and V is semisimple, the modules $V^{\otimes n}$, $\bigwedge^n V$ (the exterior power of V), and $S^n V$ (the symmetric power of V) are semisimple for all $n \geq 0$.

If the characteristic of K is $p > 0$, the tensor product is not as well behaved. Nevertheless, J.-P. Serre has established the following condition for semisimplicity:

Theorem 1. *(Serre,* [Ser94] *Théorème 1) Assume that K has characteristic $p > 0$ and that V_i, $1 \leq i \leq r$, are semisimple representations of G. If $\sum_{i=1}^{r}(\dim_K V_i - 1) < p$, then $V_1 \otimes V_2 \otimes \cdots \otimes V_r$ is again semisimple.*

Serre also proves the following:

Theorem 2. *(Serre,* [Ser94] *Théorème 2) Assume that K has characteristic $p > 0$ and that V is a semisimple representation of G of dimension n. If $n \leq \dfrac{p+3}{2}$, then $\bigwedge^2 V$ is semisimple.*

Serre finally poses the following generalization of the previous result:

Problem 1. (Serre, [Ser94]) Let V be a semisimple representation of G of dimension n. Let $m > 0$, and assume that $m(n-m) < p$. Is $\bigwedge^m V$ semisimple?

Theorem 2 provides an affirmative answer to this problem for $m = 2$. During the initial work on this paper, the author was also aware of unpublished work of Serre which gave an affirmative answer for $m = 3$.

Date: September 14, 1999.

Some time after the publication of [Ser94], Serre generalized this question a bit more, as follows:

Problem 2. (Serre, unpublished) Let $\mathbf{V} = (V_1, V_2, \ldots, V_s)$ be a sequence of semisimple representations of KG, and let $\mathbf{m} = (m_1, \ldots, m_s)$ where the m_i are integers satisfying $1 \leq m_i \leq \dim_K V_i = n_i$ for each i. Put

$$\bigwedge^{\mathbf{m}} \mathbf{V} = \bigwedge^{m_1} V_1 \otimes_K \ldots \otimes_K \bigwedge^{m_s} V_s.$$

If $\sum_i m_i(n_i - m_i) < p$, is $\bigwedge^{\mathbf{m}} \mathbf{V}$ semisimple?

We introduce some notations for convenience; let $\mathcal{M}$ denote the class of all finite sequences $\mathbf{V} = (V_1, \ldots, V_s)$ for $s \geq 1$ of semisimple KG modules. We say that $\mathbf{V}$ has type s if $\mathbf{V}$ involves s semisimple KG modules. Given $\mathbf{V} \in \mathcal{M}$ of type s, let $\mathcal{N}(\mathbf{V})$ denote the set of all integral s-tuples $\mathbf{m} = (m_1, \ldots, m_s)$ such that $0 \leq m_i \leq \dim_K V_i = n_i$ and $\sum_{i=1}^{s} m_i(n_i - m_i) < p$. Given $\mathbf{m} \in \mathcal{N}(\mathbf{V})$, we may form the module $\bigwedge^{\mathbf{m}} \mathbf{V}$ as above. In this paper, we prove:

Theorem 3. *Problem 2 has an affirmative answer. More precisely, for every* $\mathbf{V} \in \mathcal{M}$*, and for every* $\mathbf{m} \in \mathcal{N}(\mathbf{V})$*,* $\bigwedge^{\mathbf{m}} \mathbf{V}$ *is semisimple.*

Notice that the theorem implies Theorems 1 and 2, and it implies that Problem 1 has an affirmative answer.

The chronology of the solution is as follows. The author first proved that Problem 1 has an affirmative answer when V is an absolutely simple G module. Upon completion of this work, the author learned that J.-P. Serre had posed Problem 2 and, at roughly the same time, verified its validity through a quite different argument involving the notion of "G-completely reducible subgroups" of a reductive algebraic group G as described in his June 1997 lectures at the Isaac Newton Institute in Cambridge. Upon Serre's suggestion, the original techniques of the author (those used in answering Problem 1 in the absolutely simple case) were considered for Problem 2; this re-examination produced the proof of Theorem 3 given here.

The result of this paper fits into a family of results relating the dimension of a representation to its semisimplicity. The results of [Ser94] have already been pointed out. When the group G is a reductive algebraic group over K, Jantzen [Jan96] proved that any rational representation V with $\dim_K V \leq p$ is automatically semisimple; he proves the same for the finite groups of $\mathbb{F}_q$ rational points $G(\mathbb{F}_q)$ – although in this case one must exclude factors of type A_1 from G.

When G is quasisimple of rank r, the author has generalized Jantzen's result; namely he has shown [McN98] that whenever $\dim_K V \leq r.p$, V is semisimple. This work was extended in [McNb] to cover the finite groups $G(\mathbb{F}_q)$; however, there are a few more exceptions than in Jantzen's situation.

Our proof of Theorem 3 follows closely that of Theorem 1 given in [Ser94]. The basic idea is to prove the Theorem first in case G is a simply connected, connected, simple algebraic group; in this setting the argument is handled via the linkage principle combined with weight combinatorics. See §3 for the argument in this case. The result for general groups is obtained through a saturation process. In §4, we adapt the saturation procedure of Serre to obtain the desired result.

I would like to thank Jean-Pierre Serre for some valuable suggestions.

2. Preliminaries and Reductions

2.1. **Notations.** Tensor products, exterior powers, and symmetric powers are always taken over the fixed ground field K unless otherwise noted. The notation $V^{\otimes m}$ means the m-fold tensor product of V with itself. When V is a vector space, the dual vector space is denoted V^*.

2.2. **Some multilinear algebra.** If G is a group, and L is any 1 dimensional KG module, any L-valued G-equivariant non-degenerate bilinear pairing β between KG modules V and W induces a canonically defined KG isomorphism $\tilde{\beta} : V \xrightarrow{\sim} W^* \otimes_K L$. Indeed, one can canonically identify $W^* \otimes_K L$ with $\mathrm{Hom}_K(W, L)$; then $\tilde{\beta}(v)(w) = \beta(v, w)$ for all $v \in V$ and $w \in W$.

Note that in the above situation, one must have $\dim_K V = \dim_K W$; call this dimension n. For any $1 \leq m \leq n$, one has an induced G-equivariant bilinear pairing $\beta : \bigwedge^m V \times \bigwedge^m W \to L^{\otimes m}$ determined by the rule $\beta(v_1 \wedge \cdots \wedge v_m, w_1 \wedge \cdots \wedge w_m) = \det(\beta(v_s, w_t))_{s,t}$ where the determinant is computed in the tensor algebra of L. In particular, one has a KG isomorphism

$$\widetilde{\beta} : \bigwedge^m V \to (\bigwedge^m W)^* \otimes_K L^{\otimes m}. \tag{2.2.a}$$

2.2.1. For V any KG module of dimension n, write $\det(V)$ for the 1 dimensional representation $\bigwedge^n V$. For each $1 \leq m \leq n$, the pairing $\mu : \bigwedge^m V \times \bigwedge^{n-m} V \to \det(V)$ given by multiplication in the exterior algebra of V is G-equivariant and non-degenerate, hence there is a KG isomorphism

$$\widetilde{\mu} : \bigwedge^m V \to (\bigwedge^{n-m} V)^* \otimes_K \det(V).$$

2.3. **An Example.** Fix $m \geq 2$ be an integer. In this section, let K be an algebraically closed field of characteristic $p > m$, with $p \equiv -1 \pmod{m}$. Consider the group $G = \mathrm{SL}_2(K)$, and take for V the "natural" 2-dimensional G module. When $d \geq 1$, the space $S^d V$ of homogeneous polynomials of degree d in a basis of V affords a representation of G which we denote $V(d)$. This representation satisfies $\dim_K V(d) = d + 1$, and in the notation of [Jan87, II.2], one has that $V(d) = H^0(d)$ is the *induced* module with highest weight d. In particular, $V(d)$ has simple socle $L(d)$. Finally, $V(d)$ is simple if and only if $d < p$, and Steinberg's tensor product theorem 3.3.1 shows that

$$L(d) \simeq L(d_0) \otimes L(pd_1) = L(d_0) \otimes L(d_1)^{[1]}$$

if $d = d_0 + pd_1$ with $0 \leq d_0 \leq p - 1$ and $d_1 \geq 0$.

2.3.1. *With G and m as above, there is a simple G-module W, such that $m(\dim_K W - m) = p + 1$ and so that $\bigwedge^m W$ is not semisimple.*

Proof. Let $k = m^2 - m + 1$; by hypothesis, $d = \dfrac{p+k}{m}$ is an integer. Put $W = L(d)$, the simple G module with highest weight $d = \dfrac{p+k}{m}$. Since $p > \dfrac{p+k}{m}$, this simple module coincides with the module $V(d)$ and hence

(2.3.b) $$n = \dim_K W = \frac{p+k+m}{m}.$$

It follows that

(2.3.c) $$m(n-m) = p + k + m - m^2 = p + 1,$$

as desired.

The arguments given below in the proof of 3.5.3 for rank 1 show that $p+1$ is the highest weight of $\bigwedge^m W$. Since $W = H^0(d)$ is an induced module, $W^{\otimes m}$ has a good filtration (i.e. a filtration by induced modules) according to a well-known theorem of Donkin, Wang, Mathieu (see [Mat90]).

Since $p > m$, $\bigwedge^m W$ is a summand of the module $W^{\otimes m}$, hence by [Jan87, Prop II.4.16(b)], $\bigwedge^m W$ has a good filtration. Since $p+1$ is the highest weight of this module, the induced module $H^0(p+1)$ must appear as a filtration factor. By Steinberg's tensor product theorem, the socle of $H^0(p+1)$ is 4 dimensional. Since $p \geq 3$, $p+2 = \dim_K H^0(p+1)$ is at least 5, so this induced module is not semisimple and the proposition follows. □

Remark 2.1. The above generalizes the example given in [Ser94, Appendice, Remarque (1)]. One can even argue as in *loc. cit.*; one observes that, for $a \geq 0$, $V(a)$ may be identified with the space of homogeneous polynomials of degree a in the variables x and y where x and y are a weight-space basis for V. Hence one may define

$$\theta : \textstyle\bigwedge^m V(d) \to V(p+1) \text{ via } \theta(f_1 \wedge \cdots \wedge f_m) = \det\left(\dfrac{\partial^{m-1} f_i}{\partial x^{j-1} \partial y^{m-j}}\right)_{1 \leq i,j \leq m}.$$

One can show that θ is surjective and G-linear.

2.4. **Some important reductions.** We observe the following trivial but useful fact:

2.4.1. *Let $1 \leq m < n$ be positive integers. If $m(n-m) < p$, then $m < p$ and $n < p$.*

This implies in particular that if $\mathbf{V} \in \mathcal{M}$ and $\mathcal{N}(\mathbf{V})$ is non empty, then $\dim V_i < p$ for each i. Next, we observe:

2.4.2. *Theorem 3 holds provided it is verified when the field K is algebraically closed.*

Proof. Let $\mathbf{V} \in \mathcal{M}$ and $\mathbf{m} \in \mathcal{N}(\mathbf{V})$. If $K' \supseteq K$ is a field extension, one has easily

$$(\textstyle\bigwedge_K^{\mathbf{m}} \mathbf{V}) \otimes_K K' \simeq \bigwedge_{K'}^{\mathbf{m}} (\mathbf{V} \otimes_K K');$$

(where $\mathbf{V} \otimes_K K' = (V_1 \otimes_K K', \ldots, V_s \otimes_K K')$).

In particular, if $\bigwedge_{K'}^{\mathbf{m}}(\mathbf{V} \otimes_K K')$ is semisimple, then also $\bigwedge_K^{\mathbf{m}} \mathbf{V}$ is semisimple. It only remains to see that $V_j \otimes_K K'$ is semisimple for each j. Since $\dim_K V_j < p$, the argument invoked in [Ser94] Lemme 1 applies; Serre's argument shows that the center of $\mathrm{End}_G(V_j)$ is a separable field extension of K, hence that V_j is absolutely semisimple. □

We assume from now on that K is algebraically closed.

2.4.3. *Theorem 3 holds provided it is verified for those* $\mathbf{V} \in \mathcal{M}$ *for which all* V_i *are simple.*

Proof. Let $\mathcal{S}$ denote the set of all finite sequences of positive integers, and give $\mathcal{S}$ the following partial ordering. For $\alpha = (\alpha_1, \dots, \alpha_s), \beta = (\beta_1, \dots, \beta_t) \in \mathcal{S}$, we say that $\alpha \leq \beta$ provided that $s \geq t$ and $\sum_{i=1}^s \alpha_i = \sum_{j=1}^t \beta_j$.

Observe that each $\alpha \in \mathcal{S}$ lies over a minimal element in this order; namely, if $a = \sum \alpha_i$, then the tuple $\beta = (\underbrace{1, 1, \dots, 1}_{a})$ is the unique minimal element of $\mathcal{S}$ that satisfies $\beta \leq \alpha$.

If $\mathbf{V} \in \mathcal{M}$ is of type s, put

$$l = l(\mathbf{V}) = (\text{len}(V_1), \dots, \text{len}(V_s)),$$

where $\text{len}(V_j)$ denotes the length (number of composition factors) of the KG module V_j.

Consider $\mathbf{V} \in \mathcal{M}$, with corresponding $l = l(\mathbf{V}) \in \mathcal{S}$. Observe that all of the modules in $\mathbf{V}$ are simple if and only if l is minimal in $\mathcal{S}$; since there is nothing to prove in that case, assume that l is not minimal, and that the theorem is known for any $\mathbf{W} \in \mathcal{M}$ for which $l(\mathbf{W}) < l$. Without loss of generality, assume that $V_1 \simeq V_1' \oplus V_1''$ where V_1' and V_1'' are non-zero KG modules. Let d, d', d'' denote the dimensions of V_1, V_1', V_1''.

For $\mathbf{m} \in \mathcal{N}(\mathbf{V})$ one has

$$\bigwedge\nolimits^{\mathbf{m}} \mathbf{V} \simeq \bigoplus_{i+j=m_1} \bigwedge\nolimits^{\mathbf{n(i,j)}} \mathbf{W}$$

where $\mathbf{W} = (V_1', V_1'', V_2, \dots, V_s)$ and $\mathbf{n}(\mathbf{i}, \mathbf{j}) = (i, j, m_2, m_3, \dots, m_s)$ for $0 \leq j \leq m_1$. Note that $\bigwedge^{\mathbf{n(i,j)}} \mathbf{W} = 0$ unless $1 \leq i \leq d'$ and $1 \leq j \leq d''$.

It is straightforward to see that $l(\mathbf{W}) < l$; the result follows by induction provided we argue that $\mathbf{n}(\mathbf{i}, \mathbf{j}) \in \mathcal{N}(\mathbf{W})$ whenever $\bigwedge^{\mathbf{n(i,j)}} \mathbf{W} \neq 0$. The required assertion follows immediately from the inequality

$$m_1(d - m_1) = i(d' - i) + j(d'' - j) + i(d'' - j) + j(d' - i) \geq i(d' - i) + j(d'' - j)$$

□

For $\mathbf{V} \in \mathcal{M}$, put $\tilde{\mathcal{N}}(\mathbf{V}) = \{\mathbf{m} \in \mathcal{N}(\mathbf{V}) : 1 \leq m_i \leq \dim_K V_i/2 \text{ for each } i\}$.

2.4.4. *Theorem 3 holds provided it is verified for every* $\mathbf{V} \in \mathcal{M}$ *and* $\mathbf{m} \in \tilde{\mathcal{N}}(\mathbf{V})$.

Proof. A KG module W is semisimple if and only if the dual module W^* is semisimple; similarly, W is semisimple if and only if $W \otimes L$ is semisimple for any 1 dimensional representation L.

Let $\mathbf{V} \in \mathcal{M}$, and $\mathbf{m} \in \mathcal{N}(\mathbf{V})$. Suppose $\mathbf{V}$ has type s, and consider $J \subseteq \{1, 2, \dots, s\}$. Let $\mathbf{m}'$ be the s-tuple such that $m_i' = n_i - m_i$ for $i \in J$, while $m_i' = m_i$ otherwise. Define $\mathbf{V}'$ by the rule $V_i' = V_i^*$ for $i \in J$, and $V_i' = V_i$ otherwise. Evidently one has $\mathbf{m}' \in \tilde{\mathcal{N}}(\mathbf{V}')$. It follows from (2.2.1) that $\bigwedge^{\mathbf{m}} \mathbf{V} \simeq \bigwedge^{\mathbf{m}'} \mathbf{V}' \otimes_K L$ for some 1 dimensional KG module L; since $\bigwedge^{\mathbf{m}'} \mathbf{V}'$ is semisimple by assumption, the semisimplicity of $\bigwedge^{\mathbf{m}} \mathbf{V}$ is obtained. □

A KG-module V will be called tensor decomposable if $V \simeq X \otimes_K Y$ for KG modules X and Y with $\dim_K X > 1$ and $\dim_K Y > 1$; otherwise, V is tensor indecomposable.

Of course, any module of prime dimension is tensor indecomposable. A straightforward induction shows that any KG module may be written in at least one way as a tensor product of finitely many tensor indecomposable modules.

2.4.5. *Theorem 3 holds provided it is verified for those* $\mathbf{V} \in \mathcal{M}$ *for which each* V_i *is tensor indecomposable.*

Proof. Assume the conclusion of Theorem 3 is valid for those $\mathbf{V} \in \mathcal{M}$ for which each V_i is tensor indecomposable, and let $\mathbf{V} \in \mathcal{M}$ be arbitrary. According to 2.4.4, we must show that $\bigwedge^{\mathbf{m}}\mathbf{V}$ is semisimple for each $\mathbf{m} \in \tilde{\mathcal{N}}(\mathbf{V})$. Let $j \geq 0$ be the number of i such that V_i is tensor decomposable; if $j = 0$ there is nothing to do, so suppose $j > 0$ and proceed by induction on j.

Without loss of generality we may suppose that V_1 is tensor decomposable, say

$$V_1 \simeq X_1 \otimes_K X_2 \otimes_K \cdots \otimes_K X_r$$

with X_i tensor indecomposable and $r \geq 2$. Fix $\mathbf{m} \in \tilde{\mathcal{N}}(\mathbf{V})$ and put

$$\mathbf{W} = (\underbrace{X_1, \ldots, X_1}_{m_1}, \underbrace{X_2, \ldots, X_2}_{m_1}, \ldots, \underbrace{X_r, \ldots, X_r}_{m_1}, V_2, \ldots, V_s),$$

$$\mathbf{n} = (\underbrace{1, \ldots, 1}_{rm_1}, m_2, m_3, \ldots, m_s).$$

Evidently $\bigwedge^{\mathbf{m}}\mathbf{V}$ is a quotient of $\bigwedge^{\mathbf{n}}\mathbf{W}$. The list $\mathbf{W}$ has only $j-1$ tensor decomposable modules, so the result follows by induction provided $\mathbf{n} \in \mathcal{N}(\mathbf{W})$.

Let $x_i = \dim_K X_i$ for $1 \leq i \leq r$, and let $d = x_1 \cdot x_2 \cdots x_r = \dim_K V_1$. Observe that

$$\sum_i n_i(\dim_K W_i - n_i) = m_1(x_1 + x_2 + \cdots + x_r - r) + \sum_{j \geq 2} m_j(\dim_K V_j - m_j).$$

Since $\mathbf{m} \in \tilde{\mathcal{N}}(\mathbf{V})$, one has $m_1 \leq d/2$ which implies that $m_1(d - m_1) \geq \frac{1}{2}m_1 d$. So, it suffices to prove that $m_1(x_1 + x_2 + \cdots + x_r - r) \leq \frac{m_1 d}{2}$, or equivalently that

$$\frac{x_1 x_2 \cdots x_r}{2} \geq x_1 + x_2 + \cdots + x_r - r. \tag{2.4.d}$$

Since $x_i \geq 2$ for each i, we may write $x_i = 2 + y_i$ for a non-negative y_i; thus

$$\begin{aligned} \frac{x_1 \cdots x_r}{2} = \frac{1}{2}(2 + y_1) \cdots (2 + y_r) &\geq \frac{1}{2}(2^r + 2y_1 + 2y_2 + \cdots + 2y_r) \\ &= 2^{r-1} + x_1 + x_2 + \cdots + x_r - 2r. \end{aligned}$$

As $r \geq 2$, one has $2^{r-1} \geq r$ and the inequality (2.4.d) is verified. ∎

3. THE PROOF IN THE CASE OF A LINEAR ALGEBRAIC GROUP.

Let G be a linear algebraic K-group, where K is an algebraically closed field of characteristic $p > 0$. Assume that

$$[G : G^0] \not\equiv 0 \pmod{p},$$

where G^0 denotes the identity component of G. Throughout this section, we fix $\mathbf{V} \in \mathcal{M}$ and we assume that $\mathbf{V}$ is rational, i.e. that each V_i is a *rational* representation of G (i.e. that the homomorphism $G \to \mathrm{GL}(V_i)$ is a morphism of algebraic groups).

3.1. **Main result in the algebraic case.** In this section, we prove the following statement:

3.1.1. *The conclusion of Theorem 3 is valid in case G is an algebraic group for which $[G : G^0]$ is prime to p and $\mathbf{V}$ is rational.*

3.2. **Reduction to the quasisimple case.** Since the finite group G/G^0 has order prime to p, all of its representations in characteristic p are semisimple. Since G is an extension of G/G^0 by the connected algebraic group G^0, it follows from [Ser94, §3.4,Lemma 5] that $\bigwedge^{\mathbf{m}}\mathbf{V}$ is semisimple for G if and only if it is semisimple for G^0. Thus we may and shall assume that G is connected.

Let $N \lhd G$ denote the kernel of the homomorphism $G \to \prod_{i=1}^{s} \mathrm{GL}(V_i)$. Since $\bigoplus_{i=1}^{s} V_i$ is a semisimple KG module, it is well known that G/N is reductive. Since $\bigwedge^{\mathbf{m}}\mathbf{V}$ is semisimple for G if and only if it is semisimple for G/N, we may replace G by G/N and hence assume that G is connected and reductive.

Now, for connected reductive G, there is (see e.g. [Spr98, Ch. 9]) an isogeny

$$\prod_i G_i \times T \to G$$

where $\prod_i G_i$ is a finite direct product of simply connected, quasisimple algebraic groups, and T is a torus. It follows from [Jan96, §3] that a G module W is semisimple if and only if W is a semisimple module for each G_i (and for T, which is trivial).

Hence, we may assume that G is simply connected, and quasisimple.

3.3. **The simply connected, quasisimple case.** Let T be a maximal torus of G, let X denote the character group of T, and let Φ denote the set of roots of T. Choose a Borel subgroup B of G containing T; this choice determines a system of positive roots. Pick a system of simple roots Δ and for $\alpha \in \Delta$, let $\varpi_\alpha \in X$ denote the corresponding fundamental dominant weight.

A weight $\lambda = \sum_{\alpha\in\Delta} n_\alpha \varpi_\alpha \in X$ is called *dominant* if $n_\alpha \geq 0$ for every α, and a dominant weight λ is called *restricted* if $n_\alpha < p$ for every α. The subset of dominant weights is denoted X^+ and the subset of restricted weights is denoted X_1.

For each dominant weight, there is a corresponding simple rational G module denoted $L(\lambda)$; furthermore, any simple rational G module is isomorphic to a unique $L(\lambda)$.

For a dominant weight λ, we have a (finite) p-adic expansion

$$\lambda = \lambda_0 + p\lambda_1 + p^2\lambda_2 + \cdots$$

with each λ_i restricted. The importance of representing weights in this way is the following result:

3.3.1. *(Steinberg's Theorem) For λ as above, there is a G-module isomorphism*

$$L(\lambda) \simeq \bigotimes_{i\geq 0} L(\lambda_i)^{[i]}$$

where $W^{[d]}$ denotes the d-fold Frobenius twist of a rational G module W.

As a consequence, note that if $\lambda = p^i\lambda'$ for $\lambda' \in X_1$, then for any m

$$\bigwedge^m L(\lambda) \simeq \bigwedge^m \left(L(\lambda')^{[i]}\right) \simeq \left(\bigwedge^m L(\lambda')\right)^{[i]}. \tag{3.3.e}$$

According to 2.4.3 we may assume that each V_i is simple; thus there are dominant weights λ_i such that $V_i \simeq L(\lambda_i)$. By 2.4.5 we need consider only tensor indecomposable simple modules, so we may assume, in view of Steinberg's Theorem, that $\lambda_i = p^{N_i}\mu_i$ where μ_i is restricted and $N_i \geq 0$.

We will prove the following

3.3.2. *Assume that $N_i = 0$, i.e. that $\lambda_i \in X_1$, for each i. Then $\bigwedge^{\mathbf{m}}\mathbf{V}$ is semisimple and each composition factor has restricted highest weight.*

For the moment, though, let us observe that 3.3.2 suffices to prove 3.1.1. Indeed, if $s = 1$, (3.3.e) permits one to reduce to the case $\lambda_1 \in X_1$, so we may suppose $s > 1$ and proceed by induction on s.

Without loss of generality, we may suppose that $\lambda_1, \dots, \lambda_t \in X_1$ and $\lambda_{t+1}, \dots, \lambda_s \in pX$. For any $\mathbf{m} \in \mathcal{N}(\mathbf{V})$, one has

$$\bigwedge^{\mathbf{m}}\mathbf{V} \simeq \bigwedge^{\mathbf{m}'}\mathbf{V}' \otimes (\bigwedge^{\mathbf{m}''}\mathbf{V}'')^{[1]}$$

where $\mathbf{m}' = (m_1, \dots, m_t)$, $\mathbf{m}'' = (m_{t+1}, \dots, m_s)$, $\mathbf{V}' = (V_1, \dots, V_t)$, and $\mathbf{V}'' = (V_{t+1}^{[-1]}, \dots, V_s^{[-1]})$. If $t = 0$, it suffices to prove that $\bigwedge^{\mathbf{m}''}\mathbf{V}''$ is semisimple; working by induction on the minimal value of N_i, one may reduce to the case $t > 0$.

This being done, 3.3.2 shows that $\bigwedge^{\mathbf{m}'}\mathbf{V}'$ is semisimple and all its composition factors have restricted highest weight. By induction on s, the module $\bigwedge^{\mathbf{m}''}\mathbf{V}''$ is semisimple, and (3.3.e) shows that all of its composition factors have highest weight in pX. Steinberg's Theorem now shows that $\bigwedge^{\mathbf{m}}\mathbf{V}$ is itself semisimple.

In the remainder of this section, we finish the verification of 3.1.1 by proving 3.3.2.

3.4. **The linkage principle.** Let $\mathbf{C} \subset X^+$ denote the closure of the lowest dominant alcove for the dot action of the affine Weyl group W_p. Then $\mathbf{C}$ is a fundamental domain for this action of W_p. The dominant weights in this set can be described as follows:

$$\mathbf{C}^+ = \mathbf{C} \cap X^+ = \{\lambda \in X^+ : \langle \lambda + \rho, \beta^\vee \rangle \leq p\}$$

where β is the highest short root in Φ. Denote by $\hat{\mathbf{C}}$ the set $\mathbf{C}^+ \cup \{0\}$.

The following gives for us a useful criteria for membership in $\hat{\mathbf{C}}$.

3.4.1. [Ser94, Prop. 3, Prop. 5] *Let* $\lambda \in X_1$. *If* $\dim_K L(\lambda) < p$ *then* $\lambda = 0$ *or* $\langle \lambda + \rho, \beta^\vee \rangle < p$; *equivalently,* $\lambda \in \hat{\mathbf{C}}$.

The *linkage principle* (see [Jan87, II.6]) implies the following:

3.4.2. [Jan87, II.6.13,II.5.10] *If* $\lambda \in \hat{\mathbf{C}}$, *then* $\dim_K L(\lambda)$ *is equal to the value* $d(\lambda)$ *of the Weyl degree formula:*

$$d(\lambda) = \prod_{\alpha > 0} \frac{\langle \lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}.$$

Let the *character* of a G module M be the element of $\mathbb{Z}[X]$ given by $\mathrm{ch}(M) = \sum_{\mu \in X} \dim_K M_\mu e^\mu$, where M_μ denotes the μ weight space of M and the e^μ are basis elements for $\mathbb{Z}[X]$. For $\lambda \in X^+$, let $L_\mathbb{Q}(\lambda)$ denote the simple module with highest weight λ for the split simple $\mathbb{Q}$ Lie algebra $\mathfrak{g}_\mathbb{Q}$ with root system Φ; we denote $\mathrm{ch}(L_\mathbb{Q}(\lambda))$ by $\chi(\lambda)$ (the character of a $\mathfrak{g}_\mathbb{Q}$ module is defined via the weights of a maximal toral subalgebra on the module). For $m \geq 1$, it follows from the representation theory of $\mathfrak{g}_\mathbb{Q}$ that there is a finite subset $\mathcal{H}(\lambda, m) \subset X^+$ such that

(3.4.f) $$\mathrm{ch}(\textstyle\bigwedge^m L_\mathbb{Q}(\lambda)) = \sum_{\mu \in \mathcal{H}(\lambda, m)} m_\mu \chi(\mu)$$

for suitable multiplicities $m_\mu > 0$.

For $\lambda \in \hat{\mathbf{C}}$, [Jan87, II.6.13] actually shows that $\mathrm{ch}(L(\lambda)) = \mathrm{ch}(L_\mathbb{Q}(\lambda))$; it follows from [Bou72, VIII §7, exerc. 11] that:

3.4.3. *For* $\lambda \in \hat{\mathbf{C}}$ *and* $m \geq 1$, *ch*$(\bigwedge^m L(\lambda)) =$ *ch*$\bigwedge^m L_\mathbb{Q}(\lambda)$. *In particular, any weight* ν *of* $\bigwedge^m L(\lambda)$ *satisfies* $\nu \leq \mu$ *for some* $\mu \in \mathcal{H}(\lambda, m)$.

The significance of the linkage principle for semisimplicity is demonstrated by:

3.4.4. [Jan87, II.6.17,II.2.12 (1)] *If* $\lambda, \mu \in \hat{\mathbf{C}}$, *then* $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) = 0$.

After one notes $\hat{\mathbf{C}} \subset X_1$, 3.4.4 has the immediate consequence:

3.4.5. *Suppose that* $\langle \nu + \rho, \beta^\vee \rangle \leq p$ *for each weight* ν *of the* G *module* M. *Then* M *is semisimple and each composition factor of* M *has restricted highest weight.*

3.5. **Weight considerations.** Let us say that an *admissible pair* (λ, m) consists in $\lambda \in X^+$ and $1 \leq m \leq d(\lambda)/2$ such that

(3.5.g) $$\langle \nu + \rho, \beta^\vee \rangle \leq m(d(\lambda) - m)$$

for each weight $\nu \in \mathcal{H}(\lambda, m)$.

Remark 3.1. Let (λ, m) be a pair as above. Since each weight $\nu \in \mathcal{H}(\lambda, m)$ satisfies $\nu < m\lambda$, one knows that (λ, m) is admissible in case $\langle m\lambda + \rho, \beta^\vee \rangle \leq m(d(\lambda) - m)$.

Define a partial order relation on X^+ by the following simple rule: say that $\mu \to \lambda$ provided $\lambda - \mu \in X^+$.

3.5.1. *Let* $c > 0$ *be a real number. Suppose that* $d(\mu) \geq c\langle \mu + \rho, \beta^\vee \rangle$. *If* $\mu \to \lambda$, *then* $d(\lambda) \geq c\langle \lambda + \rho, \beta^\vee \rangle$.

Proof. For any positive root α, we have

$$\langle \lambda + \rho, \alpha^\vee \rangle - \langle \mu + \rho, \alpha^\vee \rangle = \langle \lambda - \mu, \alpha^\vee \rangle \geq 0$$

since $\lambda - \mu \in X^+$. Inspecting the Weyl degree formula, it is then clear that

$$d(\lambda) \geq d(\mu) \cdot \frac{\langle \lambda + \rho, \beta^\vee \rangle}{\langle \mu + \rho, \beta^\vee \rangle} \geq c\langle \mu + \rho, \beta^\vee \rangle \frac{\langle \lambda + \rho, \beta^\vee \rangle}{\langle \mu + \rho, \beta^\vee \rangle} \geq c\langle \lambda + \rho, \beta^\vee \rangle,$$

as desired. □

Remark 3.2. The numberings of the fundamental dominant weights used in the following result, and throughout this paper, are those used in the tables in [Bou72].

3.5.2. *Suppose the rank of the root system is at least 2, and let $\lambda \in X^+$. Then*

$$d(\lambda) \geq 2\langle \lambda + \rho, \beta^\vee \rangle, \tag{3.5.h}$$

unless λ is among the set of weights $\mathcal{E} = \mathcal{E}(\Phi)$ indicated in the following table:

Φ	$\mathcal{E}$
A_2	$\varpi_1, \varpi_2, 2\varpi_1, 2\varpi_2$
A_3	$\varpi_1, \varpi_2, \varpi_3$
$A_r, r \geq 4$	ϖ_1, ϖ_r
B_2	ϖ_1, ϖ_2
B_3	ϖ_1, ϖ_3
$B_r, r \geq 4$	ϖ_1

Φ	$\mathcal{E}$
$C_r, r \geq 3$	ϖ_1
$D_r, r \geq 5$	ϖ_1
D_4	$\varpi_i, i = 1, 3, 4$
G_2	ϖ_1, ϖ_2

Remark 3.3. In [McN98], the author proves a slightly stronger estimate of this sort; namely, that $\dim_K L(\lambda) \geq r\langle \lambda + \rho, \beta^\vee \rangle$ for almost all λ. However, the list of exceptional λ is larger, and the techniques used are somewhat more unwieldy than the argument given here due to the fact that $\dim_K L(\lambda) \neq d(\lambda)$ in general.

Sketch of proof. Initially, let λ be a fundamental dominant weight. In [Bou72] Table 2, the value of $d(\lambda)$ is recorded for each indecomposable root system and each fundamental dominant weight. A straightforward computation of $\langle \lambda + \rho, \beta^\vee \rangle$ in each case yields immediately the assertion that λ satisfies (3.5.h) unless λ is among the specified exceptions.

In view of 3.5.2, the assertion holds for $\Phi = E_6, E_7, E_8, F_4$. Furthermore, it suffices to prove that (3.5.h) is valid for $\lambda = \mu_1 + \mu_2$ for all possible fundamental weights μ_1 and μ_2 which fail to satisfy (3.5.h); in most cases this is true. We list below those λ for which one must check (3.5.h), and we indicate the value of $d(\lambda)$ for each such λ; it is then straightforward to verify (3.5.h). For unbounded rank, we provide references for the dimension assertions; in low rank the calculation of $d(\lambda)$ is straightforward (note that some labor may be avoided in case $\Phi = A_2, B_2, G_2$, as $d(\lambda)$ is given in closed form in [Hum80, §24.3] for those Φ).

- $\Phi = A_r$, $r \geq 4$: $\lambda = 2\varpi_1, 2\varpi_2, \varpi_1 + \varpi_r$. According to [McN98, Props. 4.2.2,4.6.8] one has (for $r \geq 1$):

$$d(2\varpi_1) = d(2\varpi_r) = \binom{r+2}{2} \text{ and } d(\varpi_1 + \varpi_r) = r(r+2).$$

- $\Phi = A_3$: $\lambda = 2\varpi_2$. $d(2\varpi_2) = 20$.
- $\Phi = A_2$: $\lambda = 3\varpi_1, 3\varpi_2, 2\varpi_1 + 2\varpi_2$. $d(\lambda) = 10, 10, 27$ respectively.
- $\Phi = B_r$, $r \geq 4$; $\Phi = C_r$, $r \geq 3$; $\Phi = D_r$, $r \geq 5$: $\lambda = 2\varpi_1$. According to [McN98, Props. 4.2.2,4.7.3,4.8.1] one has:

$$d(2\varpi_1) = \binom{2r+2}{2} - \epsilon \text{ where } \epsilon = 1, 0, 1 \text{ for } \Phi = B_r, C_r, D_r \text{ respectively}$$

- $\Phi = B_r$: $r = 2, 3$, $\lambda = 2\varpi_1, 2\varpi_r, \varpi_1 + \varpi_r$. When $r = 2$, $d(\lambda) = 14, 10, 16$ respectively. When $r = 3$, $d(\lambda) = 27, 35, 48$ respectively.
- $\Phi = D_4$: $\lambda = \varpi_2, \varpi_i + \varpi_j$ for all pairs $i, j \in \{1, 3, 4\}$. $d(\varpi_2) = 28$, $d(\varpi_i + \varpi_j) = 56$ for $i \neq j$. $d(2\varpi_i) = 35$ for each $i \in \{1, 3, 4\}$.
- $\Phi = G_2$: $\lambda = 2\varpi_1, 2\varpi_2, \varpi_1 + \varpi_2$. $d(\lambda) = 27, 77, 64$ respectively.

□

Our goal is to list those pairs (λ, m) which fail to be admissible. Towards this end, we introduce the subset $\mathcal{E}' \subset \mathcal{E}$ as follows.

Φ	$\mathcal{E}'$	Φ	$\mathcal{E}'$
A_r $(r \geq 2)$	ϖ_1, ϖ_r	D_r $(r \geq 5)$	ϖ_1
B_r $(r \geq 2)$	ϖ_1	D_4	$\varpi_1, \varpi_3, \varpi_4$
C_r $(r \geq 2)$	ϖ_1		

3.5.3. *Let $\lambda \in X^+$ and let $1 \leq m \leq d(\lambda)/2$. Then (λ, m) is admissible unless one of the following holds:*

(1) $\Phi = A_1$
(2) $m = 1$, $\lambda \in \mathcal{E}'$ *and* Φ *is one of* A_r $(r \geq 2)$, B_r $(r \geq 2)$, *or* C_r $(r \geq 2)$.
(3) $m = 2$, $\lambda = \varpi_1$ *and* $\Phi = C_2$.

If (λ, m) *is not admissible, then*

$$\langle \mu + \rho, \beta^\vee \rangle \leq m(d(\lambda) - m) + 1. \tag{3.5.i}$$

for any $\mu \in \mathcal{H}(\lambda, m)$.

Proof. We first verify (3.5.i) when Φ has rank 1. In this case X may be identified with $\mathbb{Z}$, and X^+ with $\mathbb{Z}_{\geq 0}$. For $a \in X$, one has $d(a) = a + 1$; if $1 \leq m \leq a + 1$, the $\mathfrak{g}_\mathbb{Q} = \mathfrak{sl}_2(\mathbb{Q})$ module $\bigwedge^m L_\mathbb{Q}(a)$ has highest weight given by

$$b = a + (a-2) + \cdots + (a - 2(m-1)) = ma - 2\sum_{j=1}^{m-1} j = m(a + 1 - m)$$

whence $b + 1 = \langle b + \rho, \beta^\vee \rangle = m(a + 1 - m) + 1$. The remaining assertions of 3.5.i are straightforward to verify for the indicated inadmissible pairs; we omit the details.

For the remainder of the proof, we assume that the rank of Φ is at least 2; assume first that $\lambda \notin \mathcal{E}$, i.e. that λ satisfies (3.5.h). As noted in Remark 3.1, one may test admissibility by considering $\nu = m\lambda$; for such a λ one has

$$\langle m\lambda + \rho, \beta^\vee \rangle \leq m\langle \lambda + \rho, \beta^\vee \rangle \leq \frac{m \cdot d(\lambda)}{2} \leq m(d(\lambda) - m).$$

Thus (λ, m) is admissible.

Now let $\lambda \in \mathcal{E} \setminus \mathcal{E}'$; in this case one deduces the admissibility of (λ, m) for each $1 \le m \le d(\lambda)/2$ via Remark 3.1 and the following data:

Φ	λ	$d(\lambda)$	$\langle m\lambda + \rho, \beta^\vee \rangle$
A_2	$2\varpi_1, 2\varpi_2$	6	$2m+2$
A_3	ϖ_2	6	$m+3$
B_3	ϖ_3	8	$m+5$
G_2	ϖ_1	7	$2m+5$
G_2	ϖ_2	14	$3m+5$

Finally suppose that $\lambda \in \mathcal{E}'$. It follows from constructions in [Bou72, VIII] that in the expression (3.4.f) one has $m_\mu = 1$ for each $\mu \in \mathcal{H}(\lambda, m)$, and that $\mathcal{H}(\lambda, m)$ is as specified in the following table (3.5.j). (For type A_r, B_r, and C_r, see *loc. cit.* VIII.§13 no. 1,2,3 respectively; for type D_r, see *loc. cit.* VIII.§13 no. 4 and exerc. VIII.§13.10. Note that a description of the character $\bigwedge^m L(\varpi_i)$ for type D_4 and $i = 3, 4$ is easily obtained by triality from the given description of $\bigwedge^m L(\varpi_1)$.)

(3.5.j)

Φ	λ	$d(\lambda)$	Conditions	$\mathcal{H}(\lambda, m)$
A_r	ϖ_1	$r+1$	$1 \le m \le r$	ϖ_m
A_r	ϖ_r	$r+1$	$1 \le m \le r$	ϖ_{r+1-m}
$B_r, r \ge 2$	ϖ_1	$2r+1$	$1 \le m \le r-1$	ϖ_m
$B_r, r \ge 2$	ϖ_1	$2r+1$	$m = r$	$2\varpi_r$
$C_r, r \ge 2$	ϖ_1	$2r$	$1 \le m \le r, m \equiv 0 \pmod 2$	$\varpi_m, \varpi_{m-2}, \dots, \varpi_2, 0$
$C_r, r \ge 2$	ϖ_1	$2r$	$1 \le m \le r, m \equiv 1 \pmod 2$	$\varpi_m, \varpi_{m-2}, \dots, \varpi_3, \varpi_1$
$D_r, r \ge 4$	ϖ_1	$2r$	$1 \le m \le r-2$	ϖ_m
$D_r, r \ge 4$	ϖ_1	$2r$	$m = r-1$	$\varpi_r + \varpi_{r-1}$
$D_r, r \ge 4$	ϖ_1	$2r$	$m = r$	$2\varpi_r, 2\varpi_{r-1}$

To complete the proof, fix $\lambda \in \mathcal{E}'$ and let $d = d(\lambda)$, $2 \le m \le d/2$. Suppose $\nu \in \mathcal{H}(\lambda, m)$. One has $m(d-m) \ge m(d/2) \ge d$, so in this case the admissibility of (λ, m) follows provided $\langle \nu + \rho, \beta^\vee \rangle \le d$; the data in table (3.5.j) permits one to verify this latter condition holds if $\Phi \ne C_r$ (and $m \ge 2$).

Supose now that $\Phi = C_r$; we only must consider $\lambda = \varpi_1$. Using table (3.5.j), one checks that $\langle \nu + \rho, \beta^\vee \rangle \le 2r+1$ for each $\nu \in \mathcal{H}(\lambda, m)$. Assume first that $3 \le m \le d/2 = r$; in that case $m(d-m) \ge 3r \ge 2r+1$ and the result holds. When $m = 2$ and $r \ge 3$, one gets the desired result by noting $m(d-m) = 2(2r-2) = 4(r-1) \ge 2r+1$.

The above handles $m \ge 2$. When $m = 1$, we only must consider $\Phi = D_r$ and the weight $\lambda = \varpi_1$. The table shows that $\langle \nu + \rho, \beta^\vee \rangle = 2r - 2 < 2r - 1 = d - 1$ for each $\nu \in \mathcal{H}(\lambda, 1) = \{\varpi_1\}$, whence the admissibility of $(\varpi_1, 1)$ in this case. □

We are now is a position to complete the proof of 3.3.2 (and hence of 3.1.1). Let $\mathbf{V} = (L(\lambda_1), \dots, L(\lambda_s))$ with each λ_i restricted, and let $\mathbf{m} \in \tilde{\mathcal{N}}(\mathbf{V})$. In view of (3.4.1), $\lambda_i \in \hat{\mathbf{C}}$ for each i.

Any weight ν of $\bigwedge^{\mathbf{m}} \mathbf{V}$ has the form $\nu = \nu_1 + \dots + \nu_s$ where ν_i is a weight of $\bigwedge^{m_i} L(\lambda_i)$. According to 3.4.3, there is a weight $\mu_i \in \mathcal{H}(\lambda_i, m_i)$ with $\nu_i \le \mu_i$; since $\langle \nu_i, \beta^\vee \rangle \le \langle \mu_i, \beta^\vee \rangle$, we may as well assume that $\nu_i \in \mathcal{H}(\lambda_i, m_i)$ for each i. We will verify 3.3.2; in most cases we will do this by checking that 3.4.5 holds.

After re-ordering, we may suppose that for some $1 \le i \le s+1$, (λ_j, m_j) is admissible if and only if $j < i$.

Suppose first that $i > 1$; in this case note that (3.5.i) yields $\langle \lambda_k, \beta^\vee \rangle \le m_k(d(\lambda_k) - m_k)$ for $i \le k$. Combining this with the admissibility of the first $i-1$ weights yields

$$\langle \nu + \rho, \beta^\vee \rangle \le \sum_{j=1}^{i-1} \langle \nu_j + \rho, \beta^\vee \rangle + \sum_{k=i}^{s} \langle \nu_j, \beta^\vee \rangle \le \sum_{\ell=1}^{s} m_\ell(d(\lambda_\ell) - m_\ell) < p,$$

as desired.

Now suppose that $i = 1$, i.e. that no pair (λ_j, m_j) is admissible. If $\Phi = A_1$, we have by (3.5.i)

$$b = \langle \nu, \beta^\vee \rangle \le \sum_{i=1}^{s} m_i(d(\lambda_i) - m_i) < p$$

whence $b + 1 = \langle \nu + \rho, \beta^\vee \rangle \le p$, as asserted. So we now assume that the rank of Φ is at least 2. If $s = 1$, the semisimplicity of $\bigwedge^{\mathbf{m}} \mathbf{V}$ is trivial in case $m_1 = 1$; the only situation not handled by this observation is $m_1 = 2$, $\lambda_1 = \varpi_1$, $\Phi = C_2$. A straightforward computation shows that $\bigwedge^2 L(\varpi_1)$ is semisimple with restricted composition factors unless $p = 2$; see [McN98, Lemma 4.5.3] and note that $p = 2$ is ruled out by the condition $\mathbf{m} \in \mathcal{N}(\mathbf{V})$.

Finally, suppose $s > 1$. Using (3.5.i), one has

$$\begin{aligned}\langle \nu + \rho, \beta^\vee \rangle \le \sum_{i=1}^{s} \langle \nu_i + \rho, \beta^\vee \rangle - (s-1)\langle \rho, \beta^\vee \rangle &\le \sum_{i=1}^{s} m_i(d(\lambda_i) - m_i) + s - (s-1)\langle \rho, \beta^\vee \rangle \\ &< p - (h-2)s + h - 1,\end{aligned}$$

where $h - 1 = \langle \rho, \beta^\vee \rangle \ge 2$ (h is the Coxeter number). Since $s \ge 2$, one has $s \ge \frac{h-1}{h-2}$ and 3.4.5 is verified in this case.

This completes the proof of 3.1.1.

Remark 3.4. Let $V = L(\lambda)$, $\lambda \in \mathcal{E}'$. More can be said about the G module $\bigwedge^m V$. If $\Phi = B_r$ or D_r, assume $p \ne 2$. If $\Phi = C_r$, assume $p > r$. Then one has $\bigwedge^m V \simeq \bigoplus_{\mu \in \mathcal{H}(\lambda, m)} L(\mu)$. These assertions are verified in [McN98, Prop. 4.2.2] in the following situations: $\Phi = A_r$; $\Phi = B_r$ and $m < r$; $\Phi = D_r$ and $m < r - 1$. For $\Phi = B_r$, the assertion for $\bigwedge^r V$ follows from [Sei87, 8.1]; for $\Phi = D_r$, the assertion for $\bigwedge^{r-1} V$ follows from [Sei87, 8.1]. When $\Phi = C_r$, see [McNa, Prop. 6.3.5.] where the indecomposable summands of $\bigwedge^m V$ are worked out for all p. The only remaining situation is $\bigwedge^r V$ for type D_r. As a suitable reference was not located, we sketch an argument.

Let F be a field of characteristic $p \ge 0$, $p \ne 2$, and let (V, q) be a non-degenerate quadratic F-space with $\dim_F V = 2r$, $r \ge 3$. Let $G = \mathrm{SO}(V, q)$; then G is the group of F points of an algebraic group $\mathbf{G}$ of type D_r, and V is an F-form of the rational $\mathbf{G}$-module $L(\varpi_1)$. Assume that V has an orthogonal basis $\{e_i\}$ for which $q(e_i) = \alpha_i$, and let $\Delta = \Delta(q) = (-1)^r \alpha_1 \cdots \alpha_{2r}$; of course a different choice of orthogonal basis results in a different value of Δ, but any choice yields the same element in $F^\times/(F^\times)^2$. In particular, the field extension $F' = F(\sqrt{\Delta})$ is well defined.

In [KMRT98, Proposition (10.22)], a G-automorphism τ of $\bigwedge^r V$ is constructed with the property that τ^2 is given by multiplication with $1/\Delta$. Let $V' = V \otimes_F F'$; then $\bigwedge^r V'$ is the direct sum of eigenspaces $E_\pm$ for τ with eigenvalues $\pm 1/\sqrt{\Delta}$. These eigenspaces are $F'G$ submodules of $\bigwedge^r V'$ (in fact, they are even $F'SO(V', q')$ submodules).

Write $V = Fe \oplus W$ as an orthogonal sum with e non-singular, and let $H = \mathrm{SO}(W) \le G$. Then H is the group of F points of an algebraic group of type B_{r-1}. Evidently

$$\mathrm{res}_H^G(\textstyle\bigwedge^r V) \simeq \bigwedge^{r-1} W \oplus \bigwedge^r W.$$

Using (2.2.a), we have $\bigwedge^r W \simeq \bigwedge^{r-1} W$, and we have already seen that this module is absolutely simple for FH. Since $\mathrm{res}_H^G(\bigwedge^r V')$ has length 2, it follows at once that the $F'G$ modules $E_\pm$ are simple. Working over an algebraic closure $\bar{F}$ (or over any field which splits q), one finds that the highest weights of $\bigwedge^r V \otimes_F \bar{F}$ are $2\varpi_r$ and $2\varpi_{r-1}$; since these weights are incomparable, it follows by length considerations that $\bigwedge^r V \otimes_F \bar{F} \simeq L(2\varpi_r) \oplus L(2\varpi_{r-1})$. In particular, E_+ and E_- are non-isomorphic. This gives the claimed result. We have shown that $\bigwedge^r V$ is an absolutely semisimple FG module of absolute length 2; if $\Delta \in (F^\times)^2$ then $\mathrm{End}_{FG}(\bigwedge^r V) \simeq F \times F$, otherwise $\mathrm{End}_{FG}(\bigwedge^r V) \simeq F'$ and $\bigwedge^r V$ is simple for FG.

4. THE PROOF FOR AN ARBITRARY GROUP

The argument presented in this section follows very closely that given in [Ser94]. For completeness we outline the entire argument.

4.1. **Saturation.** Let V be a vector space of dimension n over K, and let $u \in GL(V)$ be an element of order p. Then $x = u - 1$ is a nilpotent endomorphism of V satisfying $x^p = 0$.

One defines a homomorphism $\phi_s : K \to \mathrm{GL}(V)$ by using a truncated exponential. More precisely, for $t \in K$, define $\phi_u(t) = u^t \in \mathrm{GL}(V)$ to be

$$u^t = \sum_{i=0}^{p-1} \binom{t}{i} x^i = 1 + tx + \frac{t(t-1)}{2} x^2 + \cdots \tag{4.1.k}$$

4.1.1. [Ser94, §4.1] *The homomorphism $\phi_u : K \to GL(V)$ is uniquely characterized by the following properties:*

P1. $\phi_u(1) = u$.

P2. ϕ_u *has degree* $< p$, *(i.e.* $t \mapsto u^t$ *is polynomial in* t *of degree* $< p$*).*

A subgroup $H \le \mathrm{GL}(V)$ is called *saturated* if every unipotent element u of H satisfies $u^p = 1$ and $u^t \in H$ for every $t \in K$.

From our point of view, the important fact about saturated subgroups of $\mathrm{GL}(V)$ is the following:

4.1.2. [Ser94, Proposition 11] *Let $H \le GL(V)$ be an algebraic subgroup which is saturated. Then $[H : H^0] \not\equiv 0 \pmod p$, where H^0 denotes the identity component of H.*

4.2. **The proof of Theorem 3.** Let G by a group, let $\mathbf{V}$ be a sequence of semisimple G-modules of dimension n_i, and let $\mathbf{m} \in \mathcal{N}(\mathbf{V})$. Let H denote the subgroup of $\mathrm{GL}(\mathbf{V}) = \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_s)$ consisting of all elements $\mathbf{x}$ so that $\bigwedge^{\mathbf{m}}(\mathbf{x}) = \bigwedge^{m_1} x_1 \otimes \cdots \otimes \bigwedge^{m_s} x_s$ leaves stable each subspace of $\bigwedge^{\mathbf{m}}\mathbf{V}$ which is stable under G. Then H is an *algebraic* subgroup of $\mathrm{GL}(\mathbf{V})$, and $\bigwedge^{\mathbf{m}}\mathbf{V}$ is a rational H module. Furthermore, $\bigwedge^{\mathbf{m}}\mathbf{V}$ is a semisimple G module if and only if it is a semisimple module for H.

In view of the results of section 3, Theorem 3 will follow provided that we argue $[H : H^0] \not\equiv 0 \pmod{p}$. To verify this property, we invoke 4.1.2; we must verify that H is saturated.

4.2.1. *Let* $\mathbf{u} \in H$ *be unipotent. Then* $\mathbf{u}^p = 1$.

Proof. This follows (as noted in [Ser94, 4.2]) since every unipotent in $\mathrm{GL}(V_i)$ has this property when $\dim_K V_i \le p$. ☐

4.2.2. *Let* $\mathbf{u} \in H$ *be a unipotent element. Then* $\mathbf{u}^t \in H$ *for all* $t \in K$.

Proof. We must verify that $\bigwedge^{\mathbf{m}}(\mathbf{u}^t)$ leaves stable each G-invariant subspace of $\bigwedge^{\mathbf{m}}\mathbf{V}$. It is straightforward to see that $(\bigwedge^{\mathbf{m}}\mathbf{u})^t$ leaves stable each G-invariant subspace of $\bigwedge^{\mathbf{m}}\mathbf{V}$, so it suffices to show that $\bigwedge^{\mathbf{m}}(\mathbf{u}^t) = (\bigwedge^{\mathbf{m}}\mathbf{u})^t$. In view of the uniqueness in 4.1.1 and the fact that $\bigwedge^{\mathbf{m}}(\mathbf{u}^1) = (\bigwedge^{\mathbf{m}}\mathbf{u})^1$, it suffices to show that $t \mapsto \bigwedge^{\mathbf{m}}(\mathbf{u}^t)$ is polynomial of degree $f < p$.

Let f_i denote the degree of the map $t \mapsto \bigwedge^{m_i} u_i^t$; evidently $f = \sum_{i=1}^s f_i$. Thus we are reduced to showing the following:

4.2.3. *If* V *is a* K *vector space and* $u \in GL(V)$ *is unipotent, then the degree* f *of* $t \mapsto \bigwedge^m(u^t)$ *satisfies* $f \le m(\dim_K V - m)$.

Let $e_1, e_2, \ldots, e_n$ be a basis of V chosen so that the unipotent element u fixes the "standard flag"

$$E_0 = 0 \subset E_1 = Ke_1 \subset E_2 = Ke_1 + Ke_2 \subset \cdots \subset E_n = V.$$

It follows that $x = u - 1$ satisfies $x(E_i) \subseteq E_{i-1}$.

We adopt the convention that $e_i =$ if $i \le 0$ or $i > n$. For each m-tuple of integers $\vec{a}$, let

$$e(\vec{a}) = e_{a(1)} \wedge e_{a(2)} \wedge \cdots \wedge e_{a(m)} \in \textstyle\bigwedge^m V.$$

Of course $e(\vec{a}) = 0$ if any two components of $\vec{a}$ coincide, or if any $a(i)$ fails to lie between 1 and n. Put $|\vec{a}| = \sum_i a(i)$. It is straightforward to verify that:

4.2.4. *If* $\vec{a}$ *is an* m*-tuple such that* $e(\vec{a}) \neq 0$, *then* $\frac{m(m+1)}{2} \le |\vec{a}| \le mn - \frac{m(m-1)}{2}$.

Fix $\vec{a}$ with $e(\vec{a}) \neq 0$, and consider the morphism

$$f_{\vec{a}} : K \to \textstyle\bigwedge^m V$$

given by $t \mapsto \bigwedge^m(u^t) \cdot e(\vec{a})$. The degree of the polynomial map $t \mapsto \bigwedge^m(u^t)$ is equal to $\sup\{\deg(f_{\vec{a}})\}$, the $\sup$ taken over all choices of $\vec{a}$ as above.

In view of the definition of u^t and the fact that

$$\textstyle\bigwedge^m(u^t)e(\vec{a}) = (u^t e_{a(1)}) \wedge (u^t e_{a(2)}) \wedge \cdots \wedge (u^t e_{a(n)}),$$

it suffices to show the following:

4.2.5. *Let $\vec{a}$ be such that $e(\vec{a}) \neq 0$. Whenever $\vec{b}$ is an m-tuple of positive integers with $|\vec{b}| > m(n-m)$, then $e(\vec{a}-\vec{b}) = 0$.*

To prove this, we note that 4.2.4 gives an upper bound for $|\vec{a}|$, so we have

$$|\vec{a}-\vec{b}| = |\vec{a}| - |\vec{b}| < |\vec{a}| - m(n-m) \leq mn - \frac{m(m-1)}{2} - m(n-m) = \frac{m(m+1)}{2},$$

whence $e(\vec{a}-\vec{b}) = 0$ by the lower bound given in 4.2.4. We have thus verified 4.2.2. □

The fact that H is saturated now follows; as noted above, this completes the proof of Theorem 3.

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