Let $G$ be a quasisimple, simply connected algebraic group over the algebraically closed field of characteristic $p > 0$.

The main result of (McNinch 2002) – namely, Theorem B from the introduction of that paper (which is Theorem 5.4 in the body of the paper) – is incorrect when $p \leq 3$. Here is a correct formulation of that result

**Theorem 1** (Corrected version of Theorem B/Theorem 5.4). Let $p > 2$. If $L$ is an irreducible $G$-module with $\dim L \leq p$ and if $H^2(G, L) \neq 0$, then either

(a) $p = 3$ and $G = SL_3$, or

(b) $L \simeq g^{[d]}$ for some $d \geq 1$.

Here is a more precise formulation of the error(s) that led to the incorrect formulation of this main theorem:

- Prop. 4.2(2) isn’t correct when $p = 2, 3$; see Proposition 2 below.
- Lemma 5.3(2) is not correct. In fact, the Lemma contains statements (1) and (2); the given proof correctly confirms (1) and claims – incorrectly – that the proof of (2) is “the same”. See Lemma 3 below.
- The errors in Prop. 4.2(2) and Lemma 5.3(2) require a reformulation of Theorem 5.3; see Theorem 5 below.

We are going to describe corrections for these errors, and then we will sketch the proof of Theorem 1.

Before doing so, let me acknowledge that in a sequence of emails dated February–April 2009, David Stewart pointed out the error(s) just mentioned, and he communicated corrections to me. Moreover, he provided examples showing that as a result of the two errors just mentioned, the conclusion of the main Theorem of the paper sometimes fails when $p = 2$ or $p = 3$.

Here is a correct formulation of Prop. 4.2(2):

**Proposition 2.** If $p \geq h \geq 3$ then either $H^2(G_1, k)[-1] \simeq \text{Lie}(G)^\vee$ or $p = 3$ and $G$ has root system of type $A_2$.

**Sketch.** The corresponding result (McNinch 2002, Prop. 4.2(2)) is deduced from (Andersen and Jantzen 1984, Cor. 6.3). In fact, the result in loc. cit. implies the result for $p > 3$, and indeed yields a more complicated description of $H^2(G_1, k)$ when $p = 3$ and $G$ has type $A_2$ which I originally overlooked. For a more explicit argument confirming the claim of this Proposition, see (Bendel, Nakano, and Pillen 2007, Theorem 6.2)

Part (2) of lemma 5.3 should be replaced by the following:

**Lemma 3.** Let $E_2^{p,q} \rightarrow H^{p+q}$ be a convergent first quadrant spectral sequence. Assume that

$$E_2^{1,0} = E_2^{1,1} = E_2^{2,0} = E_2^{2,1} = E_2^{3,0} = 0.$$

Then $H^2 \simeq E_2^{0,2}$.

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Remark 4. The changes from the original formulation are the stipulations that the terms $E_2^{2,1}$ and $E_2^{3,0}$ vanish.

Proof. The result follows if we argue that $E_2^{0,2} \simeq E_2^{0,2}$. Of course, for $a \geq 2$, $E_2^{a,0}$ is the cohomology of the sequence

$$E_2^{-a,1+a} \to E_2^{0,2} \to E_2^{a,3-a}.$$ 

Since $E_2$ is zero outside the first quadrant, it will be enough to argue that $E_2^{a,3-a} = 0$ for all $a \geq 2$. This is immediate for $a \geq 4$. When $a = 2$ we have assumed that $E_2^{2,1} = 0$.

Finally, when $a = 3$, first note that we have assume $E_2^{3,0} = 0$. Now, $E_3^{3,0}$ is the cohomology of the sequence

$$E_2^{1,1} \to E_2^{3,0} \to E_2^{5,-1};$$

since $E_2^{1,1} = 0$ by assumption, it follows that $E_2^{3,0} = E_2^{3,0} = 0$. This completes the proof. 

Now, (McNinch 2002, Prop. 4.2(2) and Lemma 5.3) are used in the proof of Theorem 5.3. We must reformulate the statement of that Theorem as follows:

**Theorem 5** (Corrected version of Theorem 5.3). Let $p > 2$ and suppose also that $p \geq h$. Let $V$ be a $G$-module for which $H^i(G, V) = 0$ for $i = 1, 2$, and let $d \geq 1$. Then:

(a) $H^1(G, V^{[d]}) = 0$;
(b) if $p > h$, then $H^2(G, V^{[d]}) = \text{Hom}_G(g, V)$;
(c) if $p = h$ and $\dim V \leq p$, then either $H^2(G, V^{[d]}) = 0$ or $p = 3$, $G$ has root system $A_2$ and $\dim V = 3$.

Remark 6. The changes from (McNinch 2002) are the stipulation that $p > 2$ and the requirement that $p > h$ in (b). Assertion (c) is new.

**Sketch of proof of Theorem 5.** We are going to sketch the outline of the proof in order to point out where new arguments are required.

Recall that we write $E_2^{p,q} = H^p(G, H^q(G, V^{[d]}))$ for the terms of the “second page” of the Lyndon-Hochschild-Serre spectral sequence for the normal subgroup scheme $G_1$ – the first Frobenius kernel – of $G$.

The proof of (a) proceeds precisely as in (McNinch 2002).

We now proceed to prove (b) and (c). As noted in the introduction to (McNinch 2002), the hypothesis “$\dim V \leq p$” implies that $V$ is a semisimple $G$-representation, see (Jantzen 1997). Thus in the remainder of proof of the Theorem, we may and will suppose $V$ to be irreducible.

As in the original proof, since $p > 2$ we have $H^1(G_1, k) = 0$ by (McNinch 2002, Prop. 4.2(1)); since $d > 0$, it follows that (⋄) $E_2^{n,1} = 0$ for all $n \geq 0$.

In particular, $E_2^{0,1} = E_2^{1,1} = 0$, so the only possible non-zero $E_\infty$-terms of total degree 2 are precisely $E_2^{0,2}$ and $E_2^{0,0}$.

If $d > 1$, proceed as in the original proof (McNinch 2002, p.468); one gets

$$H^2(G, V^{[d]}) \simeq E_2^{2,0} \simeq H^2(G, V^{[d-1]}),$$

thus reducing the proof of (b) and (c) to the case $d = 1$.

Now suppose that $d = 1$. Then $E_2^{2,0} = 0$ by our assumption on $V$. Note that (a) shows $E_2^{1,0} = 0$. Moreover $E_2^{1,1} = E_2^{2,1} = 0$ by (⋄).

We now prove (b), so we suppose that $p > h$. This assumption excludes the situation in which $p = 3$ and the root system of $G$ is of type $A_2$; thus Proposition 2 gives a $G$-module isomorphism $H^2(G_1, k) \simeq g^\vee$.

Now suppose that

$$0 \neq E_2^{0,2} = \text{Hom}_G(g, V).$$

Since $p > h$, one knows that $g$ is an irreducible $G$-module; see e.g. the proof of (McNinch 2002, Lemma 4.1.B). Since $V$ is irreducible, there is a $G$-isomorphism $V \simeq g \simeq H^0(\tilde{x})$. In particular, $H^3(G, V) = 0$.
by (McNinch 2002, Prop. 3.4(c)) so that $E_2^{3,0} = 0$. Now the hypotheses of Lemma 3 are verified; that Lemma shows that $H^2(G, V^{[1]}) = E_2^{0,2}$ as required.

Finally, we prove (c). Since $\dim V \leq p$ and $p = h$, we find using (McNinch 2002, Prop. 5.1) that the root system of $G$ is of type $A_{p-1}$ and $V$ is a Frobenius twist of $L(\omega_1)$ or $L(\omega_{p-1})$. If $p > 3$, then Proposition 2 gives an isomorphism $H^2(G_1, k) \simeq g^\vee$. Thus $E_2^{0,2} \neq 0 \implies \text{Hom}_G(g, V) \neq 0$. But the adjoint representation $g$ has length two; it has a trivial composition factor and a composition factor $L(\tilde{\alpha})$ of dimension $p^2 - 2$. Since $V$ is irreducible of dimension $p^2$, deduce that $\text{Hom}_G(g, V) = 0$. This shows that $E_2^{0,2} = 0$ and hence $H^2(G, V^{[1]}) = E_2^{0,2} = 0$, as required.

Sketch of proof of Theorem 1. As in the given proof of (McNinch 2002, Theorem 5.4), write $L'$ for a simple $G$-module for which $(L')^{[d]} \simeq L$ for $d \geq 0$ and such that $\text{Lie}(G)$ acts non-trivially on $L'$. Recall that $p \geq h$ by (McNinch 2002, Prop. 5.1), and that $H^i(G, L') = 0$ for $i \geq 0$ by (McNinch 2002, Prop 5.2). We are finished if $d = 0$. If $d > 1$ then Theorem 5(b) applies. If $p > h$, that result shows that $H^2(G, L) = \text{Hom}_G(g, L')$ which proves the required result in this case.

If now $p = h$, Theorem 5(c) applies. It shows that $H^2(G, L) = 0$ unless $p = 3$ and $G$ has root system of type $A_2$, which completes the proof.

Remark 7. David Stewart observed that there are actually counter-examples to the original formulation of the main Theorem of (McNinch 2002) when $p = 2$ and when $p = 3$. For the $p = 2$ examples, the interested reader is referred to Stewarts paper (Stewart 2010).

Let $G = \text{SL}_3 = \text{SL}(V)$ and suppose that $p = 3$. Stewart pointed out to me that $H^2(G, L^{[d]}) \neq 0$ for $d \geq 1$ where $L$ is either the “natural” 3-dimensional $G$-module $V$ or its dual $V^\vee$.

References


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