

# Commutative Rings, Fields, and Galois Theory

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# 1 Commutative rings

## 1.1 Definitions

*Definition 1.1.1.* A ring  $R$  is an additive abelian group together with an operation of multiplication  $R \times R \rightarrow R$  given by  $(a, b) \mapsto a \cdot b$  such that the following axioms hold:

- multiplication is *associative*
- multiplication *distributes* over addition: for every  $a, b, c \in R$  we have

$$a(b + c) = ab + ac$$

and

$$(b + c)a = ba + ca.$$

We often just denote multiplication by juxtaposition: i.e. we may write  $ab$  instead of  $a \cdot b$  for  $a, b \in R$

We say that the ring  $R$  is *commutative* if the operation of multiplication is commutative; i.e. if  $ab = ba$  for all  $a, b \in R$ .

And we say that  $R$  has identity if multiplication has an identity, i.e. if there is an element  $1_R \in R$  such that  $a \cdot 1_R = 1_R \cdot a = a$  for every  $a \in R$ .

Usually we write 1 for  $1_R$ . The idea is that  $1_R$  is the multiplicative identity of  $R$ . For example, the identity matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  is the multiplicative identity  $1_R$  of the matrix ring  $R = \text{Mat}_2(\mathbf{R})$ .

In these notes, unless otherwise indicated a ring is assumed to be commutative and to have identity.

Here are some examples of commutative rings:

*Example 1.1.2.* (a)  $\mathbf{Z}$  the ring of integers,  $\mathbf{Q}$  the ring of rational numbers,  $\mathbf{R}$  the ring of real numbers,  $\mathbf{C}$  the ring of complex numbers.

(b) if  $X$  is a set and if  $R$  is a commutative ring, the set  $X^R$  of all  $R$ -valued functions on  $X$  can be viewed as a commutative ring where the sum and product of functions  $f : X \rightarrow R$  are defined “pointwise”.

## 1.2 Polynomial rings

If  $R$  is a commutative ring, the collection of all polynomials in the variable  $T$  having coefficients in  $R$  is denoted  $R[T]$ .

Notice that the set of *monomials*  $S = \{T^i \mid i \in \mathbb{N}\}$  has the following properties:

(M1) every element of  $R[T]$  is an  $R$ -linear combination of elements of  $S$ . This just amounts to the statement that every polynomial  $f(T) \in R[T]$  has the form

$$f(T) = \sum_{i=0}^N a_i T^i$$

for a suitable  $N \geq 0$  and suitable coefficients  $a_i \in R$ .

(M2) the elements of  $S$  are linearly independent i.e. if

$$\sum_{i=0}^N a_i T^i = 0 \quad \text{for} \quad a_i \in R,$$

then  $a_i = 0$  for every  $i$ .

Polynomials in  $R[T]$  can be added in a natural way. (This is just like adding vectors in a vector space).

And there is a product operation on polynomials, as follows:

if  $f(T) = \sum_{i=0}^N a_i T^i$  and  $g(T) = \sum_{i=0}^M b_i T^i$  then

$$f(T) \cdot g(T) = \sum_{i=0}^{N+M} c_i T^i \quad \text{where} \quad c_i = \sum_{s+t=i} a_s b_t.$$

**Proposition 1.2.1.**  $R[T]$  is a commutative ring with identity.

## 2 Properties of rings

### 2.1 Ring Homomorphisms

*Definition 2.1.1.* If  $R$  and  $S$  are rings, a function  $\phi : R \rightarrow S$  is called a *ring homomorphism* provided that

- (a)  $\phi$  is a homomorphism of *additive groups*,
- (b)  $\phi$  preserves multiplication; i.e. for all  $x, y \in R$  we have  $\phi(xy) = \phi(x)\phi(y)$ , and
- (c)  $\phi(1_R) = 1_S$ .

*Definition 2.1.2.* The *kernel* of the ring homomorphism  $\phi : R \rightarrow S$  is given by

$$\ker \phi = \phi^{-1}(0) = \{x \in R \mid \phi(x) = 0\};$$

thus  $\ker \phi$  is just the kernel of  $\phi$  viewed as a homomorphism of additive groups.

Here are some properties of the kernel:

- (K1)  $\ker \phi$  is an additive subgroup of  $R$
- (K2) for every  $r \in R$  and every  $x \in \ker \phi$  we have  $rx \in \ker \phi$ .

### 2.2 Ideals of a ring

For simplicity suppose that the ring  $R$  (and  $S$ ) are *commutative* rings.

*Definition 2.2.1.* A subset  $I$  of  $R$  is an *ideal* provided that

- (a)  $I$  is an additive subgroup of  $R$ , and
- (b) for every  $r \in R$  and every  $x \in I$  we have  $rx \in I$ .

We sometimes describe condition (b) by saying that “ $I$  is closed under multiplication by every element of  $R$ ”.

The proof of the following is immediate from definitions:

**Proposition 2.2.2.** *If  $\phi : R \rightarrow S$  is a ring homomorphism, then  $\ker \phi$  is an ideal of  $R$ .*

*Remark 2.2.3.* In any commutative ring  $R$  the trivial subgroup  $0 = \{0\}$  is an ideal, and the subgroup  $R$  itself is also an ideal.

### 2.3 Quotient rings

Let  $R$  be a commutative ring and let  $I$  be an ideal of  $R$ .

Since  $I$  is a subgroup of the (abelian) additive group  $R$ , we may consider the quotient group  $R/I$ . Its elements are (additive) cosets  $a + I$  for  $a \in R$ .

It follows from the definition of cosets that the  $a + I = b + I$  if and only if  $b - a \in I$ .

The additive group can be made into a commutative ring by defining the multiplication as follows:

For  $a + I, b + I \in R/I$  (so that  $a, b \in R$ ), the product is given by

$$(a + I)(b + I) = ab + I.$$

In order to make this definition, one must confirm that this rule is well-defined. Namely, if we have equalities  $a + I = a' + I$  and  $b + I = b' + I$ , we need to know that

$$(a + I)(b + I) = (a' + I)(b' + I).$$

Applying the definition, we see that we must confirm that

$$ab = I = a'b' + I.$$

For this, we need to argue that  $a'b' - ab \in I$ .

Since  $a + I = a' + I$ , we know that  $a' - a = x \in I$  and since  $b + I = b' + I$  we know that  $b' - b = y \in I$ .

Thus  $a' = a + x$  and  $b' = b + y$ . Now we see that

$$a'b' = (a + x)(b + y) = ab + ay + xb + xy$$

Since  $I$  is an ideal, we see that  $ay, xb, xy \in I$  hence  $ay + xb + xy \in I$ . Now conclude that  $a'b' + I = ab + I$  as required.

It is now straightforward to confirm that the ring axioms hold for the set  $R/I$  with these operations.

**Proposition 2.3.1.** *If  $I$  is an ideal of the commutative ring  $R$ , then  $R/I$  is a commutative ring with the addition and multiplication just described.*

## 2.4 Principal ideals

*Definition 2.4.1.* If  $R$  is a commutative ring and  $a \in R$ , the *principal ideal generated by  $a$*  – written  $Ra$  or  $\langle a \rangle$  – is defined by

$$Ra = \langle a \rangle = \{ra \mid r \in R\}.$$

**Proposition 2.4.2.** *For  $a \in R$ ,  $Ra$  is an ideal of  $R$ .*

*Example 2.4.3.* Let  $n \in \mathbf{Z}_{>0}$  and consider the principal ideal  $n\mathbf{Z}$  of the ring  $\mathbf{Z}$  generated by  $n \in \mathbf{Z}$ .

As an additive group,  $n\mathbf{Z}$  is the infinite cyclic group generated by  $n$ .

The quotient ring  $\mathbf{Z}/n\mathbf{Z}$  is the finite commutative ring with  $n$  elements; these elements are precisely the *congruence classes* of integers modulo  $n$ .

## 2.5 Isomorphism Theorem

**Theorem 2.5.1.** *Let  $R, S$  be commutative rings with identity and let  $\phi : R \rightarrow S$  be a ring homomorphism. Assume that  $\phi$  is surjective (i.e. onto). Then  $\phi$  determines an isomorphism  $\bar{\phi} : R/I \rightarrow S$  where  $I = \ker \phi$ , where  $\bar{\phi}$  is determined by the rule*

$$\bar{\phi}(a + I) = \phi(a) \quad \text{for } a \in R.$$

*Proof.* First, you must confirm that  $\bar{\phi}$  is *well-defined*; i.e. that if  $a + I = a' + I$  then  $\bar{\phi}(a + I) = \bar{\phi}(a' + I)$ .

Next, you must confirm that  $\bar{\phi}$  is a ring homomorphism (this is immediate from the definition of ring operations on  $R/I$ ).

Finally, you must confirm that  $\ker \bar{\phi} = \{0\}$ , where here 0 refers to the additive identity of the quotient ring  $R/I$ . This additive identity is of course the trivial coset  $I = 0 + I \in R/I$ .  $\square$

## 2.6 A Homomorphism from the polynomial ring to the scalars

Let  $F$  is a field and let  $a \in F$ . consider the mapping

$$\Phi : F[T] \rightarrow F$$

given by  $\Phi(f(T)) = f(a)$ . Namely, applying  $\Phi$  to a polynomial  $f(T)$  results in the value  $f(a)$  of  $f(T)$  at  $a$ .

The definition of multiplication in  $F[T]$  guarantees that  $\Phi$  is a ring homomorphism.

### 3 Polynomials over a field and the division algorithm

#### 3.1 Some general notions for commutative rings

*Definition 3.1.1.* If  $R$  is a commutative ring with 1 and if  $u \in R$  we say that  $u$  is a *unit* - or that  $u$  is *invertible* - provided that there is  $v \in R$  with  $uv = 1$ ; then  $v = u^{-1}$ .

We write  $R^\times$  for the units in  $R$ .

A commutative ring  $R$  is a *field* provided that every non-zero element is invertible. Thus  $R$  is a field if  $R^\times = R \setminus \{0\}$ .

**Proposition 3.1.2.** *If  $R$  is a commutative, then  $R^\times$  is an abelian group (with operation the multiplication in  $R$ ).*

For any commutative ring  $R$  and elements  $a, b \in R$  we say that  $a$  **divides**  $b$  - written  $a \mid b$  - if  $\exists x \in R$  with  $ax = b$ .

**Proposition 3.1.3.** *For  $a, b \in R$  we have  $a \mid b$  if and only if  $b \in \langle a \rangle$ .*

Recall that we introduced the principal ideal  $\langle a \rangle = aR$  for any commutative ring  $R$  and any  $a \in R$ . In fact, given  $a_1, \dots, a_n \in R$  we can consider the ideal

$$\langle a_1, \dots, a_n \rangle = \sum_{i=1}^n a_i R$$

defined as

$$\langle a_1, \dots, a_n \rangle = \left\{ \sum_{i=1}^n r_i a_i \mid r_i \in R \right\}.$$

It is straightforward to check that  $\langle a_1, \dots, a_n \rangle$  is indeed an ideal of  $R$ .

*Definition 3.1.4.* A non-zero element  $a \in R$  is said to be a *0-divisor* provided that there is  $0 \neq b \in R$  with  $ab = 0$ .

*Example 3.1.5.* Let  $n$  be a composite positive integer, so that  $n = ij$  for integers  $i, j > 0$ . Consider the elements  $[i] = i + n\mathbf{Z}$ ,  $[j] = j + n\mathbf{Z}$  in the quotient ring  $\mathbf{Z}/n\mathbf{Z}$ .

Then  $[i]$  and  $[j]$  are both non-zero since  $0 < i, j < n$  so that  $n \nmid i$  and  $n \nmid j$ . But  $[i] \cdot [j] = [n] = 0$  so that  $[i]$  and  $[j]$  are 0-divisors of the ring  $\mathbf{Z}/n\mathbf{Z}$ .

*Definition 3.1.6.* A commutative ring  $R$  is said to be an *integral domain* provided that it has no zero-divisors.

*Example 3.1.7.* (a) Any field is an integral domain.

(b) The ring  $\mathbf{Z}$  of integers is an integral domain.

(c) Any subring of an integral domain is an integral domain.

For example, the ring  $\mathbf{Z}[i] = \{a + bi \mid a, b \in \mathbf{Z}\}$  of gaussian integers is an integral domain.

(d)  $\mathbf{Z}/n\mathbf{Z}$  is not an integral domain whenever  $n$  is composite.

(e) If  $R$  and  $S$  are commutative rings, the direct product  $R \times S$  is *never* an integral domain.

Indeed, the elements  $(1, 0)$  and  $(0, 1)$  are 0-divisors.

**Lemma 3.1.8. (Cancellation)** *Let  $R$  be an integral domain and let  $a, b, c \in R$  with  $c \neq 0$ . If  $ac = bc$  then  $a = b$ .*

*Proof.* The equation  $ac = bc$  implies that  $ac - bc = 0$  so that  $(a - b)c = 0$  by the distributive property. Since  $R$  has no zero divisors and since  $c \neq 0$  by assumption, conclude that  $a - b = 0$  i.e. that  $a = b$ .  $\square$

**Proposition 3.1.9.** *Let  $R$  be an integral domain and let  $d, d' \in R \setminus \{0\}$ . If  $\langle d \rangle = \langle d' \rangle$  then  $d$  and  $d'$  are associate.*

*Proof.* Since  $d \in \langle d' \rangle$  we may write  $d = xd'$  and since  $d' \in \langle d \rangle$  we may write  $d' = yd$ . Now we see that  $d = xd' = xyd$ . Since  $d \neq 0$  cancellation (Lemma 3.1.8) implies that  $xy = 1$ . Thus  $x, y \in R^\times$  and indeed  $d, d'$  are associate.  $\square$

### 3.2 An important result on polynomial rings

**Proposition 3.2.1.** *Let  $R$  and  $S$  be rings, let  $\phi : R \rightarrow S$  be a ring homomorphism, and let  $\alpha \in S$  be an element. There is a unique ring homomorphism*

$$\Psi : R[T] \rightarrow S$$

*such that  $\Psi(T) = \alpha$  and such that  $\Psi|_R = \phi$ .*

*Proof.* Let  $f, g \in R[T]$ , say

$$f = \sum_{i=0}^n a_i T^i \quad \text{and} \quad g = \sum_{i=0}^m b_i T^i$$

be elements of  $R[T]$ .

To see that  $\Psi$  is an additive homomorphism, note that  $f + g = \sum_{i=0}^{\max(n,m)} (a_i + b_i) T^i$  so that

$$\Psi(f + g) = \sum_{i=0}^{\max(n,m)} (a_i + b_i) \alpha^i = \sum_{i=0}^n a_i \alpha^i + \sum_{i=0}^m b_i \alpha^i = \Psi(f) + \Psi(g)$$

Similarly, to see that  $\Psi$  is multiplicative, note that  $fg = \sum_{i=0}^{n+m} c_i T^i$  where  $c_i = \sum_{s+t=i} a_s b_t$ . Now,

$$\Psi(fg) = \sum_{i=0}^{n+m} \phi(c_i) \alpha^i = \left( \sum_{i=0}^n \phi(a_i) \alpha^i \right) \left( \sum_{i=0}^m \phi(b_i) \alpha^i \right) = \Psi(f) \cdot \Psi(g)$$

$\square$

### 3.3 The degree of a polynomial

Let  $F$  be a field and consider the ring of polynomials  $F[T]$ .

**Definition 3.3.1.** The *degree* of a polynomial  $f = f(T) \in F[T]$  is defined to be  $\deg(f) = -\infty$  if  $f = 0$ , and otherwise  $\deg(f) = n$  where

$$f = \sum_{i=0}^n a_i T^i \quad \text{with each } a_i \in F \text{ and } a_n \neq 0.$$

We have some easy and familiar properties of the degree function:

**Proposition 3.3.2.** *Let  $f, g \in F[T]$ .*

(a)  $\deg(fg) = \deg(f) + \deg(g)$ .

(b)  $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$  and equality holds if  $\deg(f) \neq \deg(g)$ .

(c)  $f \in F[T]^\times$  if and only if  $\deg(f) = 0$ . In particular,  $F[T]^\times = F^\times$ .

**Corollary 3.3.3.** *For a field  $F$ , the polynomial ring  $F[T]$  is an integral domain.*

*Proof.* Let  $f, g \in F[T]$  and suppose that  $fg = 0$ . We must argue that either  $f = 0$  or  $g = 0$ .  $\square$

**Proposition 3.3.4.** *Let  $f, g \in F[T]$ . If  $g \neq 0$  and  $\deg g < \deg f$  then  $[g] = g + \langle f \rangle$  is a non-zero element of  $F[T]/\langle f \rangle$ .*

### 3.4 The division algorithm

**Theorem 3.4.1.** *Let  $F$  be a field, and let  $f, g \in F[T]$  with  $0 \neq g$ . Then there are polynomials  $q, r \in F[T]$  for which*

$$f = qg + r$$

*and  $\deg r < \deg g$ .*

*Proof.* First note that we may suppose  $f$  to be non-zero. Indeed, if  $f = 0$ , we just take  $q = r = 0$ . Clearly  $f = qg + r$ , and  $\deg(r) = -\infty < \deg(g)$  since  $g$  is non-zero.

We now proceed by induction on  $\deg(f) \geq 0$ .

For the base case in which  $\deg(f) = 0$ , we note that  $f = c$  is a constant polynomial; here  $c \in F^\times$ .

If  $\deg(g) = 0$  as well, then  $g = d \in F^\times$  and then  $c = (c/d)d + 0$  so we may take  $q = c/d$  and  $r = 0$ . Now  $\deg(r) = -\infty < \deg(g)$  as required.

If  $\deg(g) > 0$ , we simply take  $q = 0$  and  $r = f$ : we then have  $f = 0 \cdot g + f$  and  $\deg(f) = 0 < \deg(g)$  as required.

We have now confirmed the Theorem holds when  $\deg(f) = 0$ .

Proceeding with the induction, we now suppose  $n > 0$  and that the Theorem holds whenever  $f$  has degree  $< n$ . We must prove the Theorem holds when  $f$  has degree  $n$ .

Since  $f$  has degree  $n$ , we may write  $f = a_n T^n + f_0$  where  $a_n \in F^\times$  and  $f_0 \in F[T]$  has  $\deg(f_0) < n$ .

Let us write  $g = \deg(g)$ ; we may write  $g = b_m T^m + g_0$  where  $b_m \in F^\times$  and  $g_0 \in F[T]$  has  $\deg(g_0) < m$ .

If  $n < m$  we take  $q = 0$  and  $r = f$  to find that  $f = qg + r$  and  $\deg(r) < \deg(g)$ .

Finally, if  $m \leq n$  we set

$$f_1 = f - (a_n/b_m)T^{n-m}g = a_n T^n + f_0 - \left( \frac{a_n}{b_m} b_m T^n + \frac{a_n}{b_m} T^{n-m} g_0 \right) = f_0 - \frac{a_n}{b_m} T^{n-m} g_0.$$

We have  $\deg(f_0) < n$  by assumption, and  $\deg\left(\frac{a_n}{b_m} T^{n-m} g_0\right) < n$  by the Proposition together with the fact that  $\deg(g_0) < m$ .

Thus  $\deg(f_1) < n$ . Now we apply the induction hypothesis to write

$$f_1 = q_1 g + r_1 \quad \text{with } \deg(r_1) < \deg(g).$$

Finally, we have

$$f = f_1 + (a_n/b_m)T^{n-m}g = q_1 g + r_1 + (a_n/b_m)T^{n-m}g = (q_1 + (a_n/b_m)T^{n-m})g + r_1$$

so we have indeed written  $f = qg + r$  in the required form.  $\square$

**Corollary 3.4.2.** *Let  $F$  be a field and let  $f \in F[T]$ . For  $a \in F$ , there is a polynomial  $q \in F[T]$  for which*

$$f = q(T - a) + f(a).$$

**Corollary 3.4.3.** *For  $f \in F[T]$  an element  $a \in F$  is a root of the polynomial  $f$  if and only if  $T - a \mid f$  in  $F[T]$ . In particular, if  $d = \deg(f)$ ,  $f$  has no more than  $d$  distinct roots in  $F$ .*

*Proof.* The first statement is clear from Corollary 3.4.2. Now consider the distinct roots

$$\alpha_1, \dots, \alpha_e \in F$$

of  $f$ . Then  $T - \alpha_1$  divides  $f$  so that  $f = (T - \alpha_1)f_1$  for some  $f_1 \in F[T]$ . Since  $\alpha_2$  is a root of  $f$  we see that

$$0 = f(\alpha_2) = (\alpha_2 - \alpha_1)f_1(\alpha_2)$$

which shows that  $\alpha_2$  is a root of  $f_1$  since  $\alpha_1 \neq \alpha_2$ . Thus we find that

$$f = (T - \alpha_1)(T - \alpha_2)f_2$$

for some  $f_2 \in F[T]$ . Continuing in this way we find that  $\prod_{i=1}^e (T - \alpha_i)$  divides  $f$ , so that  $e \leq \deg f$  by Proposition 3.3.2.  $\square$

## 4 Ideals of the polynomial ring

### 4.1 Description of ideals of the polynomial ring

**Corollary 4.1.1.** *Let  $F$  be a field and let  $I$  be an ideal of the ring  $F[T]$ . Then  $I$  is a principal ideal; i.e. there is  $g \in I$  for which*

$$I = \langle g \rangle = g \cdot F[T].$$

*Proof.* If  $I = 0$  the results is immediate. Thus we may suppose  $I \neq 0$ .

Consider the set  $\{\deg(g) \mid 0 \neq g \in I\}$ . This is a non-empty set of natural numbers, hence it contains a minimal element by the *well-ordering principle*.

Choose  $g \in I$  such that  $\deg(g)$  is this minimal degree; we claim that  $I = \langle g \rangle$ .

Clearly  $\langle g \rangle \subseteq I$ . To complete the proof, it remains to establish the inclusion  $I \subseteq \langle g \rangle$ . Let  $f \in I$  and use the **Division Algorithm** to write  $f = qg + r$  for  $q, r \in F[T]$  with  $\deg r < \deg g$ .

Observe that  $f - qg \in I$  so that  $r \in I$ . Since  $\deg r < \deg g$  conclude that  $r = 0$ . This shows that  $f = qg \in \langle g \rangle$  as required, completing the proof.  $\square$

Let  $F$  be a field,  $F[T]$  be the ring of polynomials with coefficients in  $F$ , let  $f, g \in F[T]$  be polynomials which are not both 0.

**Definition 4.1.2.** The **greatest common divisor**  $\gcd(f, g)$  of the pair  $f, g$  is a monic polynomial  $d$  such that

- (a)  $d \mid f$  and  $d \mid g$ ,
- (b) if  $e \in F[T]$  satisfies  $e \mid f$  and  $e \mid g$ , then  $e \mid d$ .

**Remark 4.1.3.** If  $d, d'$  are two gcds of  $f, g$  then  $d \mid d'$  and  $d' \mid d$ . In particular,  $\deg(d) = \deg(d')$  and  $d' = \alpha d$  for some  $\alpha \in F^\times$ . It is then clear that there is no more than one monic polynomial satisfying i. and ii.

Note that  $f, g$  are not both 0 if and only if the ideal  $\langle f, g \rangle$  is not 0.

**Proposition 4.1.4.** (a)  $\langle f, g \rangle$  is an ideal. Since  $F[T]$  is a principal ideal domain by Corollary 4.1.1, there is a monic polynomial  $d \in F[T]$  with

$$\langle d \rangle = \langle f, g \rangle.$$

Then  $d = \gcd(f, g)$

(b) In particular,  $d = \gcd(f, g)$  may be written in the form  $d = uf + vg$  for  $u, v \in F[T]$ .

*Proof.* For a., write  $I = \langle f, g \rangle = \langle d \rangle$ . Since  $f, g \in I$ , the definition of  $\langle d \rangle$  shows that  $d \mid f$  and  $d \mid g$ .

Now suppose that  $e \in F[T]$  and that  $e \mid f$  and  $e \mid g$ . Then  $f, g \in \langle e \rangle$  which shows that  $\langle f, g \rangle \subseteq \langle e \rangle$ .

But this implies that  $\langle d \rangle \subseteq \langle e \rangle$  so that  $e \mid d$  as required. Thus we see that  $d$  is indeed equal to  $\gcd(f, g)$ .

Since  $d \in \langle d \rangle = \langle f, g \rangle$ , assertion b. follows from the definition of  $\langle f, g \rangle$ .  $\square$

## 4.2 Principal ideal domains (PIDs)

*Definition 4.2.1.* An integral domain  $R$  is said to be a **principal ideal domain** (abbreviated PID) provided that every ideal  $I$  of  $R$  has the form

$$I = \langle a \rangle \quad \text{for some } a \in R;$$

i.e. provided that every ideal of  $R$  is principal.

*Example 4.2.2.* (a) The ring  $\mathbf{Z}$  of integers is a PID.

(b) For any field  $F$ , the ring  $F[T]$  of polynomials is a PID - this follows from the Corollary to the division algorithm, above.

(c) The rings  $\mathbf{Z}[i]$  and  $\mathbf{Z}[\sqrt{2}]$  are PIDs - to see this one can argue that these rings are Euclidean domains and then one proves that any Euclidean domain is a PID.

## 4.3 PIDs and greatest common divisors

Let  $R$  be a PID.

The results about gcd in the polynomial ring proved in Section 4.1 actually hold in the generality of the PID  $R$ . We quickly give the statements:

*Definition 4.3.1.* Let  $a, b \in R$  such that  $\langle a, b \rangle \neq 0$ . A gcd of  $a$  and  $b$  is an element  $d \in R$  such that

- (i)  $d \mid a$  and  $d \mid b$  (in words: “ $d$  is a common divisor of  $a$  and  $b$ ”)
- (ii) if  $e \mid a$  and  $e \mid b$  then  $e \mid d$ . (in words: “any common divisor of  $a$  and  $b$  divides  $d$ ”)

**Lemma 4.3.2.** *If  $R$  is a PID and if  $d$  and  $d'$  are gcds of  $a$  and  $b$  then  $d$  and  $d'$  are associates.*

*Proof.* This follows from Proposition 3.1.9 □

*Proof.* Using the definition of gcd we see that  $d \mid d'$  and  $d' \mid d$ . Thus  $d' = dv$  and  $d = d'u$  for  $u, v \in R$ .

This shows that  $d' = dv = d'uv$ . Using cancellation, find that  $1 = uv$  so that  $u, v \in R^\times$ . □

*Remark 4.3.3.* This definition of course covers the cases when  $R = \mathbf{Z}$  and when  $R = F[T]$ . The main thing to point out is that when  $R = \mathbf{Z}$ , there is a unique **positive** gcd for any pair  $a, b \in \mathbf{Z}$  and when  $R = F[T]$  there is a unique **monic** gcd for any pair  $f, g \in F[T]$ .

For a general PID there need not be a natural choice of gcd, so for  $x, y \in R$  we can only speak of  $\gcd(x, y)$  up to multiplication by a unit of  $R$ .

**Proposition 4.3.4.** *Let  $R$  be a PID and let  $x, y \in R$  with  $\langle x, y \rangle \neq 0$ .*

(a) *Since  $R$  is a PID, we may write find  $d \in R$  with*

$$\langle d \rangle = \langle x, y \rangle.$$

*Then  $d = \gcd(x, y)$ .*

(b) *In particular,  $d = \gcd(x, y)$  may be written in the form  $d = ux + vy$  for  $u, v \in R$ .*

To prove Proposition 4.3.4 proceed as in the proof of Proposition 4.1.4.

**Proposition 4.3.5.** *Let  $R$  be a PID and let  $a, b \in R$  not both 0. Put  $d = \gcd(a, b)$ , so that  $\frac{a}{d}, \frac{b}{d} \in R$ . Then  $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$ .*

*Proof.* According to Proposition 4.3.4 (b), we may write  $d = ax + by$  for suitable  $x, y \in R$ . Since  $d \mid a$  we know that  $\frac{a}{d} \in R$ ; similarly  $\frac{b}{d} \in R$ . We now see that

$$d = d\frac{a}{d}x + d\frac{b}{d}y = d\left(\frac{a}{d}x + \frac{b}{d}y\right);$$

now applying *cancellation* – i.e. Lemma 3.1.8 – we conclude that

$$1 = \frac{a}{d}x + \frac{b}{d}y.$$

This shows that  $1 \in \left\langle \frac{a}{d}, \frac{b}{d} \right\rangle$ , the ideal generated by  $\frac{a}{d}$  and  $\frac{b}{d}$ . But this implies that  $R \subseteq \left\langle \frac{a}{d}, \frac{b}{d} \right\rangle$  so that  $\langle 1 \rangle = R = \left\langle \frac{a}{d}, \frac{b}{d} \right\rangle$ . According to Proposition 4.3.4 this proves that  $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$  as required.  $\square$

## 5 Prime elements and unique factorization

### 5.1 Irreducible elements

Let  $R$  be a principal ideal domain.

*Definition 5.1.1.* A non-zero element  $p \in R$  is said to be *irreducible* provided that  $p \notin R^\times$  and whenever  $p = xy$  for  $x, y \in R$  then either  $x \in R^\times$  or  $y \in R^\times$ .

*Remark 5.1.2.* Assume that  $p, a \in R$  with  $p$  irreducible. Then either  $\gcd(p, a) = 1$  or  $\gcd(p, a) = p$ .

**Proposition 5.1.3.**  $p \in R$  is irreducible if and only if ( $\clubsuit$ ): whenever  $a, b \in R$  and  $p \mid ab$  then either  $p \mid a$  or  $p \mid b$ .

*Proof.* ( $\Rightarrow$ ): Assume that  $p$  is irreducible, suppose that  $a, b \in R$  and that  $p \mid ab$ . We must show that  $p \mid a$  or  $p \mid b$ .

For this, we may as well suppose that  $p \nmid a$ ; we must then prove that  $p \mid b$ . Since  $p \nmid a$ , we see that  $\gcd(a, p) = 1$  by the Remark above. Then  $ua + vp = 1$  for elements  $u, v \in R$ .

Now we see that

$$b = 1 \cdot b = (ua + vp) \cdot b = uab + vpb.$$

Since  $p \mid ab$  we see that  $p \mid uab + vpb$  which proves that  $p \mid b$ , as required.

( $\Leftarrow$ ): Assume that condition ( $\clubsuit$ ) holds for  $p$ . We must show that  $p$  is irreducible. For this, assume  $p = xy$  for  $x, y \in R$ ; we must show that either  $x \in R^\times$  or  $y \in R^\times$ .

Since  $p = xy$ , in particular  $p \mid xy$  and we may apply ( $\clubsuit$ ) to conclude without loss of generality that  $p \mid x$ .

Write  $x = pa$ . We now see that  $p = xy = pay$ ; by cancellation, find that  $1 = ay$  so that  $y \in R^\times$ . We conclude that  $p$  is irreducible, as required.  $\square$

*Remark 5.1.4.* For any integral domain  $R$ , we can speak of *irreducible elements* defined as in Definition 5.1.1. And we can speak of *prime elements*, where an element  $p \in R$  is *prime* if it satisfies condition ( $\clubsuit$ ) of Proposition 5.1.3. In this language, Proposition 5.1.3 shows that in a PID, an element is prime iff it is irreducible.

**Corollary 5.1.5.** Let  $R$  be a PID, let  $p, a_1, \dots, a_n \in R$  with  $p$  prime, and suppose that  $p \mid a_1 a_2 \cdots a_n = \prod_{i=1}^n a_i$ . Then  $p \mid a_i$  for some  $1 \leq i \leq n$ .

*Example 5.1.6.* Let  $F$  a field and let  $f \in F[T]$  be a non-constant polynomial; i.e.  $\deg(f) > 0$ . If  $f$  is reducible there are polynomials  $g, h \in F[T]$  for which  $f = gh$  and  $\deg(g), \deg(h) > 0$ .

*Example 5.1.7.* If  $f \in F[T]$  is reducible (i.e. not irreducible) then the quotient ring  $F[T]/\langle f \rangle$  is not an integral domain.

Indeed, write  $f = gh$  for  $g, h \in F[T]$  non-units. Thus  $\deg f > \deg g, \deg h > 0$  by Proposition 3.3.2. According to Proposition 3.3.4, the classes  $[g], [h] \in F[T]$  are non-zero, but  $[g] \cdot [h] = [f] = 0$ . Thus  $F[T]/\langle f \rangle$  has zero divisors and is not an integral domain.

### 5.2 Unique factorization in a PID

The Fundamental Theorem of Arithmetic says that any integer  $n > 1$  may factored uniquely as a product of primes. This result holds for any PID, as follows:

**Theorem 5.2.1.** Let  $R$  be a PID, let  $0 \neq a \in R$ , and suppose that  $a$  is not a unit.

- (a) There are irreducible elements  $p_1, p_2, \dots, p_n \in R$  such that  $a = p_1 \cdot p_2 \cdots p_n$ .
- (b) if  $q_1, \dots, q_m \in R$  are irreducibles such that  $a = q_1 \cdots q_m$  then  $n = m$  and – after possibly reordering the  $q_i$  – there are units  $u_i \in R^\times$  for which  $q_i = u_i p_i$  for each  $i$ .

*Proof.* We first prove (a). For this, we first prove the following claim:

(\*) : if the conclusion of (a) fails, there is a sequence of elements  $a_1, a_2, \dots \in R \setminus R^\times$  with the property that for each  $i \geq 1$  we have: (i)  $a_{i+1} \mid a_i$  and (ii)  $a_{i+1}$  and  $a_i$  are not associate.

To prove (\*), let  $x_1 = a$ . Now suppose we have found elements  $a_1, a_2, \dots, a_n$  such that for each  $1 \leq i \leq n$  conditions (i) and (ii) hold, and such that the conclusion of (a) fails for  $a_n$ . In particular,  $a_n$  is reducible, so we may write  $a_n = xy$  with  $x, y \in R$  and  $x, y \notin R^\times$ . Without loss of generality, we may suppose that the conclusion of (a) fails for  $x$  and we set  $a_{n+1} = x$ . By construction,  $a_{n+1} \mid a_n$ ; moreover  $a_{n+1}$  and  $a_n$  are not associates. Thus we have proved by induction that (\*) holds.

To prove (a), we will now show that (\*) leads to a contradiction.

Let  $\{a_i\}$  be a sequence of elements as in (\*) and let  $I$  be given by

$$I = \bigcup_{i \geq 1} \langle a_i \rangle.$$

Since

$$\langle a_1 \rangle \subseteq \langle a_2 \rangle \subseteq \langle a_3 \rangle \subseteq \cdots$$

it is straightforward to see that  $I$  is an ideal. Since  $R$  is a PID, we may write  $I = \langle d \rangle$  for some  $d \in R$ . By the definition of  $I$ , we may find an index  $N$  for which  $d \in \langle a_j \rangle$  for each  $j \geq N$ .

Fix  $j \geq N$ . We may write  $d = x \cdot a_j$  for  $x \in R$ .

On the other hand,  $\langle a_j \rangle \subseteq \langle d \rangle$ , we may write  $a_j = y \cdot d$  for  $y \in R$ .

We now see that  $d = x \cdot a_j = xy d$  so that  $x, y \in R^\times$  by cancellation (Lemma 3.1.8). Thus  $d$  and  $a_j$  are *associates* so that  $\langle d \rangle = \langle a_j \rangle$ . In particular, we have proved that

$$\langle d \rangle = \langle a_N \rangle = \langle a_{N+1} \rangle = \langle a_{N+2} \rangle = \cdots$$

contradicting the assumption (ii) that  $a_{j+1}$  and  $a_j$  are not associates. This contradiction proves (a).

We now prove (b). We are given an equality

$$p_1 \cdots p_n = q_1 \cdots q_m$$

with  $p_i, q_j$  irreducible and  $n, m \geq 1$ .

We proceed by induction on the minimum  $\min(n, m)$ , and without loss of generality we suppose that  $n \leq m$  so that  $n = \min(n, m)$ .

In case  $n = 1$ , our assumption is  $p_1 = q_1 \cdots q_m$ . Applying Corollary 5.1.5 we see that  $p_i \mid q_j$  for some  $1 \leq j \leq m$ . Since  $p_i$  and  $q_j$  are irreducible, we see that  $q_j = u \cdot p_1$  for some unit  $u \in R^\times$ . Thus

$$p_1 = u \cdot p_1 \cdot \prod_{i \neq j} q_i.$$

Applying cancellation (Lemma 3.1.8) we see  $u \cdot \prod_{i \neq j} q_i = 1$  so that  $q_i \in R^\times$  for  $i \neq j$ . Thus  $m = 1$  and  $p_1$  and  $q_1$  are associates, as required. This confirms the base-case of the induction.

Now suppose that  $n > 1$  and that the result is known when the element has an expression as a product of  $< n$  irreducibles.

Thus we have

$$p_1 \cdots p_n = q_1 \cdots q_m$$

and  $m \geq n$ . Now  $p_n \mid q_1 \cdots q_m$  and as before we see for some  $1 \leq j \leq m$  that  $q_j = up_n$  for a unit  $u \in R^\times$ . Without loss of generality we may suppose that  $j = m$ . We find

$$p_1 \cdots p_{n-1} \cdot p_n = u \cdot p_n \cdot q_1 \cdots q_{m-1}$$

Applying cancellation (Lemma 3.1.8) we find that

$$p_1 \cdots p_{n-1} = uq_1 \cdots q_{m-1}$$

Replacing  $q_1$  by the irreducible  $uq_1$ , we can view the right-hand side as a product of  $m - 1$  irreducibles. Since  $m - 1 \geq n - 1$  we may apply the induction hypothesis to find that  $m - 1 = n - 1$  and that after re-ordering we have  $p_i$  associate to  $q_i$  for  $1 \leq i \leq m - 1$ . Since  $p_n$  and  $q_m$  are associate as well, this proves (b).  $\square$

## 6 The Field of fractions of an Integral Domain

Recall Example 3.1.7 that any subring of a field is an integral domain. We now want to argue that the *converse* to this statement is true, as well. Namely, an integral domain  $R$  is a subring of a field. In fact, we are essentially going to give a *construction* of such a field from  $R$ .

Let's fix an integral domain  $R$ . To confirm the suggested converse to the above Corollary, we must construct a field  $F$  and an inclusion  $i : R \subseteq F$ .

Of course, if we have such a mapping  $i$ , then for any  $0 \neq b \in R$ , the element  $i(b)$  is non-zero in  $F$  and hence  $i(b)^{-1} = \frac{1}{i(b)}$  should be an element of  $F$  (even though  $i(b)^{-1}$  is possibly not an element of  $R$ ). For any  $a \in R$  we should be able to multiply  $i(a)$  and  $\frac{1}{i(b)}$  in  $F$  to form the *fraction*  $\frac{i(a)}{i(b)}$ . If we choose to identify  $R$  with the image  $i(R)$ , we might simply write  $\frac{a}{b} = \frac{i(a)}{i(b)}$  for this *fraction*.

So if the field  $F$  *exists*, it must contain all fractions  $\frac{a}{b}$  for  $a, b \in R$  with  $0 \neq b$ .

In fact, we are going to construct a field  $F$  by formally introducing such fractions.

Consider the set  $W = \{(a, b) \mid a, b \in R, b \neq 0\}$  and define a relation  $\sim$  on the set  $W$  by the condition

$$(a, b) \sim (s, t) \iff at = bs.$$

This relation is motivated by the observation that for *fractions* in a field  $F$  we have

$$\frac{a}{b} = \frac{s}{t} \iff at = bs.$$

One needs to check the following:

**Proposition 6.0.1.**  $\sim$  defines an equivalence relation on  $W$ .

*Proof.* We must confirm properties of  $\sim$ :

(*reflexive*) if  $(a, b) \in W$ , then  $ab = ba \implies (a, b) \sim (a, b)$ .

(*symmetric*) if  $(a, b), (s, t) \in W$  then

$$(a, b) \sim (s, t) \implies at = bs \implies sb = ta \implies (s, t) \sim (a, b).$$

(*transitive*) Let  $(a, b), (s, t), (u, v) \in W$  and suppose that  $(a, b) \sim (s, t)$  and  $(s, t) \sim (u, v)$ . The assumptions mean that  $at = bs$  and  $sv = tu$ .

Multiplying the equation  $at = bs$  by  $v$  on each side, we see that

$$atv = bsv \implies atv = btu \implies (av)t = (bu)t;$$

since  $t \neq 0$  and since the cancellation law holds in an integral domain – see Lemma 3.1.8, conclude  $av = bu$ . Hence  $(a, b) \sim (u, v)$  which confirms the transitive law.

□

We are now going to show that the fractions - i.e. the equivalence classes in  $W$  - form a field. We define  $Q = Q(R)$  to be the set of equivalence classes of  $W$  under the equivalence relation  $\sim$ .

We write  $\frac{a}{b} = [(a, b)]$  for the equivalence class of  $(a, b) \in W$ . Thus  $Q$  is the set of (formal) fractions of elements of  $R$ , and

$$\frac{a}{b} = \frac{s}{t} \iff (a, b) \sim (s, t) \iff at = bs$$

It remains to argue that  $Q$  has the structure of a field. To do this, we must define binary operations  $+$  and  $\cdot$  on the set  $Q$  and check that they satisfy the correct axioms.

Define addition of fractions: for  $a, b, s, t \in R$  with  $b, t \neq 0$ ,

$$(\clubsuit) \quad \frac{a}{b} + \frac{s}{t} = \frac{at + bs}{bt}.$$

And define multiplication of fractions:

$$(\diamond) \quad \frac{a}{b} \cdot \frac{s}{t} = \frac{as}{bt}.$$

**Theorem 6.0.2.** *For an integral domain  $R$ , the set  $Q(R)$  of fractions of  $R$  forms a field with the indicated addition and multiplication.*

*Sketch of proof.* What must be checked??

- must first confirm that  $(\clubsuit)$  is *well-defined*! i.e. if  $a', b', s', t' \in R$  with  $\frac{a}{b} = \frac{a'}{b'}$  and  $\frac{s}{t} = \frac{s'}{t'}$ , we must check that  $\frac{a}{b} + \frac{s}{t} = \frac{a'}{b'} + \frac{s'}{t'}$ ; i.e. that

$$\frac{at + bs}{bt} = \frac{a't' + b's'}{b't'}.$$

This is straightforward if a bit tedious.

- One readily checks that  $0 = \frac{0}{1}$  is an identity for the binary operation  $+$  on  $Q$ .
- One readily checks that  $+$  is commutative for  $Q$ .
- One readily checks that  $\frac{-a}{b}$  is an additive inverse for  $\frac{a}{b}$ .
- With some more effort, one confirms that  $+$  is *associative* on  $Q$ ; i.e. for  $\alpha, \beta, \gamma \in Q$

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma).$$

Thus  $(Q, +)$  is an abelian group. Now consider the operation  $\diamond$  of multiplication.

- must again confirm that  $(\diamond)$  is *well-defined*! i.e. if  $a', b', s', t' \in R$  with  $\frac{a}{b} = \frac{a'}{b'}$  and  $\frac{s}{t} = \frac{s'}{t'}$ , we must check that  $\frac{a}{b} \cdot \frac{s}{t} = \frac{a'}{b'} \cdot \frac{s'}{t'}$ ; i.e. that

$$\frac{as}{bt} = \frac{a's'}{b't'}.$$

- One readily checks that  $1 = \frac{1}{1}$  is an identity for the binary operation  $\cdot$  on  $Q$ .
- One readily checks that  $\cdot$  is commutative for  $Q$ .
- With some more effort, one confirms that  $\cdot$  is *associative* on  $Q$ ; i.e. for  $\alpha, \beta, \gamma \in Q$

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

- Next, one must confirm the *distributive law*: for  $\alpha, \beta, \gamma \in Q$ ,

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma.$$

*Phew!* □

*Remark 6.0.3.* Despite the details of the preceding proof, all that is happening is confirming properties of operations of fractions that you have used since grade-school...

Now, we want to emphasize a crucial property of the field of fractions of an integral domain.

Let  $Q(R)$  be the field constructed above, and note that there is a natural ring homomorphism  $i : R \rightarrow Q(R)$  given by  $r \mapsto i(r) = \frac{r}{1}$  for  $r \in R$ . This homomorphism is one-to-one: indeed, if  $\frac{r}{1} = 0 = \frac{0}{1}$ , then  $r \cdot 1 = 0 \cdot 1 \implies r = 0$ . Thus, we may identify  $R$  with a subring of  $Q(R)$ .

**Proposition 6.0.4.** *Let  $R$  be an integral domain, let  $\phi : R \rightarrow S$  be any ring homomorphism, and suppose that for all  $0 \neq d \in R$ ,  $\phi(d) \in S^\times$  - i.e.  $\phi(d)$  is a unit in  $S$ . Then there is a unique homomorphism  $\tilde{\phi} : Q(R) \rightarrow S$  with the property that  $\tilde{\phi}|_R = \phi$ .*

*Proof.* Let  $x \in Q(R)$  be any element. Thus  $x = \frac{a}{b} = \frac{a}{1} \cdot \frac{1}{b}$  for  $a, b \in R$  with  $b \neq 0$ .

Let's first argue that uniqueness of  $\tilde{\phi}$ . If  $\tilde{\phi}$  is a ring homomorphism, then

$$1 = \tilde{\phi}(1) = \tilde{\phi}(b \cdot \frac{1}{b}) = \phi(b)\tilde{\phi}(\frac{1}{b}) \implies \tilde{\phi}(\frac{1}{b}) = \phi(b)^{-1}$$

Since  $\tilde{\phi}$  is a ring homomorphism, we must have

$$(\clubsuit) \quad \tilde{\phi}(x) = \tilde{\phi}(\frac{a}{1})\tilde{\phi}(\frac{1}{b}) = \phi(a) \cdot \phi(b)^{-1}$$

which confirms the uniqueness.

It now only remains to check that the rule  $(\clubsuit)$  determines a ring homomorphism, which is straightforward. □

*Example 6.0.5.* The field of rational functions

Let  $F$  be a field, and consider  $R = F[T]$  the ring of polynomials. This is an integral domain, and its field of fractions  $Q(R)$  is usually written  $F(T)$  and is known as the field of rational functions over  $F$ .

Note that

$$F(T) = \left\{ \frac{f}{g} \mid f, g \in F[T], g \neq 0 \right\};$$

thus elements of  $F(T)$  are fractions  $\frac{f}{g}$  whose numerator and denominator are *polynomials*; we usually call such expressions *rational functions*.

## 7 Irreducible polynomials over a field

### 7.1 Fields as quotient rings

**Proposition 7.1.1.** *Let  $R$  be a PID and let  $p \in R$  be an irreducible element. Then the quotient ring  $A = R/\langle p \rangle$  is a field.*

*Proof.* Let  $\alpha \in A$  be non-zero. To prove that  $A$  is a field, we must show that  $\alpha$  has a multiplicative inverse. Thus  $\alpha$  has the form  $h + \langle p \rangle$  and since  $\alpha \neq 0$  we know that  $p \nmid h$ . Since  $p$  is irreducible, Remark 5.1.2 shows that  $\gcd(p, h) = 1$ .

Thus according to Proposition 4.3.4 there are elements  $x, y \in R$  for which

$$1 = xp + yh$$

Let  $\beta = y + \langle p \rangle \in A$ . Then

$$\alpha\beta = yh + \langle p \rangle = 1 + \langle p \rangle$$

since  $yh \equiv 1 \pmod{p}$ . Thus  $\beta$  is the multiplicative inverse of  $\alpha$  in  $A$ .  $\square$

*Example 7.1.2.* •  $\mathbf{Z}/p\mathbf{Z}$  is a field for a prime number  $p$ .

As a special case of Proposition 7.1.1, we have:

**Corollary 7.1.3.** *Let  $F$  be a field and let  $f$  be an irreducible polynomial in  $F[T]$ . Then  $A = F[T]/\langle f \rangle$  is a field.*

For small degree polynomials, one can confirm irreducibility just by considering roots, as follows:

**Proposition 7.1.4.** *Let  $F$  be a field and let  $f \in F[T]$  be a polynomial with  $\deg(f) \leq 3$ . If  $f$  has no root in  $F$  then  $f$  is irreducible.*

*Proof.* Suppose that  $f$  is reducible, say  $f = gh$  with  $\deg(g), \deg(h) > 0$ . Since  $\deg(f) \leq 3$  and since  $\deg(g) + \deg(h) = \deg(f)$  by Proposition 3.3.2, we see that at least one of  $g$  or  $h$  must have degree 1; without loss of generality we suppose  $\deg(g) = 1$ .

Thus  $g = aT + b$  for  $a, b \in F$  with  $a \neq 0$ . Set  $\alpha = \frac{-b}{a} \in F$  and observe that  $f(\alpha) = g(\alpha)h(\alpha) = 0$ ; thus  $f$  has the root  $\alpha \in F$ .  $\square$

*Example 7.1.5.* Let  $p$  be a prime number. Then the polynomial  $T^2 - p \in \mathbf{Q}[T]$  is *irreducible*. In particular,

$$\mathbf{Q}(\sqrt{p}) = \mathbf{Q}[T]/\langle T^2 - p \rangle$$

is a field.

### 7.2 The rational roots test

**Theorem 7.2.1.** *Let  $R$  be a PID with field of fractions  $F$  and let  $f \in R[T]$ , say*

$$f = a_0 + a_1T + \cdots + a_nT^n$$

*with  $a_i \in R$  and  $a_n \neq 0$ .*

*If  $\alpha = \frac{x}{y} \in F$  is a root of  $f$  for  $x, y \in R$  and  $y \neq 0$  and  $\gcd(x, y) = 1$  then  $x \mid a_0$  and  $y \mid a_n$ .*

*Proof.* Since  $\alpha$  is a root of  $f$  we have the equation

$$0 = f(\alpha) = a_0 + a_1 \left(\frac{x}{y}\right) + \cdots + a_n \left(\frac{x}{y}\right)^n = \sum_{i=0}^n a_i \left(\frac{x}{y}\right)^i$$

in the field  $F$ . Multiplying by the non-zero element  $y^n \in R$  we find the equation

$$0 = a_0 y^n + a_1 x y^{n-1} + \cdots + a_n x^n = \sum_{i=0}^n a_i x^i y^{n-i}$$

in  $R$ .

Thus we see that

$$a_0 y^n = -(a_1 x y^{n-1} + \cdots + a_n x^n) = - \sum_{i=1}^n a_i x^i y^{n-i} = -x \sum_{i=1}^n a_i x^{i-1} y^{n-i}$$

which shows that  $x \mid a_0 y^n$ . Since  $\gcd(x, y) = 1$  also  $\gcd(x, y^n) = 1$ . Now conclude that  $x \mid a_0$ .

Similarly, we see that

$$a_n x^n = - \sum_{i=0}^{n-1} a_i x^i y^{n-i} = -y \sum_{i=0}^{n-1} a_i x^i y^{n-i-1}$$

which shows that  $y \mid a_n x^n$ . Since  $\gcd(x^n, y) = 1$  we conclude that  $y \mid a_n$  as required.  $\square$

*Remark 7.2.2.* Let  $f = \sum_{i=0}^n a_i T^i \in R[T]$  as in the statement of Theorem 7.2.1. According to the theorem, to find a root of  $f$  in the field of fractions  $F$  of  $R$ , we must consider all fractions  $\alpha = \frac{x}{y}$  where  $\gcd(x, y) = 1$ , where  $x$  is a divisor of  $a_0$  and where  $y$  is a divisor of  $a_n$ .

Writing  $a_0 = p_1 p_2 \cdots p_n$  and  $a_n = q_1 q_2 \cdots q_m$  for irreducibles  $p_i$  and  $q_j$ , we see that it is possible in principle to make a list of all possible  $\alpha$  and then check for each candidate whether or not  $\alpha$  is a root of  $f$ .

*Example 7.2.3.* Consider the polynomial  $f = T^3 - 3T^2 + 2T - 6 \in \mathbf{Z}[T]$ . For any root  $\alpha = \frac{x}{y} \in \mathbf{Q}$  with  $\gcd(x, y) = 1$  we must have that  $x \mid 6$  and  $y \mid 1$ . Thus according to Theorem 7.2.1, the possible rational roots are  $\alpha = \pm 1, \pm 2, \pm 3, \pm 6$ .

Notice that if  $x \in \mathbf{R}$  is negative, then  $f(x) < 0$ . Thus the possible rational roots are simple  $\alpha = 1, 2, 3, 6$ . We notice that  $f(1) = -6$ ,  $f(2) = -6$  and  $f(3) = 0$ . Using the division algorithm we see that

$$T^3 - 3T^2 + 2T - 6 = (T^2 + 2)(T - 3)$$

It is now clear that 6 is not a root and that  $T^2 + 2$  is irreducible. We  $f$  has exactly one rational root, namely  $\alpha = 3$ .

### 7.3 The Gauss Lemma

Let  $R$  be a PID with field of fractions  $F$ . The polynomial ring  $R[T]$  is the subring of  $F[T]$  consisting of polynomials whose coefficients lie in  $R$ . In particular  $R[T]$  is itself an integral domain.

*Remark 7.3.1.* Note that in the case where  $R$  is *already* a polynomial ring  $F[X]$ , we introduce a *new* variable  $T$  different from  $X$ .

*Definition 7.3.2.* The *content*  $\text{content}(f)$  of the element  $f = \sum_{i=0}^N a_i T^i \in R[T]$  where  $a_i \in R$  is defined to be

$$\text{content}(f) = \gcd(a_0, a_1, \dots, a_N).$$

We say that the polynomial  $f \in R[T]$  is *primitive* if  $\text{content}(f) = 1$ .

**Lemma 7.3.3.** *Let  $f \in R[T]$  be a non-zero polynomial and let  $c = \text{content}(f) \in R$ . Then  $f$  may be written  $f = cf_0$  where  $f_0 \in R[T]$  is primitive.*

*Proof.* Write  $f = \sum_{i=0}^n a_i T^i$  with  $a_i \in R$ . Then by definition we have  $c = \gcd(a_0, \dots, a_n)$ . Note that  $c \mid a_i$  for each  $i$ ; we write  $b_i = \frac{a_i}{c} \in R$ .

We set  $f_0 = \sum_{i=0}^n b_i T^i \in R[T]$  and notice that

$$c \cdot f_0 = \sum_{i=0}^n c \cdot b_i T^i = \sum_{i=0}^n a_i T^i = f$$

as required. Finally,

$$\text{content}(f_0) = \gcd(b_0, \dots, b_n) = \gcd\left(\frac{a_0}{c}, \dots, \frac{a_n}{c}\right) = 1$$

by Proposition 4.3.5. Thus  $f_0$  is indeed primitive.  $\square$

**Lemma 7.3.4.** *Let  $p \in R$  be irreducible and consider the assignment*

$$h \mapsto \bar{h} : R[T] \rightarrow (R/\langle p \rangle)[T]$$

*defined as follows: for  $h = \sum_{i=0}^N c_i T^i \in R[T]$  with  $c_i \in R$ , the polynomial  $\bar{h} \in (R/\langle p \rangle)[T]$  is given by*

$$\bar{h} = \sum_{i=0}^N [c_i] T^i$$

*where  $[c_i] = c_i + pR$  is the class of  $c_i$  modulo  $pR$ .*

(a) *This assignment is a ring homomorphism.*

(b) *For  $h \in R[T]$ ,  $\bar{h} = 0$  if and only if  $p \mid \text{content}(h)$ .*

*Proof.* (a) follows from Proposition 3.2.1. For (b), just observe that  $\bar{h} = 0$  if and only if  $p \mid c_i$  for every  $i$ .  $\square$

**Proposition 7.3.5.** (*“The Gauss Lemma”*) *If  $f, g \in R[T]$  are primitive, then the product  $fg$  is primitive.*

*Proof.* Suppose on the contrary that there are primitive polynomials  $f, g \in R[T]$  for which  $fg$  is not primitive. Writing  $d = \text{content}(fg)$  for the content of the product, we know that  $\langle d \rangle \neq R$  so that  $d$  is divisible by some prime  $p \in R$ .

Consider the ring homomorphism  $h \mapsto \bar{h}$  of Lemma 7.3.4.

Now,  $p \mid \text{content}(fg) \implies 0 = \overline{fg} = \bar{f} \cdot \bar{g}$ . Since  $R/pR$  is a field, the ring  $(R/pR)[T]$  is an integral domain, so we may conclude that either  $\bar{f} = 0$  or  $\bar{g} = 0$ .

But according to Lemma 7.3.4 (b),  $\bar{f} = 0 \implies p \mid \text{content}(f)$  and  $\bar{g} = 0 \implies p \mid \text{content}(g)$ . This contradicts our assumption that  $1 = \text{content}(f) = \text{content}(g)$ . Thus indeed  $\text{content}(fg) = 1$ .  $\square$

**Theorem 7.3.6.** Suppose that  $f \in R[T]$  is a primitive polynomial, and that  $g, h \in K[T]$  are polynomials for which  $f = gh$  in  $K[T]$ . Then there are polynomials  $g_1, h_1 \in R[T]$  with  $\deg g = \deg g_1$  and  $\deg h = \deg h_1$  for which  $f = g_1 h_1$  in  $R[T]$ .

*Proof.* Using Lemma 7.3.3, we may write  $g = \frac{x}{y}g_1$  and  $h = \frac{z}{w}h_1$  where  $g_1, h_1 \in R[T]$  are primitive and  $x, y, z, w \in R$  with  $y, w \neq 0$ . We now see that

$$(\heartsuit) \quad yw \cdot f = xz \cdot g_1 h_1.$$

Since  $f$  is primitive, notice that  $yw = \text{content}(yw f)$ . Moreover, the Gauss Lemma – i.e. Proposition 7.3.5 – shows that  $g_1 h_1$  is primitive; thus, we have  $\text{content}(xz g_1 h_1) = xz$ .

It follows that

$$\langle yw \rangle = \langle xz \rangle$$

i.e. that  $(\clubsuit) \quad u \cdot yw = xz$  for a unit  $u \in R^\times$  – see Proposition 3.1.9.

But then  $(\heartsuit)$  and  $(\clubsuit)$  together show that  $yw \cdot f = u \cdot yw \cdot g_1 h_1$  and now the cancellation law Lemma 3.1.8 in the integral domain  $R[T]$  implies  $f = (u g_1) \cdot h_1$  which proves the Theorem.  $\square$

## 7.4 Eisenstein's irreducibility criterion

**Theorem 7.4.1.** Let  $p \in R$  be irreducible, and let

$$f = \sum_{i=0}^n a_i T^i \in R[T], \quad (\text{where } a_i \in R, \ 0 \leq i \leq n)$$

be a polynomial with  $a_n \neq 0$ . Suppose that  $p \nmid a_n$ , that  $p \mid a_i$  for  $0 \leq i \leq n-1$  and that  $p^2 \nmid a_0$ . Then  $f$  is irreducible when viewed as an element of  $F[T]$ .

*Proof.* Let  $c = \text{content}(f)$ . Then  $c \not\equiv 0 \pmod{p}$  since  $p \nmid a_n$ . Observe now that the polynomial  $\tilde{f} = \frac{1}{c}f \in R[T]$  still satisfies the assumptions of the Theorem. Since  $\tilde{f}$  is irreducible in  $K[T]$  if and only if the same is true for  $f$ , it suffices to prove the Theorem when  $f = \tilde{f}$  is primitive.

Now, according to Theorem 7.3.6 the irreducibility of  $f \in F[T]$  will follow once we show that if  $f = gh$  for  $g, h \in R[T]$  then either  $\deg g = 0$  or  $\deg h = 0$ . So suppose  $f = gh$  for  $g, h \in R[T]$ .

Consider the ring homomorphism  $f \mapsto \bar{f} : R[T] \rightarrow (R/pR)[T]$  as in Lemma 7.3.4. Assumptions on the coefficients  $a_i$  show  $\bar{f} = \bar{g}\bar{h}$  to be a non-zero multiple of  $T^n$ . Using unique factorization in the principal ideal domain  $(R/pR)[T]$  – see Theorem 5.2.1 –, it follows that  $\bar{g}$  is a non-zero multiple of  $T^i$  and  $\bar{h}$  is a non-zero multiple of  $T^j$  where  $i + j = n$  and  $0 \leq i, j \leq n$ . Moreover  $i = \deg g$  and  $j = \deg h$ .

Now the Theorem follows since if  $i, j > 0$  then  $p$  divides the constant term of both  $g$  and  $h$ , and then  $p^2 \mid a_0$  contradicting our assumption.  $\square$

*Example 7.4.2.* (a) Let  $p$  be a prime integer, let  $n \geq 1$  and let  $f = T^n - p$ . Then Theorem 7.4.1 shows that  $f \in \mathbf{Q}[T]$  is irreducible.

(b) Let  $K$  be a field and consider the ring  $K[X]$  of polynomials over  $K$ . The field of fractions of  $K[X]$  is the field  $F = K(X)$  of rational functions.

Let  $n \geq 1$  and consider the polynomial  $f = T^n - X \in F[T] = K(X)[T]$ . Then  $f$  is irreducible in  $K(X)[T]$  by Theorem 7.4.1.

## 7.5 Irreducibility of certain cyclotomic polynomials

For a prime number  $p$  consider the polynomial

$$F(T) = F_p(T) = \frac{T^p - 1}{T - 1} = T^{p-1} + T^{p-2} + \cdots + T + 1 \in \mathbf{Q}[T].$$

Applying the change of variables  $U = T - 1$  we see that

$$\begin{aligned} F(U+1) &= \frac{(U+1)^p - 1}{(U+1) - 1} = \frac{\sum_{i=1}^p \binom{p}{i} U^i}{U} \\ &= \frac{U^p + \binom{p}{p-1} U^{p-1} + \cdots + \binom{p}{2} U^2 + \binom{p}{1} U}{U} \\ &= U^{p-1} + \binom{p}{p-1} U^{p-2} + \cdots + \binom{p}{2} U + p \end{aligned}$$

In particular,  $g(U) = F(U+1) = \sum_{i=0}^{p-1} c_i U^i \in \mathbf{Q}[U]$  has degree  $p-1$  and the coefficients are given by the formulae

$$c_i = \binom{p}{i+1}, \quad 0 \leq i \leq p-1.$$

**Proposition 7.5.1.** *For a prime number  $p > 0$ , the polynomial*

$$F(T) = \frac{T^p - 1}{T - 1} = T^{p-1} + T^{p-2} + \cdots + T + 1 \in \mathbf{Q}[T]$$

*of degree  $p-1$  is irreducible.*

*Proof.* Clearly  $F(T) \in \mathbf{Q}[T]$  is irreducible if and only if  $g(U) \in \mathbf{Q}[U]$  is irreducible. Now,  $g(U) \in \mathbf{Z}[U]$  since binomial coefficients  $\binom{n}{m}$  are always integers. We are going to apply Eisenstein's criteria to show the irreducibility of  $g(U)$ . For this, we first note that  $c_{p-1} = 1$  is not divisible by  $p$  and that  $c_0 = p$  is divisible by  $p$  but not by  $p^2$ .

The irreducibility will now follow from Theorem 7.4.1 once we argue that  $(\clubsuit) : p \mid \binom{p}{i}$  for each  $1 \leq i \leq p-1$ .

To prove  $(\clubsuit)$  just note that

$$\binom{p}{i} = \frac{p!}{i!(p-i)!}.$$

Since  $0 < i < p$ , neither  $i!$  nor  $(p-i)!$  is divisible by  $p$ . On the other hand

$$p! = p \cdot (p-1) \cdot (p-2) \cdots 2 \cdot 1$$

is divisible by  $p$ .

Since one knows that  $\binom{p}{i} \in \mathbf{Z}$ , unique factorization – see Section 5.2 – implies that  $p \mid \binom{p}{i}$  as required.  $\square$

*Example 7.5.2.* For example,  $f(T) = T^4 + T^3 + T^2 + T + 1 \in \mathbf{Q}[T]$  is an irreducible since  $f(T) = \frac{T^5 - 1}{T - 1}$  and since  $p = 5$  is prime.

## 8 Some recollections of Linear Algebra

Let  $F$  be a field. Much of what you learned in a course on linear algebra remains valid for vector spaces over  $F$  and not just for vector spaces over  $\mathbf{R}$  or  $\mathbf{C}$ .

### 8.1 Vector Spaces

*Definition 8.1.1.* A *vector space* over  $F$  is an additive abelian group  $V$  together with a mapping

$$F \times V \rightarrow V$$

denoted by

$$(\alpha, v) \mapsto \alpha v$$

called *scalar multiplication* that is required to satisfy several axioms:

(VS1) the multiplicative identity  $1 = 1_F \in F$  satisfies  $1 \cdot v = v$  for all  $v \in V$ .

(VS2) scalar multiplication is associative: for all  $\alpha, \beta \in F$  and all  $v \in V$ , we have  $\alpha(\beta v) = (\alpha\beta)v$ .

(VS3) scalar multiplication distributes over addition in  $V$ : for all  $\alpha, \beta \in F$  and for all  $v, w \in V$ , we have

$$\alpha \cdot (v + w) = \alpha \cdot v + \alpha \cdot w$$

and

$$(\alpha + \beta) \cdot v = \alpha v + \beta v.$$

You should compare these requirements with axioms you may have seen in a course in linear algebra. The present list is probably shorter – that is because one needs axioms governing the behavior of addition, which we have handled by requiring  $V$  to be an additive abelian group.

### 8.2 Linear Transformations, subspaces and quotient vector spaces

*Definition 8.2.1.* Let  $V$  be a vector space over  $F$ . A subset  $W \subseteq V$  is called a **subspace** (or more precisely, an  $F$ -subspace) provided that

- (a)  $W$  is an additive subgroup of  $V$ , and
- (b)  $W$  is closed under scalar multiplication by  $F$  – i.e.

$$\alpha w \in W \quad \text{for all } \alpha \in F \text{ and all } w \in W.$$

*Definition 8.2.2.* If  $V$  and  $W$  are vector spaces over  $F$ , a function  $T : V \rightarrow W$  is a *linear transformation* (or more precisely, an  $F$ -linear transformation) if

- (a)  $T$  is a homomorphism of additive groups  $V \rightarrow W$ , and
- (b)  $T$  commutes with scalar multiplication – i.e.  $T(\alpha v) = \alpha T(v)$  for all  $\alpha \in F$  and all  $v \in V$ .

*Definition 8.2.3.* If  $V, W$  are vector spaces, a linear transformation  $T : V \rightarrow W$  is an *isomorphism* if there is a linear transformation  $S : W \rightarrow V$  such that  $T \circ S = 1_W$  and  $S \circ T = 1_V$ .

If  $T$  is an isomorphism, one says that  $V$  and  $W$  are isomorphic vector spaces.

**Proposition 8.2.4.** *Let  $V, W$  be  $F$ -vector spaces and let  $T : V \rightarrow W$  be a linear transformation. Then  $T$  is an isomorphism if and only if  $T$  is bijective.*

*Proof.* Suppose that  $T$  is bijective. Then we know that  $T$  is an isomorphism of additive groups, and hence there is an inverse isomorphism  $S : W \rightarrow V$ . It only remains to show that  $S$  is a linear transformation (rather than simply a group homomorphism).

So let  $\alpha \in F$  and  $w \in W$ . Since  $T$  is onto, we may write  $w = T(v)$  for some  $v \in V$ . Now,

$$S(\alpha w) = S(\alpha T(v)) = S(T(\alpha v)) = 1_W(\alpha v) = \alpha v = \alpha S(T(v)) = \alpha S(w).$$

On the other hand, if  $T$  is an isomorphism, then the inverse isomorphism  $S$  is an inverse function to  $T$  so in particular  $T$  is one-to-one and onto.  $\square$

**Proposition 8.2.5.** *If  $T : V \rightarrow W$  is a linear transformation, then*

- (a)  $\ker(T)$  is a subspace of  $V$ , and
- (b) the image  $T(V) = \{T(v) \mid v \in V\}$  is a subspace of  $W$ .

*Proof.* Exercise!  $\square$

**Proposition 8.2.6.** *Let  $W$  be a subspace of the  $F$ -vector space  $V$ . The quotient group  $V/W$  has the structure of an  $F$ -vector space, and the natural quotient mapping  $\pi : V \rightarrow V/W$  given by  $\pi(v) = v + W$  is an  $F$ -linear transformation.*

*Proof.* We must define a scalar multiplication for the additive group  $V/W$ . Given  $\alpha \in F$  and an element  $v + W \in V/W$ , define

$$\alpha \cdot (v + W) = (\alpha v) + W.$$

We must confirm that this rule is independent of the choice of coset representative  $v$  for  $v + W$ . Thus, we must suppose that

$$v + W = v' + W$$

and we must show that  $\alpha \cdot (v + W) = \alpha \cdot (v' + W)$  i.e. that  $\alpha v + W = \alpha v' + W$ .

The assumption that  $v + W = v' + W$  means that  $v - v' \in W$ . Since  $W$  is a  $F$ -subspace, we find that  $\alpha(v - v') \in W$  and using the distributive law we conclude that  $\alpha v - \alpha v' \in W$ . This shows that  $\alpha v + W = \alpha v' + W$  as required. This proves that we've given a well-defined operation of scalar multiplication.

It now remains to check that the associative and distributive laws hold for this operation. Since these properties hold for the scalar multiplication in  $V$ , the verification is straightforward; details are left to the reader.  $\square$

**Proposition 8.2.7.** *If  $T : V \rightarrow W$  is a linear transformation, there is an isomorphism  $\tilde{T} : V/\ker(T) \rightarrow T(V)$  given by  $\tilde{T}(v + \ker T) = T(v)$  for  $v \in V$ .*

*Proof.* The first isomorphism theorem for groups tells us that the rule  $\tilde{T}$  is an isomorphism of groups. In view of @prop:inv-iso, it remains to argue that  $\tilde{T}$  is a linear transformation.

Thus, let  $\alpha \in F$  and  $x \in V/\ker T$ . We may write  $x = v + \ker T$  for some  $v \in V$ . Now, by definition we have

$$\alpha x = \alpha(v + \ker T) = \alpha v + \ker T.$$

Thus, since  $T$  is a linear transformation we find the following:

$$\tilde{T}(\alpha x) = \tilde{T}(\alpha v + \ker T) = T(\alpha v) = \alpha T(v) = \alpha \tilde{T}(v + \ker T).$$

This confirms that  $\tilde{T}$  commutes with scalar multiplication and is thus a linear transformation.  $\square$

### 8.3 Bases and dimension

You are probably familiar with the notions of *spanning set* and of *linear independence*. One issue to be aware of is how to handle possibly-infinite sets in this setting.

To quote from Michael Artin's algebra text (Artin 2011):

In algebra it is customary to speak only of linear combinations of finitely many vectors. Therefore, the span of an infinite set  $S$  must be interpreted as the set of those vectors  $V$  which are linear combinations of finitely many elements of  $S$ ...

**Definition 8.3.1.** If  $S \subseteq V$  is a set of elements, the span of  $S$  is defined to be

$$\text{span}(S) = \left\{ \sum_{i=1}^r a_i x_i \mid r \in \mathbf{Z}_{\geq 0}, a_i \in F, x_i \in V (1 \leq i \leq r) \right\}$$

It is clear that  $\text{span}(S)$  is a *subspace* of  $V$ .

**Definition 8.3.2.** A subset  $S \subseteq V$  of the vector space  $V$  is said to be *linearly independent* if whenever  $n \in \mathbf{Z}_{\geq 0}$ , whenever  $x_1, \dots, x_n \in V$  are *distinct* elements of  $V$ , and whenever  $\alpha_1, \dots, \alpha_n \in F$  then

$$\sum_{i=1}^n \alpha_i x_i = 0 \implies \alpha_j = 0 \quad \text{for each } 1 \leq j \leq n.$$

**Remark 8.3.3.** We say that the vector space is *finitely generated* if there is a *finite* set  $S \subseteq V$  for which  $V = \text{span}(S)$ . In fact,  $V$  is then *finite dimensional* (see Definition 8.3.6 below).

**Definition 8.3.4.** Let  $V$  be a vector space over the field  $F$ . A *basis* for  $V$  is a subset  $S \subseteq V$

- (a)  $S$  spans  $V$ ; i.e.  $V = \text{span}(S)$ , and
- (b)  $S$  is linearly independent.

**Proposition 8.3.5.** *Let  $V$  be an  $F$ -vector space.*

- (a) *There is a basis  $\mathcal{B}$  for  $V$ .*
- (b) *If  $W \subseteq V$  is a subspace of  $V$ , and if  $\mathcal{C}$  is a basis for  $W$ , there is a basis  $\mathcal{B}$  for  $V$  with  $\mathcal{C} \subseteq \mathcal{B}$ .*
- (c) *If  $V = \text{span}(S)$  then there is a basis of  $V$  contained in  $S$ .*
- (d) *If  $S \subseteq V$  is a linearly independent subset, there is a basis of  $V$  containing  $S$ .*
- (e) *Any two bases of  $V$  have the same cardinality.*

*Proof.* When  $V$  is finitely generated, results (a)-(e) can be found in (Hoffman and Kunze 1971), §2.2 and 2.3, and in (Friedberg, Insel, and Spence 2002) §1.6.

For the general case of (a)-(d) see (Friedberg, Insel, and Spence 2002) §1.7.

A proof of (e) in case  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are *infinite* bases for  $V$  requires the Schroeder-Bernstein Theorem; we won't need this result in the course.  $\square$

*Definition 8.3.6.* If  $V$  is a vector space with basis  $\mathcal{B}$ , the *dimension* of  $V$

- written  $\dim V$  or  $\dim_F V$  - is equal to the cardinality of the set  $\mathcal{B}$ .

It follows from Proposition 8.3.5 (e) that the dimension of  $V$  doesn't depend on the choice of basis.

**Proposition 8.3.7.** *Let  $V, W$  be  $F$ -vector spaces, let  $\mathcal{B}$  be a basis for  $V$ , and let  $x_b \in W$  for each  $b \in \mathcal{B}$ . Then there is a unique linear transformation  $T : V \rightarrow W$  such that  $T(b) = x_b$  for each  $b \in \mathcal{B}$ .*

*Example 8.3.8.* Let  $F[T]$  be the polynomial ring over the field  $F$ . Then  $F[T]$  is in particular a vector space over  $F$  with countably infinite basis given by  $\{T^i \mid i \geq 0\}$ .

The linear independence of this basis precisely means that if  $f = \sum_{i=0}^N a_i T^i \in F[T]$  for  $a_i \in F$ , then  $f = 0$  if and only if all  $a_i = 0$ .

**Proposition 8.3.9.** *Let  $T : V \rightarrow W$  be a linear transformation of  $F$ -vector spaces with  $\dim V < \infty$ . Then*

$$\dim_F V = \dim_F T(V) + \dim_F \ker(V).$$

## 9 Field extensions

*Definition 9.0.1.* Let  $F$  and  $E$  be fields and suppose that  $F \subseteq E$  is a *subring*. We say that  $F$  is a *subfield* of  $E$  and that  $E$  is a *field extension* of  $F$ .

Throughout this discussion, let  $F \subseteq E$  be an extension of fields.

### 9.1 Algebraic extensions of fields

*Definition 9.1.1.* An element  $\alpha \in E$  is said to be *algebraic* over  $F$  provided that there is some polynomial  $0 \neq f \in F[T]$  for which  $\alpha$  is a root – i.e. for which  $f(\alpha) = 0$ .

If  $\alpha$  is not algebraic over  $F$ , we say that  $\alpha$  is *transcendental* over  $F$ .

*Example 9.1.2.* • it is a fact that  $\pi, e \in \mathbf{R}$  are transcendental over  $\mathbf{Q}$ .

- Of course,  $\pi, e$  are algebraic over  $\mathbf{R}$ .
- Any element  $\alpha = a + bi \in \mathbf{C}$  (for  $a, b \in \mathbf{R}$ ) is algebraic over  $\mathbf{R}$ . Indeed,  $\alpha$  is a root of the polynomial

$$\begin{aligned} f(T) &= (T - \alpha)(T - \bar{\alpha}) \\ &= T^2 - 2\operatorname{Re}(\alpha)T + |\alpha|^2 \\ &= T^2 - 2aT + (a^2 + b^2) \in \mathbf{R}[T] \end{aligned}$$

where  $\operatorname{Re}(\alpha) = a$  denotes the *real part* of the complex number  $\alpha$ .

### 9.2 The minimal polynomial

**Proposition 9.2.1.** *Let  $\alpha \in E$  and suppose that  $\alpha$  is algebraic over  $F$ . Then there is a unique monic irreducible polynomial  $p \in F[T]$  for which  $\alpha$  is a root.*

Moreover,

(a)  *$p$  is the monic polynomial of smallest degree for which  $\alpha$  is a root.*

(b) *if  $f \in F[T]$  is any polynomial with  $f(\alpha) = 0$ , then  $p \mid f$ .*

*Proof.* Let  $I = \{f \in F[T] \mid f(\alpha) = 0\}$ . It is straightforward to check that it is an additive subgroup, and it is closed under multiplication with any polynomial in  $F[T]$ ; thus  $I$  is an ideal of  $F[T]$ .

Since  $\alpha$  is algebraic,  $I \neq \{0\}$ . Thus  $I$  coincides with the principal ideal  $I = \langle p \rangle$  for some monic  $0 \neq p \in F[T]$ , and  $p$  is the unique monic element of smallest degree in  $I$ .

It only remains to argue that  $p$  is irreducible. Suppose that  $f, g \in F[T]$  and that  $p \mid fg$ . We need to argue that  $p \mid f$  or  $p \mid g$ . Well, since  $fg = pq$  for  $q \in F[T]$ , we see that

$$0 = (pq)(\alpha) = (fg)(\alpha) = f(\alpha) \cdot g(\alpha).$$

Since  $f(\alpha), g(\alpha)$  are elements of the field  $E$ , the only way their product can be 0 is for at least one factor to be zero – i.e. either  $f(\alpha) = 0$  or  $g(\alpha) = 0$ . But then either  $f \in I$  or  $g \in I$  and thus  $p \mid f$  or  $p \mid g$ .  $\square$

**Corollary 9.2.2.** *Let  $\alpha \in E$ . If  $p \in F[T]$  is irreducible and monic, and if  $p(\alpha) = 0$ , then  $p$  is the minimal polynomial of  $\alpha$  over  $F$ .*

*Definition 9.2.3.* Let  $\alpha \in E$  be algebraic over  $F$ .

- The irreducible polynomial  $p$  of the proposition is known as the *minimal polynomial* of  $\alpha$  over  $F$ .
- The *degree* of  $\alpha$  over  $F$  is defined to be the degree of the minimal polynomial  $p$ .

*Example 9.2.4.* An element  $\alpha \in F$  has degree 1 over  $F$ , since it is the root of the irreducible degree 1 polynomial  $T - \alpha \in F[T]$ .

*Example 9.2.5.* Consider the complex number  $z = a + bi \in \mathbf{C}$  with  $a, b \in \mathbf{R}$ . Then  $z$  has degree  $\leq 2$  over  $\mathbf{R}$ , and that degree is 2 if and only if  $b \neq 0$ .

Indeed, if  $b = 0$ , then  $z = a \in \mathbf{R}$  is a root of  $T - a \in \mathbf{R}[T]$  so  $z$  has degree 1 over  $\mathbf{R}$ . Otherwise,  $z$  is a root of

$$p = (T - z)(T - \bar{z}) = T^2 - 2aT + (a^2 + b^2) \in \mathbf{R}[T].$$

Since  $p$  has roots  $z, \bar{z}$ , it has no real roots; since it has degree 2,  $p$  is irreducible over  $\mathbf{R}$ . Now the Corollary shows that  $p$  is the minimal polynomial of  $z$ .

*Example 9.2.6.* Let  $F$  be a field and let  $F(X)$  be the field of fractions  $Q(F[X])$  of the polynomial ring  $F[X]$ .

$F(X)$  is often called the field of rational functions over  $F$ ; its elements have the form

$$\frac{f}{g} = \frac{f(X)}{g(X)} \quad \text{for } f, g \in F[X]$$

Then the element  $X \in F(X)$  is *transcendental* over  $F$ .

Indeed, given any non-zero polynomial  $f(T) \in F[T]$ , we wonder: is  $f(X) = 0$ ? and of course, the answer is “no” because  $f(X)$  is just the polynomial  $f(T)$  after the substitution  $T \mapsto X$ .

In particular, the degree of  $X$  over  $F$  is undefined (or we could define it to be  $\infty$ ).

*Example 9.2.7.* Consider the field  $F = \mathbf{Q}(\sqrt{2})$  defined by adjoining to  $\mathbf{Q}$  a root of  $T^2 - 2$ . We identify  $F$  with a subfield of  $\mathbf{R}$ .

Consider the polynomial  $p(T) = T^4 - 2$  and write  $\alpha = 2^{1/4}$  for the positive real root of  $p(T)$ . Since  $p \in \mathbf{Q}[T]$  is irreducible,  $\alpha$  has degree 4 over  $\mathbf{Q}$ .

On the other hand,  $\alpha$  has degree 2 over  $F$ . Indeed, note that in  $F[T]$ ,

$$p(T) = T^4 - 2 = (T^2 - \sqrt{2})(T^2 + \sqrt{2}).$$

Since  $\alpha$  is a root of  $T^2 - \sqrt{2} \in F[T]$ , the degree of  $\alpha$  over  $F$  is  $\leq 2$ . To see that equality holds, we must argue that  $T^2 - \sqrt{2}$  is irreducible over  $F$ .

To establish this irreducibility, we will argue that  $T^2 - \sqrt{2}$  has no root in  $F$ .

A typical element of  $F$  has the form  $x = a + b\sqrt{2}$  for  $a, b \in \mathbf{Q}$ .

Suppose that

$$(\diamond) \quad \sqrt{2} = x^2 = (a + b\sqrt{2})^2 = a^2 + 2b^2 + 2ab\sqrt{2}.$$

But then comparing coefficients we see that  $a^2 + 2b^2 = 0$  and  $2ab = 1$ .

Now

$$a^2 + 2b^2 = 0 \implies a = b = 0 \implies 2ab \neq 1.$$

Thus the assumption ( $\diamond$ ) is impossible and so

$$T^2 - \sqrt{2} \in F[T] = \mathbf{Q}(\sqrt{2})[T]$$

is indeed irreducible.

We repeat for emphasis:

- the minimal polynomial of  $\alpha$  over  $\mathbf{Q}$  is  $T^4 - 2$  and has degree 4,
- the minimal polynomial of  $\alpha$  over  $\mathbf{Q}(\sqrt{2})$  is  $T^2 - \sqrt{2}$  and has degree 2.

### 9.3 Generation of extensions and primitive extensions

*Definition 9.3.1.* Let  $S \subseteq E$  be a subset. The smallest subfield of  $E$  containing  $F$  and  $S$  is denoted by  $F(S)$ . If  $S = \{u_1, u_2, \dots, u_n\}$  is a finite set, we often write  $F(S) = F(u_1, \dots, u_n)$  for this field.

If  $E = F(u_1, \dots, u_n)$  we say that the elements  $u_i$  *generate* the extension  $E$  of  $F$ .

If  $n = 1$ , the extension  $F(u) = F(u_1)$  of  $F$  is said to be a *primitive extension* (or sometimes: a *simple extension*).

*Remark 9.3.2. Remark:* Note that  $F(S)$  is equal to the intersection

$$F(S) = \bigcap_{K \in \mathcal{E}} K$$

of the collection

$$\mathcal{E} = \{K \subseteq E \mid K \text{ a subfield of } E \text{ containing } F \text{ and } S\}.$$

Since the intersection of subfields is again a subfield (check!), the notation  $F(S)$  is meaningful.

*Remark 9.3.3.* Note that by definition

$$F(u_1, u_2, \dots, u_n) = F(u_1, u_2, \dots, u_{n-1})(u_n).$$

So to “describe” the extension  $F \subseteq F(u_1, \dots, u_n)$  we can focus on describing primitive extensions. Given a description of primitive extensions, we can first describe the extension  $F \subset F(u_1)$  of  $F$ , next we can describe the extension  $F(u_1) \subset F(u_1)(u_2)$  of  $F(u_1)$ , and so on.

**Proposition 9.3.4.** *Let  $\alpha \in E$ .*

*a. If  $\alpha$  is algebraic over  $F$  with minimal polynomial  $p \in F[T]$  over  $F$ , then*

$$F(\alpha) \simeq F[T]/\langle p \rangle,$$

*where  $\alpha$  identifies with  $T + \langle p \rangle$ .*

*In particular,  $F(\alpha)$  has as an  $F$ -basis the elements*

$$1, \alpha, \dots, \alpha^{n-1}$$

*where  $n = \deg p = \deg \alpha$ .*

b. If  $\alpha$  is transcendental over  $F$ , then  $F(\alpha) \simeq F(T)$  where  $F(T)$  is the field of fractions of the polynomial ring  $F[T]$ .

*Proof.* Construct the homomorphism

$$\phi : F[T] \rightarrow E \quad \text{such that } \phi|_F \text{ is the identity, and } \phi(T) = \alpha.$$

We are going to argue in both case (a) and (b) that  $\phi$  induces the desired isomorphism.

First consider case (a). Suppose that  $\alpha$  is algebraic with minimal polynomial  $p$ . The previous Proposition now shows that  $\ker \phi = \langle p \rangle$ .

Since  $p$  is irreducible, the quotient  $F[T]/\langle p \rangle$  is a *field*. According to the first isomorphism theorem,  $\phi$  induces an isomorphism between  $F[T]/\langle p \rangle$  and its image  $K$ . Thus  $K \subseteq E$  is a subfield containing  $F$  and  $\alpha$ , so by definition  $F(\alpha) \subseteq K$ .

On the other hand,  $\alpha$  identifies with the class  $T + \langle p \rangle$ , and so we've seen that the elements  $1, \alpha, \dots, \alpha^{n-1}$  form an  $F$ -basis for  $K$  viewed as a vector space over  $F$ . Now, any subfield  $K_1$  of  $E$  containing  $F$  and  $\alpha$  must contain all  $F$ -linear combinations of the elements  $\alpha^i$ ; thus  $K \subseteq K_1$  and this proves that

$$K \subset F(\alpha) = \bigcap_{K_1 \in \mathcal{E}} K_1.$$

We now conclude that  $K = F(\alpha)$  as required.

Now consider case (b). The condition that  $\alpha$  is transcendental is equivalent to the requirement that  $\ker \phi = \{0\}$ .

Thus for any non-zero polynomial  $f \in F[T]$ ,  $\phi(f) = f(\alpha)$  is a non-zero element of  $F(\alpha)$ . In particular,  $f(\alpha)^{-1} \in E$ .

Now the *defining property* of the field of fractions gives a unique ring homomorphism  $\tilde{\phi} : F(T) \rightarrow E$  for which  $\tilde{\phi}|_{F[T]} = \phi$ .

Since  $F(T)$  is a field,  $\tilde{\phi}$  is one-to-one, and its image is a subfield of  $E$  containing  $\alpha$ . On the other hand, any subfield of  $E$  containing  $\alpha$  must contain the image of  $\tilde{\phi}$  and statement (b) follows at once.  $\square$

*Example 9.3.5.* For any transcendental number  $\gamma \in \mathbf{R}$ , the subfield  $\mathbf{Q}(\gamma)$  of  $\mathbf{R}$  is isomorphic to the field  $\mathbf{Q}(T)$  of rational functions.

In particular, Proposition 9.3.4 shows that there is an isomorphism  $\mathbf{Q}(e) \simeq \mathbf{Q}(\pi)$ .

*Remark 9.3.6.* Here is a question we'll answer in an upcoming lecture. As before, let  $F \subseteq E$  be a field extension.

If  $\alpha, \beta \in E$  are algebraic over  $F$ , is  $\alpha + \beta$  algebraic over  $F$ ? How about  $\alpha \cdot \beta$ ?

*Example 9.3.7.* Let  $E = \mathbf{Q}[T]/\langle T^3 - 2 \rangle$  and let  $\gamma = T + \langle T^3 - 2 \rangle$ . Of course,  $E \simeq \mathbf{Q}(\sqrt[3]{2})$  and under this isomorphism,  $\gamma$  is mapped to  $\sqrt[3]{2}$ . Put another way,  $\gamma$  is a root of  $T^3 - 2$  in  $F$ .

We recall that since  $T^3 - 2$  has degree 3,  $E$  has dimension 3 as a  $\mathbf{Q}$ -vector space, and  $\{1, \gamma, \gamma^2\}$  is a  $\mathbf{Q}$ -basis for  $E$ .

For an element  $\alpha = a + b\gamma + c\gamma^2$  consider the  $\mathbf{Q}$ -linear mapping

$$\lambda_\alpha : E \rightarrow E$$

given by the left multiplication with  $\alpha$ ; i.e. by the rule  $\lambda_\alpha(\beta) = \alpha \cdot \beta$  for  $\beta \in E$ .

We are going to compute the *matrix* of  $\lambda_\alpha$  in the above basis for  $E$ . For this, note that the choice of basis determines a linear isomorphism  $\phi : E \rightarrow \mathbf{Q}^3$  given by  $\phi(s + t\gamma + u\gamma^2) = \begin{bmatrix} s \\ t \\ u \end{bmatrix}$ .

So we are looking for a  $3 \times 3$  matrix  $M = M_\alpha$  with the property that

$$\phi(\lambda_\alpha(\beta)) = M \cdot \phi(\beta).$$

- $\lambda_\alpha(1) = \alpha$  so that  $\phi(\lambda_\alpha(1)) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ . This is the first column of  $M$ .

- $\lambda_\alpha(\gamma) = \alpha\gamma = a\gamma + b\gamma^2 + c\gamma^3 = a\gamma + b\gamma^2 + 2c = 2c + a\gamma + b\gamma^2$  so that  $\phi(\lambda_\alpha(\gamma)) = \begin{bmatrix} 2c \\ a \\ b \end{bmatrix}$ .

This is the second column of  $M$ .

- $\lambda_\alpha(\gamma^2) = \alpha\gamma^2 = a\gamma^2 + b\gamma^3 + c\gamma^4 = a\gamma^2 + 2b + 2c\gamma = 2b + 2c\gamma + a\gamma^2$  so that  $\phi(\lambda_\alpha(\gamma^2)) = \begin{bmatrix} 2b \\ 2c \\ a \end{bmatrix}$ .

This is the third column of  $M$ .

Thus

$$M = M_\alpha = M_{a+b\gamma+c\gamma^2} = \begin{bmatrix} a & c & 2b \\ b & a & 2c \\ c & b & a \end{bmatrix}$$

We claim for  $\alpha_1, \alpha_2 \in E$  that  $M_{\alpha_1+\alpha_2} = M_{\alpha_1} + M_{\alpha_2}$  and  $M_{\alpha_1 \cdot \alpha_2} = M_{\alpha_1} \cdot M_{\alpha_2}$ . Since  $M_\alpha$  is the matrix determined by the linear transformation  $\lambda_\alpha$ , our claim will follow if we just observe that  $\lambda_{\alpha_1} + \lambda_{\alpha_2} = \lambda_{\alpha_1+\alpha_2}$  and  $\lambda_{\alpha_1} \circ \lambda_{\alpha_2} = \lambda_{\alpha_1 \cdot \alpha_2}$  (where  $\circ$  denotes the *composition* of linear transformations). But for  $\beta \in E$  notice that  $\lambda_{\alpha_1} \circ \lambda_{\alpha_2}(\beta) = \lambda_{\alpha_1}(\alpha_2\beta) = \alpha_1\alpha_2\beta = \lambda_{\alpha_1\alpha_2}(\beta)$ ; the other verification is similarly straightforward.

This proves that  $\alpha \mapsto M_\alpha$  determines a *ring homomorphism*

$$E \rightarrow \text{Mat}_{3 \times 3}(\mathbf{Q})$$

Consider the element  $1 + \gamma \in E$  and notice that  $M_{1+\gamma} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ .

Now, we can compute the inverse matrix  $M_{1+\gamma}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$  which we recognize

as the matrix  $M_{(1-\gamma+\gamma^2)/3}$ .

Thus we see that

$$\frac{1}{1+\gamma} = \frac{1}{3}(1-\gamma+\gamma^2)$$

\

## 9.4 The degree of a field extension

*Definition 9.4.1.* We write  $[E : F] = \dim_F E$  and say that  $[E : F]$  is the *degree* of the extension  $F \subseteq E$ .

If  $E$  is *not* a finite dimensional vector space over  $F$ , then  $[E : F] = \dim_F E = \infty$ .

**Proposition 9.4.2.** *Let  $\alpha \in E$ . Then  $\alpha$  is algebraic over  $F$  if and only if  $[F(\alpha) : F] < \infty$ .*

*Remark 9.4.3.* If  $\alpha$  is transcendental, the cardinality of an  $F$ -basis for  $F(\alpha)$  fails to be countable if  $F$  is uncountable. Indeed, you can show that the elements

$$\left\{ \frac{1}{T-a} \in F(T) \mid a \in F \right\}$$

are linearly independent.

**Proposition 9.4.4.** *Let  $E$  be an extension of the field  $F$  and let  $\alpha \in E$ . The following are equivalent:*

- a.  $\alpha$  is algebraic over  $F$ .*
- b. the primitive extension  $F(\alpha)$  is a finite extension of  $F$ .*
- c.  $\alpha \in E_1$  for some subfield  $E_1 \subseteq E$  with  $F \subseteq E_1$  which is a finite extension of  $F$ .*

*Proof.* a.  $\implies$  b: If  $\alpha$  is algebraic, let  $d = \deg \alpha$  be the degree of  $\alpha$  over  $F$ . We have seen that  $1, \alpha, \dots, \alpha^{d-1}$  form an  $F$ -basis for  $F(\alpha)$ , so  $[F(\alpha) : F] = d$  and thus  $F(\alpha)$  is indeed a finite extension of  $F$ .

b.  $\implies$  c: This is immediate; just take  $E_1 = F(\alpha)$ .

c.  $\implies$  a.: Assume  $\dim_F E_1 = d$ . Since  $\alpha \in E_1$  and  $E_1$  is a field, also  $\alpha^i \in E_1$  for all  $i \in \mathbf{Z}_{\geq 0}$ . Since  $E_1$  has dimension  $d$  over  $F$ , it follows from linear algebra that the  $d+1$  elements

$$1, \alpha, \dots, \alpha^{d-1}, \alpha^d$$

are linearly dependent over  $F$ . Let  $c_0, c_1, \dots, c_d \in F$  not all zero be such that

$$\sum_{i=0}^d c_i \alpha^i = 0$$

and consider the *polynomial*

$$f(T) = \sum_{i=0}^d c_i T^i \in F[T].$$

Since not all of the coefficients  $c_i$  are 0,  $f(T) \neq 0$ . Since  $f(\alpha) = 0$ , we have proved that  $\alpha$  is algebraic over  $F$  as required.  $\square$

**Proposition 9.4.5.** *Let  $F \subseteq E \subseteq K$  be fields where  $K$  is a finite extension of  $E$  and  $E$  is a finite extension of  $F$ . Then  $K$  is a finite extension of  $F$  and moreover:*

$$[K : F] = [K : E] \cdot [E : F].$$

*Proof.* Let

$$a_1, \dots, a_N \in E \quad \text{be an } F\text{-basis for } E$$

and let

$$b_1, \dots, b_M \in K \quad \text{be an } E\text{-basis for } K$$

Multiplying in the field  $K$ , we consider the elements  $a_s b_t$ , and we assert:

$$\mathcal{B} = \{a_s b_t \mid 1 \leq s \leq N, 1 \leq t \leq M\} \quad \text{is an } F\text{-basis for } K$$

- $\mathcal{B}$  spans  $K$  over  $F$ : indeed, let  $x \in K$ . We must express  $x$  as a linear combination of the vectors  $\mathcal{B}$ .

Since the  $\{b_t\}$  span  $K$  over  $E$ , we may write

$$x = u_1 b_1 + \cdots + u_M b_M \quad \text{for } u_t \in E.$$

Since the  $\{a_s\}$  span  $E$  over  $F$ , for each  $1 \leq t \leq M$  we may write

$$u_t = v_{1,t} a_1 + \cdots + v_{N,t} a_N \quad \text{for } v_{s,t} \in F$$

Now

$$x = \sum_{t=1}^M u_t b_t = \sum_{t=1}^M \left( \sum_{s=1}^N v_{s,t} a_s \right) b_t = \sum_{1 \leq s \leq N, 1 \leq t \leq M} v_{s,t} \cdot a_s b_t$$

- $\mathcal{B}$  is linearly independent over  $F$ .

Suppose that

$$0 = \sum_{1 \leq s \leq N, 1 \leq t \leq M} v_{s,t} \cdot a_s b_t = \sum_{t=1}^M \left( \sum_{s=1}^N v_{s,t} a_s \right) b_t$$

for coefficients  $v_{s,t} \in F$ .

Now use the fact that  $\{b_t\}$  are linearly independent over  $E$  to conclude for each  $1 \leq t \leq M$  that

$$0 = \sum_{s=1}^N v_{s,t} a_s$$

For any  $1 \leq t \leq M$ , use the fact that  $\{a_s\}$  are linearly independent over  $F$  to conclude for each  $1 \leq s \leq N$  that  $v_{s,t} = 0$ .

□

**Corollary 9.4.6.** *Let  $E$  be a finite extension of  $F$ . If  $\alpha \in E$  then the degree of  $\alpha$  over  $F$  is a divisor of  $[E : F]$ :*

$$\deg_F(\alpha) \mid [E : F].$$

*Proof.* Apply Proposition 9.4.5 to the tower of field extensions

$$F \subseteq F(\alpha) \subseteq E$$

to deduce that

$$[E : F] = [E : F(\alpha)] \cdot [F(\alpha) : F]$$

and the result follows since  $[F(\alpha) : F] = \deg_F \alpha$ .

□

## 9.5 Examples of finite extensions

*Example 9.5.1.*  $[\mathbf{Q}(\sqrt{2}, \sqrt{3}) : \mathbf{Q}] = 4$ .

The polynomials  $T^2 - 2, T^2 - 3 \in \mathbf{Q}$  are known to be irreducible over  $\mathbf{Q}$  (can you give a quick argument?)

We claim that  $T^2 - 3$  remains irreducible over  $\mathbf{Q}(\sqrt{2})$  –i.e. that  $T^2 - 3 \in \mathbf{Q}(\sqrt{2})[T]$  is irreducible.

If we verify the claim, it follows that

$$[\mathbf{Q}(\sqrt{2}, \sqrt{3}) : \mathbf{Q}(\sqrt{2})] = 2$$

and thus

$$[\mathbf{Q}(\sqrt{2}, \sqrt{3}) : \mathbf{Q}] = [\mathbf{Q}(\sqrt{2}, \sqrt{3}) : \mathbf{Q}(\sqrt{2})] \cdot [\mathbf{Q}(\sqrt{2}) : \mathbf{Q}] = 2 \cdot 2 = 4$$

as required.

Let's now prove the claim. Since  $T^2 - 3$  has degree 2, the irreducibility will follow provided we argue that  $T^2 - 3$  has no root in  $\mathbf{Q}(\sqrt{2})$ .

So: suppose that  $3 = (a + b\sqrt{2})^2$  for  $a, b \in \mathbf{Q}$ . Thus

$$3 + 0 \cdot \sqrt{2} = 3 = a^2 + 2b^2 + 2ab\sqrt{2}$$

and comparing coefficients we find that

$$3 = a^2 + 2b^2 \quad \text{and} \quad 0 = 2ab.$$

Now  $2ab = 0 \implies a = 0$  or  $b = 0$  and the equation  $3 = a^2 + 2b^2$  is then impossible (since neither 3 nor  $3/2$  is a square in  $\mathbf{Q}$ ). This completes the proof that  $T^2 - 3$  is irreducible over  $\mathbf{Q}(\sqrt{2})$ .

*Example 9.5.2.*  $[\mathbf{Q}(\sqrt{2} + \sqrt{3}) : \mathbf{Q}] = 4$ .

To prove the claim, we argue that

$$\mathbf{Q}(\sqrt{2} + \sqrt{3}) = \mathbf{Q}(\sqrt{2}, \sqrt{3});$$

the assertion then follows from the previous example.

Write  $K = \mathbf{Q}(\sqrt{2} + \sqrt{3})$ . To confirm this equality, first note that trivially we have

$$K \subset \mathbf{Q}(\sqrt{2}, \sqrt{3})$$

so it is enough to argue

$$\sqrt{2}, \sqrt{3} \in K.$$

**(Why?)**

In fact, it is easy to see that  $\sqrt{2} \in K \iff \sqrt{3} \in K$  (since  $\sqrt{2} + \sqrt{3} \in K$  by construction!).

So it only remains to argue e.g. that  $\sqrt{3} \in K$ .

Let's observe that

$$\frac{1}{\sqrt{2} + \sqrt{3}} = \frac{1}{\sqrt{2} + \sqrt{3}} \cdot \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} - \sqrt{2}} = \frac{\sqrt{3} - \sqrt{2}}{1} \in K$$

and since  $K$  is a field,

$$\frac{1}{\sqrt{2} + \sqrt{3}} + \sqrt{2} + \sqrt{3} = (\sqrt{3} - \sqrt{2}) + (\sqrt{2} + \sqrt{3}) = 2\sqrt{3} \in K$$

so indeed  $\sqrt{3} \in K$ .

The preceding calculation confirms (for example) that  $\sqrt{2}$  may be written in the form

$$\begin{aligned}\sqrt{2} &= a + b\alpha + c\alpha^2 + d\alpha^3 \\ &= a + b(\sqrt{2} + \sqrt{3}) + c(\sqrt{2} + \sqrt{3})^2 + d(\sqrt{2} + \sqrt{3})^3\end{aligned}$$

for some coefficients  $a, b, c, d \in \mathbf{Q}$ , though we'd need to do some work to find  $a, b, c, d$ .

## 9.6 Algebraic extensions

Let  $F \subseteq E$  be any extension of fields. We are going to argue that

$$E_{\text{alg}} = \{u \in E \mid u \text{ is algebraic over } F\}$$

is a subfield of  $E$ .

For example, this requires us to know that if  $x, y \in E_{\text{alg}}$  then  $x - y \in E_{\text{alg}}$ . It is not completely clear how to find an algebraic equation satisfied by  $x - y$ , so we use a more indirect argument.

Our main tool is the following:

**Lemma 9.6.1.** *Let  $\alpha, \beta \in E$  be algebraic. Then  $[F(\alpha, \beta) : F]$  is a finite extension. In particular,  $\alpha \pm \beta$  and  $\alpha \cdot \beta$  are algebraic over  $F$ ; if  $0 \neq \alpha$ , then also  $\alpha^{-1} = \frac{1}{\alpha}$  is algebraic over  $F$ .*

*Proof.* Indeed,  $\beta$  is algebraic over  $F$  hence algebraic over  $F(\alpha)$  so

$$[F(\alpha, \beta) : F(\alpha)] < \infty$$

since  $F(\alpha, \beta) = F(\alpha)(\beta)$ .

Since  $\alpha$  is algebraic over  $F$ ,  $[F(\alpha) : F] < \infty$  and thus

$$[F(\alpha, \beta) : F] = [F(\alpha, \beta) : F(\alpha)] \cdot [F(\alpha) : F]$$

is finite. The result now follows from Proposition 9.4.4.  $\square$

**Corollary 9.6.2.** *Let  $E$  be an extension field of  $F$ . The set of all elements of  $E$  which are algebraic over  $F$  forms a subfield  $E_{\text{alg}}$  of  $E$ .*

*Proof.* We first observe that  $E_{\text{alg}}$  is an additive subgroup of  $E$ . For this, note that  $0 \in E_{\text{alg}}$  so it just remains to show that if  $x, y \in E_{\text{alg}}$  then  $x - y \in E_{\text{alg}}$ . But this statement follows from the Lemma 9.6.1.

It now remains to argue that  $E_{\text{alg}}$  is closed under multiplication and contains the inverse of its non-zero elements. These statements again follow from Lemma 9.6.1.  $\square$

**Definition 9.6.3.** An extension field  $E$  of  $F$  is *algebraic* over  $F$  if each element of  $E$  is algebraic over  $F$ .

**Proposition 9.6.4.** *Every finite extension of fields is algebraic.*

*Proof.* Let  $F \subseteq E$  be a finite extension and let  $\alpha \in E$  be an arbitrary element of  $E$ . Since  $[F(\alpha) : F]$  is a divisor of  $[E : F]$ ,  $[F(\alpha) : F]$  is finite and hence  $\alpha$  is algebraic by Proposition 9.4.4. This shows that  $E$  is algebraic over  $F$  as required.  $\square$

**Lemma 9.6.5.** *Let  $F \subseteq E$  be an algebraic extension, and let  $\alpha_1, \dots, \alpha_n \in E$ . Then*

$$[F(\alpha_1, \dots, \alpha_n) : F] < \infty.$$

*Proof.* Proceed by induction on  $n \geq 1$ .

First consider the case  $n = 1$ . Since  $E$  is algebraic over  $F$ ,  $\alpha = \alpha_1$  is algebraic over  $F$  and  $[F(\alpha) : F]$  is finite by previous results.

Now suppose  $n > 1$  and write  $E_i = F(\alpha_1, \dots, \alpha_i)$  for  $1 \leq i \leq n$ . The induction hypothesis is then:  $[E_i : F] < \infty$  for  $i < n$ . Note that  $E_n = E_{n-1}(\alpha_n)$ , and – since  $\alpha_n$  is algebraic over  $F$  –  $\alpha_n$  is algebraic over  $E_{n-1}$ . Thus

$$[E_n : E_{n-1}] = [E_{n-1}(\alpha_n) : E_{n-1}] < \infty$$

by Proposition 9.4.4 and it follows by induction that

$$[E_n : F] = [E_n : E_{n-1}] \cdot [E_{n-1} : F] < \infty$$

as required.  $\square$

**Proposition 9.6.6.** *Let  $E$  be an algebraic extension of  $F$  and let  $K$  be an algebraic extension of  $E$ . Then  $K$  is an algebraic extension of  $F$ .*

*Proof.* Let  $\alpha \in K$ . We must argue that  $\alpha$  is algebraic over  $F$ . Since  $\alpha$  is algebraic over  $E$ , it is the root of some polynomial

$$f(T) = a_0 + a_1T + a_2T^2 + \dots + a_NT^N \quad a_i \in E.$$

Now, form the extension  $E_1 = F(a_0, a_1, \dots, a_N)$ . Since  $E$  is algebraic over  $F$ , all  $a_i$  are algebraic over  $F$ . It follows from Lemma 9.6.5 that  $[E_1 : F] < \infty$ . Since  $\alpha$  is algebraic over  $E_1$  we know that  $[E_1(\alpha) : E_1] < \infty$  by Proposition 9.4.4. It now follows that

$$[E_1(\alpha) : F] = [E_1(\alpha) : E_1][E_1 : F] < \infty$$

so that  $\alpha$  is algebraic over  $F$  by Proposition 9.6.4.  $\square$

## 9.7 Another example

Consider the field  $K = \mathbf{Q}(T)$  where  $T$  is transcendental over  $\mathbf{Q}$ . It follows from Theorem 7.4.1 that

$$X^n - T - a \in K[X] = \mathbf{Q}(T)[X]$$

is irreducible for  $n = 2, 3$  for any  $a \in \mathbf{Q}$ .

These irreducibility statements mean that

$$[K(\sqrt{T-a}) : K] = 2 \quad \text{and} \quad [K(\sqrt[3]{T-a}) : K] = 3$$

(or writing everything out in full detail, that

$$[\mathbf{Q}(T, \sqrt{T-a}) : \mathbf{Q}(T)] = 2 \quad \text{and} \quad [\mathbf{Q}(T, \sqrt[3]{T-a}) : \mathbf{Q}(T)] = 3.)$$

**Lemma 9.7.1.**  $K(\sqrt{T-a}, \sqrt[3]{T-a}) = \mathbf{Q}(T, \sqrt{T-a}, \sqrt[3]{T-a})$  has degree 6 over  $K = \mathbf{Q}(T)$ .

*Proof.* Let  $L = K(\sqrt{T-a}, \sqrt[3]{T-a})$ . The claim will follow if we show that

$$(\clubsuit) \quad [L : K(\sqrt{T-a})] = 3$$

since then

$$[L : K] = [L : K(\sqrt{T-a})] \cdot [K(\sqrt{T-a}) : K] = 3 \cdot 2 = 6.$$

Now,  $(\clubsuit)$  follows if we argue that  $f(X) = X^3 - T - a \in K(\sqrt{T-a})[X]$  is irreducible; since  $f$  has degree 3, it suffices to argue that  $f$  has no root in  $K(\sqrt{T-a})$ .

But were  $\alpha \in K(\sqrt{T-a})$  a root of  $f$ , we know that  $\alpha$  has degree 3 over  $K$ . But this is impossible since

$$\alpha \in K(\sqrt{T-a}) \implies \deg_K \alpha \mid [K(\sqrt{T-a}) : K] = 2.$$

This completes the proof that  $f$  is irreducible over  $K(\sqrt{T-a})$  and thus the Lemma is verified.  $\square$

## 10 Constructible real numbers

As an example of the utility of field theory, we are going to describe a field-theory-based answer to a “geometric-constructions/geometric” question about numbers. Loosely put, we are going to answer the question: “can one trisect an angle using ruler and compass?”

### 10.1 Ruler and compass constructions

As a starting point, we are given two points at *unit distance* in the Euclidean plane.

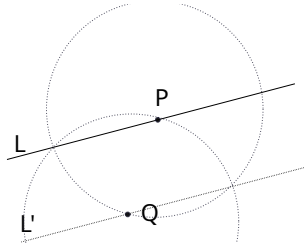
Given any two distinct known points  $P$  and  $Q$ , one can construct:

- the *line* through  $P$  and  $Q$  (this uses a *straightedge*)
- the circle with center  $P$  which passes through  $Q$  (this uses a *compass*)

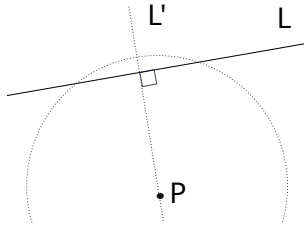
One views the points of intersection of lines and circles that have been constructed as *constructible* (i.e. known) points.

Here are some useful constructions that we are going to use without further argumentation:

**Lemma 10.1.1.** ( $\clubsuit$ ) *Given a point  $P$  on a line  $L$ , and a second point  $Q$  not on  $L$ , we can construct a line  $L'$  parallel to  $L$  passing through  $Q$ .*

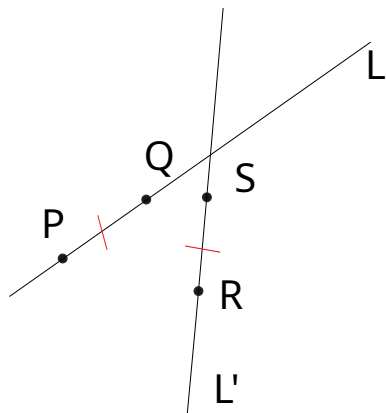


**Lemma 10.1.2.** ( $\heartsuit$ ) *Given a line  $L$  and a point  $P$  not lying on  $L$ , one can construct a line  $L'$  containing  $P$  and perpendicular to  $L$ .*



**Lemma 10.1.3.** ( $\spadesuit$ ) *Given two points  $P \neq Q$  on a line  $L$ , a second line  $L'$ , and a point  $R$  on  $L'$ , we can construct a point  $S$  on  $L'$  such that*

$$|\overline{PQ}| = |\overline{RS}|.$$



## 10.2 Constructions

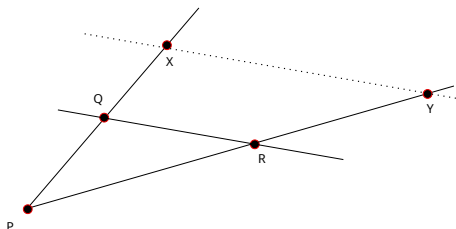
*Definition 10.2.1.* A real number  $r$  is *constructible* if one can construct a line segment of length  $|r|$  using straightedge and compass.

**Proposition 10.2.2.** *The set of constructible real numbers forms a subfield  $C \subseteq \mathbf{R}$ .*

*Sketch of proof.* First, use Lemma 10.1.3 to show that  $C$  forms an additive subgroup of  $\mathbf{R}$ .

To argue that  $C$  is closed under multiplication, proceed as follows:

- Given positive constructible numbers  $y, z, w$  construct a diagram with points  $P, Q, R, Y$  as follows with  $|PQ| = z$ ,  $|PR| = w$  and  $|PY| = y$ .



- Now use ( $\clubsuit$ ) to construct the line through  $Y$  parallel to the line through  $Q$  and  $R$ .
- Writing  $X$  for the (constructible) point of intersection of the indicated lines, write  $x = |PX|$  and notice that  $x/y = z/w$ .
- Now let  $a, b > 0$  be constructible and let  $y = a$ ,  $z = b$  and  $w = 1$ ; the above argument shows that  $x = yz = ab$  is constructible.

Similar arguments give the constructibility of  $a/b$  where  $a, b > 0$  are constructible.  $\square$

Let's observe that according to the Proposition, every rational number is constructible.

We may and will suppose that the points  $(1, 0)$  and  $(0, 1)$  in the plane are constructible. In particular, the coordinates  $r, s$  of any constructible point  $P = (r, s)$  are constructible real numbers.

### 10.3 Lines and Circles over a field

Of course, any line may be described as the set of solutions to an equation

$$aX + bY + c = 0$$

for  $a, b, c \in \mathbf{R}$ , and any circle may be described as the solutions to an equation

$$X^2 + Y^2 + aX + bY + c = 0$$

for  $a, b, c \in \mathbf{R}$ .

If  $F$  is a subfield of  $\mathbf{R}$ , a *line over  $F$*  means a line with equation  $aX + bY + c = 0$  where  $a, b, c \in F$ .

Similarly, a *circle over  $F$*  means a circle with equation

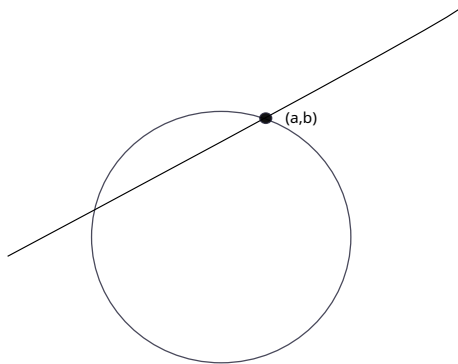
$$X^2 + Y^2 + aX + bY + c = 0 \quad \text{where } a, b, c \in F.$$

**Lemma 10.3.1.** • If the points  $P \neq Q$  both have coordinates in  $F$ , the line through  $P$  and  $Q$  is a line over  $F$ .

- If  $C$  is circle for which both the radius and the coordinates of its center are all in  $F$ , then  $C$  is a circle over  $F$ .

Constructing points via ruler and compass amounts to finding the intersections of lines and circles. We record the following fact about such intersections:

**Proposition 10.3.2.** Let  $F \subseteq \mathbf{R}$  be a subfield. The coordinates of the points of intersection of lines over  $F$  and circles over  $F$  belong to the field  $F(\sqrt{u})$  for some  $u \in F$ .



If in this diagram the line and the circle are “over  $F$ ”, the conclusion is that  $a, b \in F(\sqrt{u})$  for a suitable  $u \in F$ .

### 10.4 Characterizing constructible numbers

Using Proposition 10.3.2, we can give an important characterization of constructible real numbers:

**Theorem 10.4.1.**  $u \in \mathbf{R}$  is constructible  $\iff$  there are  $u_1, \dots, u_n \in \mathbf{R}$  such that:

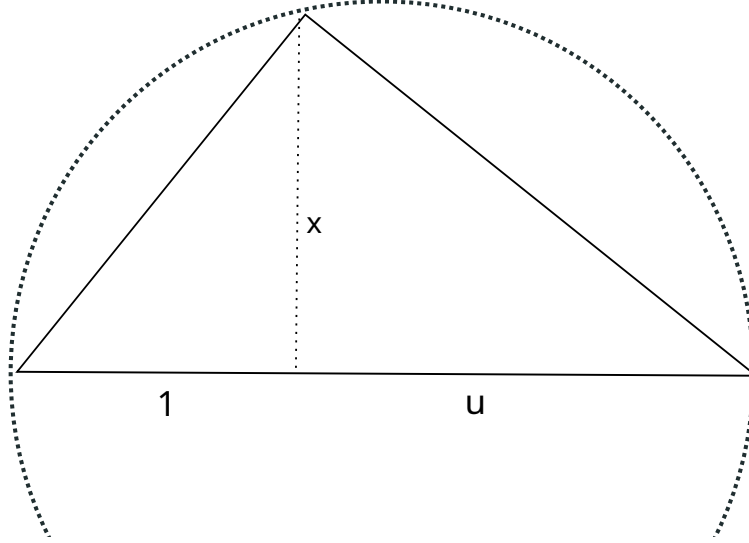
- $u_1^2 \in \mathbf{Q}$ ,

b.  $u_i^2 \in \mathbf{Q}(u_1, \dots, u_{i-1})$  for  $2 \leq i \leq n$ , and

c.  $u \in \mathbf{Q}(u_1, \dots, u_n)$ .

*Proof.*  $\Rightarrow$ : This follows from the Proposition.

$\Leftarrow$ : Use the following: if  $F$  is any subfield of the field of constructible numbers, then  $\sqrt{u}$  is constructible for each positive  $u \in F$ . For this, construct a circle of diameter  $1 + u$ , and a line perpendicular to the diameter, intersecting the diameter 1 unit from the west pole:



Then  $x = \sqrt{u}$ . □

**Corollary 10.4.2.** *If  $u$  is a constructible real number, then  $u$  is algebraic over  $\mathbf{Q}$  and  $\deg(u)$  is a power of 2.*

## 10.5 Angle trisection

**Lemma 10.5.1.** a. *For any angle  $\theta$ , we have the following identities:*

$$4 \cos^3(\theta) - 3 \cos(\theta) - \cos(3\theta) = 0.$$

b. *Let  $\alpha = \cos\left(\frac{\pi}{9}\right)$ .  $\alpha$  is a root of the irreducible polynomial*

$$f(T) = 8T^3 - 6T - 1 \in \mathbf{Q}[T].$$

*In particular, the degree of  $\alpha$  over  $\mathbf{Q}$  is 3.*

c.  $\alpha$  is not a constructible number.

*Proof.* Recall the trigonometric identities:

$$\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta) \tag{10.1}$$

and

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta). \tag{10.2}$$

Taking  $\alpha = \beta$  we get

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

and

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha).$$

For a real number  $\theta$ , we find that “double angle formula”

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= \cos^2(\theta) - (1 - \cos^2(\theta)) \\ &= 2\cos^2(\theta) - 1\end{aligned}$$

This shows that

$$2\cos^2(\theta) - \cos(2\theta) - 1 = 0 \tag{10.3}$$

To prove (a), let  $\alpha = 2\theta$  and  $\beta = \theta$  in (10.2); we get

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta + \theta) \\ &= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) \\ &= (2\cos^2(\theta) - 1)\cos(\theta) - (2\sin(\theta)\cos(\theta))\sin(\theta) \\ &= 2\cos^3(\theta) - \cos(\theta) - 2\cos(\theta)\sin^2(\theta) \\ &= 2\cos^3(\theta) - \cos(\theta) - 2\cos(\theta)(1 - \cos^2(\theta)) \\ &= 4\cos^3(\theta) - 3\cos(\theta).\end{aligned}$$

This shows that  $4\cos^3(\theta) - 3\cos(\theta) - \cos(3\theta) = 0$  as required.

We now prove (b). If  $\theta = \frac{\pi}{9}$ , then of course  $\cos(3\theta) = \frac{1}{2}$ , so (a) shows  $\theta$  to be a root of the equation  $4T^3 - 3T - \frac{1}{2} \in \mathbf{Q}[T]$ . Multiplying this polynomial by 2 gives  $8T^3 - 6T - 1$  and we can use the *rational roots test* Theorem 7.2.1 to confirm the that this polynomial has no root in  $\mathbf{Q}$  and is thus irreducible in  $\mathbf{Q}[T]$ .

Now (c) follows from Corollary 10.4.2, since  $3 \nmid 2^m$  for any  $m \geq 1$ .  $\square$

**Theorem 10.5.2.** *It is impossible to find a general construction for trisecting an angle.*

*Proof.* Since  $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$  and  $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ , one can construct points  $Q = \frac{1}{2}(1, \sqrt{3})$ ,  $P = (0, 0)$ ,  $R = (1, 0)$  and then  $\angle QPR$  is  $\frac{\pi}{3}$ .

We claim that one can't construct further points  $S, T$  such that the  $\angle QPS$ ,  $\angle SPT$  and  $\angle TPR$  are all equal.

Indeed, if it were so, the coordinates of  $T$  would be  $(\cos\left(\frac{\pi}{9}\right), \sin\left(\frac{\pi}{9}\right))$ , and then  $\cos\left(\frac{\pi}{9}\right)$  would be a constructible number, contrary to Lemma 10.5.1.  $\square$

## 11 Splitting fields

### 11.1 The notion of a splitting field

Let  $F$  be a field and consider a polynomial

$$f = a_0 + a_1T + \cdots + a_nT^n \in F[T]$$

of degree  $n \geq 1$ .

*Definition 11.1.1.* If  $E$  is an extension field of  $F$ , we say that  $f$  *splits over*  $E$  provided that there are elements  $r_1, \dots, r_n \in E$  such that

$$f = (T - r_1)(T - r_2) \cdots (T - r_n) = \prod_{i=1}^n (T - r_i) \in E[T].$$

*Definition 11.1.2.* If  $f$  splits over the field extension  $E$  of  $F$ , and if  $r_1, \dots, r_n \in E$  are the roots of  $f$ , we say that  $E$  is a *splitting field* for  $f$  over  $F$  if moreover  $E = F(r_1, \dots, r_n)$ .

Thus a splitting field  $E$  is somehow a minimal field extension over which  $f$  splits.

*Example 11.1.3.*  $E = \mathbf{Q}(i)$  is a splitting field over  $\mathbf{Q}$  for the polynomial  $f = T^2 - 2T + 2$  since

$$f = (T - 1 - i)(T - 1 + i) \in \mathbf{Q}(i)[T]$$

and since  $\mathbf{Q}(i) = \mathbf{Q}(1 + i, 1 - i)$ .

**Theorem 11.1.4.** *Let  $f \in F[T]$  has degree  $n \geq 1$ . Then there exists a splitting field  $E$  for  $f$  over  $F$  with  $[E : F] \leq n!$ .*

*Proof.* Proceed by induction on  $n \geq 1$ . The result holds when  $n = 1$ , since then  $f$  splits over  $E = F$ .

Now suppose that the result is known for all fields  $F$  and all polynomials of degree  $\leq n - 1$ .

Now, choose an irreducible factor  $p$  of  $f$  in  $F[T]$ , say of degree  $d \leq n$ . Choose a root of  $p$  in some field extension of  $F$ , and consider the field  $K = F(\alpha)$ . We know that  $[K : F] = [F(\alpha) : F] = d = \deg p$ .

Since  $\alpha$  is a root of  $p$ , it is also a root of  $f$ ; thus by the *remainder theorem* – see Corollary 3.4.2 –, we may write

$$f = (T - \alpha) \cdot g \quad \text{for } g \in K[T] \text{ with } \deg g = n - 1.$$

Now use the *induction hypothesis* to construct a splitting field  $E$  for  $g$  over  $K$  with  $[E : K] \leq (n - 1)!$ .

Thus  $E = K(r_2, \dots, r_n)$  and

$$g = \prod_{i=2}^n (T - r_i) \in E[T].$$

We now have

$$f = (T - \alpha) \cdot g = (T - \alpha) \cdot \prod_{i=2}^n (T - r_i) \in E[T];$$

thus,  $f$  splits over  $E$ . Moreover,  $E = K(r_2, \dots, r_n) = F(\alpha, r_2, \dots, r_n)$  which confirms that  $E$  is a splitting field of  $f$  over  $F$ .

Finally, note that

$$[E : F] = [E : K][K : F] \leq (n - 1)! \cdot d \leq n!$$

since  $d \leq n$ .  $\square$

$\square$

## 11.2 More examples of splitting fields

### 11.2.1 Fourth root of 2

The field  $E = \mathbf{Q}(i, \sqrt[4]{2})$  is a splitting field for  $f = T^4 - 2$  over  $\mathbf{Q}$ , and  $[E : \mathbf{Q}] = 8$ .

First, if we write  $\alpha = \sqrt[4]{2}$  for the *real* fourth root of 2, the roots of  $f$  are precisely  $\pm\alpha, \pm i\alpha$ . Indeed,

$$(T - \alpha)(T + \alpha)(T - i\alpha)(T + i\alpha) = (T^2 - \sqrt{2})(T^2 + \sqrt{2}) = f.$$

Now,  $E = \mathbf{Q}(i, \sqrt[4]{2}) = \mathbf{Q}(\pm\alpha, \pm i\alpha)$ .

Finally, to see that  $[E : \mathbf{Q}] = 8$ , first note that  $[\mathbf{Q}(\alpha) : \mathbf{Q}] = 4$  since  $T^4 - 2$  is irreducible over  $\mathbf{Q}$ .

Now  $\alpha \in \mathbf{R} \implies \mathbf{Q}(\alpha) \subset \mathbf{R}$ , so  $\mathbf{Q}(\alpha)$  does not contain a root of  $T^2 + 1$ . Thus  $T^2 + 1$  is irreducible over  $\mathbf{Q}(\alpha)$ .

This shows that

$$[E : \mathbf{Q}] = [E : \mathbf{Q}(\alpha)] \cdot [\mathbf{Q}(\alpha) : \mathbf{Q}] = 2 \cdot 4 = 8.$$

### 11.2.2 Transcendental extension

$E = \mathbf{C}(X, \sqrt[4]{X+1})$  is a splitting field over  $\mathbf{C}(X)$  for  $T^4 - (X+1)$ , and  $[E : \mathbf{C}(T)] = 4$ .

### 11.2.3 Finite field example

Let  $F = \mathbf{F}_7$  be the field with 7 elements.

Let's describe the splitting field for  $f = T^3 - 3 \in F[T]$  over  $F$ .

First, note that the cubes mod 7 are as follows:

```
return [ (n,n**3 % 7) for n in range(7) ]
```

```
[(0, 0), (1, 1), (2, 1), (3, 6), (4, 1), (5, 6), (6, 6)]
```

In particular,  $f = T^3 - 3$  has no root in  $F = \mathbf{F}_7$ . So if  $\alpha$  denotes a root of  $f$  in some extension field, then  $F(\alpha)$  is a degree 3 extension of  $F$ .

Now let's notice that the multiplicative order of (the class of) 2 in  $\mathbf{F}_7^\times$  is 3: indeed  $2^3 = 8 \equiv 1 \pmod{7}$  but  $2, 2^2 \not\equiv 1 \pmod{7}$ . So we can observe that also  $2\alpha$  and  $4\alpha$  are also roots of  $T^3 - 3$ . Thus

$$f = (T - \alpha)(T - 2\alpha)(T - 4\alpha) \in F(\alpha)[T] = \mathbf{F}_7(\alpha)[T].$$

This shows that  $F(\alpha) = \mathbf{F}_7(\alpha)$  is a splitting field over  $F$  of  $f = T^3 - 3$ .

Observe that  $|F(\alpha)| = 7^3 = 343$ ; elements of  $F(\alpha)$  all have the form

$$a + b\alpha + c\alpha^2 \quad a, b, c \in \mathbf{F}_7.$$

## 11.3 Uniqueness of splitting fields

We are going to argue that a splitting field for a polynomial  $f$  over  $F$  is *essentially unique*.

Let us first make an observation: if  $\theta : F \rightarrow F_1$  is an isomorphism of fields, then  $\theta$  may be extended to an isomorphism

$$\theta : F[T] \rightarrow F_1[T]$$

with the property that  $\theta(T) = T$ . Note that polynomials satisfy

$$p \in F[T] \text{ is irreducible} \iff \theta(p) \in F_1[T] \text{ is irreducible.}$$

**Lemma 11.3.1.** *Let  $\theta : F \rightarrow F_1$  be an isomorphism of fields, let  $E = F(u)$  where  $u$  is algebraic over  $F$  with minimal polynomial  $p \in F[T]$ , and let  $p_1 = \theta(p)$ . If  $v$  is a root of  $p_1$  in an extension field of  $F_1$ , there is a unique way of extending  $\theta$  to an isomorphism  $\phi : F(u) \rightarrow F_1(v)$  subject to the conditions (i)  $\phi(u) = v$ , and (ii)  $\phi|_F = \theta$ , i.e. the restriction of  $\phi$  to  $F$  is given by  $\theta$ .*

This diagram might be useful for visualizing the situation:

$$\begin{array}{ccc} F(u) & \xrightarrow{\phi} & F_1(v) \\ \uparrow & \circ & \uparrow \\ F & \xrightarrow{\theta} & F_1 \end{array}$$

*Proof.* We first observe that  $\phi$  is uniquely determined by the indicated conditions. Indeed,  $F(u)$  is spanned as  $F$ -vector space by elements of the form  $u^i$ , and since  $\phi$  is a ring homomorphism it must satisfy  $\phi(u^i) = v^i$ .

We now prove the existence of  $\phi$ . We first note that –according to Proposition 9.3.4 – there are isomorphisms  $\gamma : F[T]/\langle p \rangle \xrightarrow{\sim} F(u)$  and  $\psi : F_1[T]/\langle p_1 \rangle \xrightarrow{\sim} F_1(v)$  with

$$\gamma(T + \langle p \rangle) = u \quad \text{and} \quad \psi(T + \langle p_1 \rangle) = v$$

such that  $\gamma|_F = \text{id}$  and  $\psi|_{F_1} = \text{id}$ .

Now, consider the ring homomorphism  $F[T] \xrightarrow{\theta} F_1[T] \xrightarrow{\pi_1} F_1[T]/\langle p_1 \rangle$  where  $\pi_1$  is the quotient mapping. This mapping  $\pi_1 \circ \theta$  is onto and has kernel  $\langle p \rangle$ ; according to the First Isomorphism Theorem – see Theorem 2.5.1 – it induces an isomorphism

$$\Phi : F[T]/\langle p \rangle \xrightarrow{\sim} F_1[T]/\langle p_1 \rangle$$

such that  $\Phi|_F = \theta$  and such that  $\Phi(T + \langle p \rangle) = T + \langle p_1 \rangle$ .

Now  $\psi \circ \Phi \circ \gamma^{-1} : F(u) \xrightarrow{\sim} F_1(v)$  has the required properties.  $\square$

*Remark 11.3.2.* Using the notations of the preceding proof, the isomorphism  $F(u) \rightarrow F_1(v)$  is given by

$$F(u) \xrightarrow{\gamma^{-1}} F[T]/\langle p \rangle \xrightarrow{\Phi} F_1[T]/\langle p_1 \rangle \xrightarrow{\psi} F_1(v).$$

*Example 11.3.3.* Consider the field  $F = \mathbf{Q}(i)$ . Write  $\sigma : \mathbf{Q}(i) \rightarrow \mathbf{Q}(i)$  for *complex conjugation*; thus  $\sigma(a + bi) = \overline{a + bi} = a - bi$  for  $a, b \in \mathbf{Q}$ . The mapping  $\sigma$  is an *automorphism* of the field  $F = \mathbf{Q}(i)$ .

We claim that the polynomials  $f_1 = T^2 - (1 + i)$  and  $f_2 = T^2 - (1 - i)$  in  $F[T]$  are irreducible. Note that  $f_2 = \sigma(f_1)$  so it is sufficient to argue that  $f_1$  is irreducible.

According to Proposition 7.1.4 it is enough to argue that the degree 2 polynomial  $f_1$  has no roots in  $F = \mathbf{Q}(i)$ .

If  $\alpha \in \mathbf{Q}(i)$  is a root of  $f_1$  then  $\alpha^2 = 1 + i$  so that

$$\alpha^2 \cdot \sigma(\alpha^2) = (1 + i) \cdot \sigma(1 + i) = (1 + i)(1 - i) = 2$$

But then  $(\alpha\sigma(\alpha))^2 = 2$ , and it is easy to see that  $\alpha \cdot \sigma(\alpha) = \alpha\bar{\alpha} \in \mathbf{Q}$ . Since  $\sqrt{2} \notin \mathbf{Q}$  this contradiction proves that there is no root  $\alpha \in F$  of  $f_1$ . Thus indeed  $f_1$  and  $f_2$  are irreducible.

In particular  $F(\sqrt{1+i}) = \mathbf{Q}(i, \sqrt{1+i})$  and  $F(\sqrt{1-i}) = \mathbf{Q}(i, \sqrt{1-i})$  are degree 2 extensions of the field  $F = \mathbf{Q}(i)$ .

Now Lemma 11.3.1 shows that there is an isomorphism  $\phi : \mathbf{Q}(i, \sqrt{1+i}) \rightarrow \mathbf{Q}(i, \sqrt{1-i})$  such that  $\phi(\sqrt{1+i}) = \sqrt{1-i}$  and such that  $\phi|_{\mathbf{Q}(i)} = \sigma$ ; in particular,  $\phi(i) = -i$ .

**Proposition 11.3.4.** *Let  $E$  be a splitting field over  $F$  for  $f \in F[T]$ , let  $\theta : F \rightarrow F_1$  be a field isomorphism, and let  $g = \theta(f) \in F_1[T]$ . Let  $E_1$  be a splitting field for  $g$  over  $F_1$ . Then there is an isomorphism  $\phi : E \rightarrow E_1$  such that  $\phi|_F = \theta$ .*

*Proof.* We use induction on  $n = \deg f$ . If  $n = 1$ , then  $E = F$ ,  $E_1 = F_1$  and we can simply take  $\phi = \theta$ .

Now suppose that  $n > 1$  and that the result holds for all field  $F$  and all polynomials of degree  $< n$ .

Let  $p \in F[T]$  be an irreducible factor of  $f$ , so that  $q = \theta(p)$  is an irreducible factor of  $g$ .

Since  $f$  splits over  $E$ , also  $p$  splits over  $E$ . Choose a root  $u \in E$  of  $p$ . Thus  $F \subseteq F(u) \subseteq E$ .

Choose also a root  $v \in E_1$  of  $q$ , so that  $F_1 \subseteq F_1(v) \subseteq E_1$ .

According to the preceding Lemma, there is an isomorphism  $\hat{\theta} : F(u) \rightarrow F_1(v)$  such that  $\hat{\theta}|_F = \theta$  and such that  $\hat{\theta}(u) = v$ .

Write

$$f = (T - u)s \in F(u)[T] \quad \text{for } s \in F(u)[T]$$

and

$$g = (T - v)s_1 \in F_1(v)[T] \quad \text{for } s_1 \in F_1(v)[T]$$

Now,  $E$  is a splitting field for  $s$  over  $F(u)$  and  $E_1$  is a splitting field for  $s_1$  over  $F_1(v)$ . And since  $\theta(f) = g$  and  $\hat{\theta}(u) = v$  it is easy to see that  $\hat{\theta}(s) = s_1$ .

Thus the induction hypothesis gives an isomorphism  $\phi : E \rightarrow E_1$  such that  $\phi|_{F(u)} = \hat{\theta}$ . This isomorphism  $\phi$  has the required properties.  $\square$

We find the following theorem as an immediate consequence:

**Theorem 11.3.5.** *Let  $f \in F[T]$  be a polynomial with  $\deg f > 0$ . If  $E$  and  $E_1$  are splitting fields for  $f$  over  $F$ , there is an isomorphism  $\phi : E \rightarrow E_1$  such that  $\phi(a) = a$  for each  $a \in F$  – i.e. such that  $\phi|_F$  is the identity mapping.*

*Proof.* In the Proposition, just take  $\theta$  to be the identity map!  $\square$

*Remark 11.3.6.* Observe that the proof of Proposition 11.3.4 requires us to prove the statement involving  $\theta$ , even though in Theorem 11.3.5 we are interested in only in the case  $\theta = \text{id}$ .

### 11.3.1 Example: automorphisms of a splitting field

The ideas behind the results Proposition 11.3.4 and Theorem 11.3.5 will be really important as we start talking about Galois theory. So, it seems useful to first do a non-trivial example.

Let's give an example of automorphisms of a splitting field.

Let's fix a prime number  $p$ , consider the polynomial  $f = T^3 - p \in \mathbf{Q}[T]$ , and let  $E$  be a splitting field for this polynomial over  $\mathbf{Q}$ .

The Theorem 11.3.5 tells us that any splitting field of  $f$  over  $\mathbf{Q}$  is isomorphic to  $E$ . Let's try to understand what this statement could mean about automorphisms of  $E$ .

First, let's make some observations. Notice that if  $\beta$  and  $\beta'$  are roots of  $f$ , then  $\left(\frac{\beta}{\beta'}\right)^3 = 1$  i.e.  $\frac{\beta}{\beta'}$  is a root of  $T^3 - 1$ . Moreover,  $\frac{\beta}{\beta'} = 1$  if and only if  $\beta = \beta'$ .

Let's exclude the "trivial" cube root of unity; observe that

$$\frac{T^3 - 1}{T - 1} = T^2 + T + 1 \in \mathbf{Q}[T]$$

has roots  $\omega, \omega^2 \in \mathbf{C}$  where

$$\omega = \exp\left(\frac{2\pi i}{3}\right) = \cos\left(\frac{2\pi i}{3}\right) + i \sin\left(\frac{2\pi i}{3}\right) \in \mathbf{C};$$

Notice that  $\omega \neq 1$  and  $\omega^3 = 1$  so viewed as an element of the group  $\mathbf{C}^\times$ ,  $\omega$  has order 3.

Neither  $\omega$  nor  $\omega^2$  is rational, so  $T^2 + T + 1$  is *irreducible* over  $\mathbf{Q}$ .

We can now construct a splitting field  $E$  of  $f$  over  $\mathbf{Q}$  *abstractly*. Take  $E = \mathbf{Q}(\alpha, \omega)$  where  $\alpha$  is a root of  $T^3 - p$  and  $\omega$  is a root of  $T^2 + T + 1$ .

First notice that

$$E = \mathbf{Q}(\alpha, \omega) = \mathbf{Q}(\alpha, \omega\alpha, \omega^2\alpha)$$

so that  $E$  is a splitting field. Now notice that  $\deg_{\mathbf{Q}} \alpha = 3$  and  $\deg_{\mathbf{Q}} \omega = 2$  so  $T^2 + T + 1$  remains irreducible over  $\mathbf{Q}(\alpha)$ . Thus we may conclude that

$$[E : \mathbf{Q}] = [\mathbf{Q}(\alpha, \omega) : \mathbf{Q}(\alpha)] \cdot [\mathbf{Q}(\alpha) : \mathbf{Q}] = 6.$$

Now observe that this argument actually shows that if we fix *any* root  $\beta$  of  $f$  in  $E$ , and *any* root  $\zeta$  of  $T^2 + T + 1$  in  $E$  then

$$f = (T - \beta)(T - \zeta\beta)(T - \zeta^2\beta).$$

E.g. if we choose  $\zeta = \omega^2$  and  $\beta = \omega\alpha$ , then

$$f = (T - \beta)(T - \zeta\beta)(T - \zeta^2\beta) = (T - \omega\alpha)(T - \omega^2(\omega\alpha))(T - \omega^4(\omega\alpha))$$

since

$$\{\omega\alpha, \omega^2(\omega\alpha), \omega^4(\omega\alpha)\} = \{\omega\alpha, \omega^3\alpha, \omega^5\alpha\} = \{\omega\alpha, \alpha, \omega^2\alpha\}.$$

The thing to take home from all this is that there are some choices to be made in describing the roots of  $f$ . In this case, you could pin things down more precisely e.g. by taking for  $\alpha$  the "real" cube root of  $P$  and for  $\omega$  the complex root of  $T^2 + T + 1$  which is in "quadrant 2". But a more systematic way of keeping track of choices is through study of automorphisms of the splitting field  $E$ .

Notice that  $\alpha$  and  $\beta = \omega\alpha$  are roots of the irreducible polynomial  $T^3 - p \in \mathbf{Q}[T]$ . Thus, there is an isomorphism of fields

$$\theta : \mathbf{Q}(\alpha) \rightarrow \mathbf{Q}(\beta)$$

such that  $\theta$  is the identity on  $\mathbf{Q}$  and  $\theta(\alpha) = \beta = \omega\alpha$ .

Notice that  $\theta(T^2 + T + 1) = T^2 + T + 1$  is irreducible over  $\mathbf{Q}(\alpha)$  and over  $\mathbf{Q}(\beta)$ .

Now, Lemma 11.3.1 tells us that there is an isomorphism

$$\Theta : \mathbf{Q}(\alpha, \omega) \rightarrow \mathbf{Q}(\beta, \zeta)$$

such that  $\Theta|_{\mathbf{Q}(\alpha)} = \theta$  - i.e. for which  $\Theta(\alpha) = \beta$  - and for which  $\Theta(\omega) = \zeta$ .

This  $\Theta$  is an isomorphism between splitting fields of  $f$ . Since we took  $\beta = \omega\alpha$  and  $\zeta = \omega^2$ , we have

$$E = \mathbf{Q}(\alpha, \omega) = \mathbf{Q}(\beta, \zeta)$$

so in fact  $\Theta : E \rightarrow E$  is an *automorphism* of  $E$ .

Note that  $\Theta$  is not the identity mapping on the roots of  $f$ :

$$(\Theta(\alpha), \Theta(\omega\alpha), \Theta(\omega^2\alpha)) = (\omega\alpha, \zeta\omega\alpha, \zeta^2\omega\alpha) = (\omega\alpha, \alpha, \omega^2\alpha).$$

Also note that upon restriction to  $\mathbf{Q}(\omega)$ ,  $\Theta|_{\mathbf{Q}(\omega)}$  is *complex conjugation*, since

$$\Theta(\omega) = \omega^2 = \overline{\omega}.$$

## 12 Finite fields

### 12.1 The prime subfield of a field

First let's recall for any field  $F$  that there is always a ring homomorphism  $\mathbf{Z} \rightarrow F$  for which  $n \mapsto n \cdot 1_F$ .

**Proposition 12.1.1.** *Let  $F$  be a field.*

- a. *If the homomorphism  $\mathbf{Z} \rightarrow F$  is one-to-one, then  $F$  contains a copy of the field  $\mathbf{Q}$  of rational numbers.*
- b. *If the homomorphism  $\mathbf{Z} \rightarrow F$  is not one-to-one, then  $F$  contains a copy of the field  $\mathbf{Z}/p\mathbf{Z}$  for some prime number  $p$ .*

*Remark 12.1.2.* In case a., we say that  $F$  has *characteristic 0*. Note in that case that the *additive order* of any non-zero element of  $F$  is  $\infty$ .

In case b., we say that  $F$  has *characteristic  $p$* . In that case, the additive order of any non-zero element of  $F$  is  $p$ .

*Definition 12.1.3.* The *prime subfield* of  $F$  is the smallest subfield containing the image of the homomorphism  $\mathbf{Z} \rightarrow F$ ; thus when  $F$  has characteristic 0, the prime subfield identifies with  $\mathbf{Q}$ , and when  $F$  has characteristic  $p > 0$ , the prime subfield identifies with  $\mathbf{Z}/p\mathbf{Z}$ .

*Proof of the Proposition.* If the homomorphism  $\phi : \mathbf{Z} \rightarrow F$  is injective, it maps non-zero elements of  $\mathbf{Z}$  to *invertible* elements of  $F$ . Thus by the *defining property* of the field of fractions  $\mathbf{Q} = Q(\mathbf{Z})$ , the homomorphism  $\phi$  extends to a homomorphism  $\tilde{\phi} : \mathbf{Q} \rightarrow F$ ; see Proposition 6.0.4. Thus  $F$  indeed contains a copy of  $\mathbf{Q}$ .

Suppose on the other hand that the homomorphism  $\phi$  is not one-to-one; thus  $\ker \phi = n\mathbf{Z}$  for some  $n \neq 0$ . The *First Isomorphism Theorem* Theorem 2.5.1 now implies that the image of  $\phi$  is a subring of  $F$  isomorphic to the finite ring  $\mathbf{Z}/n\mathbf{Z}$ . Since  $F$  is a field, this subring must be an integral domain – see Example 3.1.7 (c); thus by Example 3.1.7 (d) we see that  $n = p$  must be a prime number.  $\square$

### 12.2 Some properties of finite fields

We've met some finite fields already, namely  $\mathbf{Z}/p\mathbf{Z}$  for a prime number  $p$ .

We've can construct finite extensions of  $\mathbf{Z}/p\mathbf{Z}$  to get fields  $F$  for which  $|F|$  is not prime. Let's first make an observation about  $|F|$ , as follows:

**Proposition 12.2.1.** *Let  $F$  be a finite field. Then  $F$  has characteristic  $p > 0$  for some prime number  $p$ . The number of elements of  $F$  is  $p^m$  for some whole number  $m \geq 1$ .*

*Proof.* Since  $\mathbf{Q}$  is not finite, the previous proposition shows that  $F$  must have characteristic  $p > 0$  for a prime number  $p$ .

Write  $F_0 \subseteq F$  where  $F_0$  is the prime subfield; thus  $F_0 \simeq \mathbf{Z}/p\mathbf{Z}$ .

Now,  $F$  may be viewed as an  $F_0$ -vector space. A basic theorem in linear algebra says that  $F$  must have a *basis*  $\mathcal{B}$  as an  $F_0$ -vector space; see Proposition 8.3.5. Since  $F$  is finite, this basis must be finite; say  $|\mathcal{B}| = m$ .

Write  $\mathcal{B} = \{b_1, b_2, \dots, b_m\}$ . Then an element  $x$  of  $F$  may be written uniquely in the form

$$x = t_1 b_1 + t_2 b_2 + \cdots + t_m b_m$$

for  $t_i \in F_0$ ; see e.g. Section 8.3. Since  $F_0 \simeq \mathbf{Z}/p\mathbf{Z}$ , there are  $p$  choices for each  $t_i$ ; this shows that the number of elements of  $F$  is

$$|F| = p \cdot p \cdot \dots \cdot p = p^m$$

as required.  $\square$

### 12.3 Finite fields as splitting fields over the prime field

**Proposition 12.3.1.** *Let  $F$  be a finite field with  $p^m$  elements for some prime number  $p$ . Then  $F$  is the splitting field over the prime subfield  $F_0 \simeq \mathbf{Z}/p\mathbf{Z}$  of the polynomial*

$$T^{p^m} - T \in F_0[T].$$

*Proof.* Since  $F$  has  $p^m$  elements, the multiplicative group  $F^\times$  has  $p^m - 1$  elements. This means that every element  $x \in F^\times$  satisfies the condition

$$x^{p^m-1} = 1.$$

It is then immediate that every element  $x \in F$  satisfies

$$x^{p^m} = x.$$

Put another way, every element of  $F$  is a root of the polynomial

$$f = T^{p^m} - T \in F_0[T].$$

Since  $f$  can have no more than  $p^m$  roots in an extension field, it follows that  $F$  contains all roots of  $f$ . Since  $F$  is generated by these roots,  $F$  is a splitting field for  $f$  over  $F_0$ .  $\square$

*Remark 12.3.2.* The proof shows that the identity

$$f = T^{p^m} - T = \prod_{\alpha \in F} (T - \alpha)$$

holds in  $F[T]$ .

**Corollary 12.3.3.** *Two finite fields  $F$  and  $E$  are isomorphic if and only if  $|F| = |E|$ .*

*Proof.* If  $F$  and  $E$  are isomorphic, there is a one-to-one onto function  $\phi : F \rightarrow E$  and thus  $|F| = |E|$ .

On the other hand, if  $|F| = |E|$ , we know that  $|F| = p^m$  and  $|E| = q^n$  for some primes  $p, q$  and some  $m, n \geq 1$ . By unique factorization of integers – see Theorem 5.2.1 –,  $p = q$  and  $m = n$ . Now the Proposition shows that  $E, F$  are splitting fields of  $T^{p^m} - T$  over  $\mathbf{Z}/p\mathbf{Z}$ .

Now the existence of an isomorphism  $F \simeq E$  is a consequence of the uniqueness of splitting fields.  $\square$

## 12.4 Existence of a finite field of any prime-power order

Let  $p$  be a prime number. One might see the following Lemma in a class in elementary number theory:

**Lemma 12.4.1.** *For  $x, y \in \mathbf{Z}$ , we have:*

- a.  $x^p \equiv x \pmod{p}$
- b.  $(x + y)^p \equiv x^p + y^p \equiv x + y \pmod{p}$ .

We are going to prove a slightly more general version of this result that is valid for elements of *any* field of characteristic  $p > 0$ , as follows:

**Lemma 12.4.2.** *Let  $F$  be a field of char.  $p > 0$ , let  $x, y \in F$ , and let  $n \in \mathbf{Z}_{>0}$ . Then:*

- a.  $(x + y)^{p^n} = x^{p^n} + y^{p^n}$ .
- b.  $\{x \in F \mid x^{p^n} = x\}$  is a subfield of  $F$ .

*Proof.* For  $0 < i < p$ , the binomial coefficients  $\binom{p}{i} = \frac{p!}{i! \cdot (p-i)!}$  satisfy the congruence

$$\binom{p}{i} \equiv 0 \pmod{p}.$$

Indeed,  $p$  divides the numerator  $p!$  but  $p$  does not divide the denominator  $i! \cdot (p-i)!$  and the result follows since the quotient is integral.

Since  $\binom{p}{0} = \binom{p}{p} = 1$ , it follows that

$$(x + y)^p = \sum_{i=0}^p \binom{p}{i} x^i y^{p-i} = x^p + y^p \quad (12.1)$$

for elements  $x, y \in F$ . To prove a., proceed by induction on  $n \geq 1$ . The case  $n = 1$  is just (12.1). Assuming the result is valid for  $n - 1$ , we see that

$$(x + y)^{p^n} = \left( (x + y)^{p^{n-1}} \right)^p = \left( x^{p^{n-1}} + y^{p^{n-1}} \right)^p = x^{p^n} + y^{p^n};$$

we used the induction hypothesis for the second equality, and we used (12.1) applied to  $x^{p^{n-1}}$  and  $y^{p^{n-1}}$  for the final equality. This proves a.

For b., write

$$F_1 = \{x \in F \mid x^{p^n} = x\}.$$

To see that  $F_1$  is an additive subgroup of  $F$ , first note that  $0 \in F_1$ . Now, the result from a. shows that if  $x, y \in F_1$  then  $x + y \in F_1$ .

Next we argue that  $F_1$  is closed under multiplication. This follows since if  $x, y \in F_1$  then

$$(xy)^{p^n} = x^{p^n} y^{p^n} = xy.$$

Finally, if  $x \in F_1$  is non-zero, then

$$1 = 1^{p^n} = (x \cdot x^{-1})^{p^n} = x^{p^n} x^{-p^n} = x x^{-p^n}$$

which shows that  $(x^{-1})^{p^n} = x^{-p^n} = x^{-1}$  hence  $x^{-1} \in F_1$ . □

**Lemma 12.4.3.** *Let  $F$  be a field of characteristic  $p > 0$  and let  $\mathcal{F} : F \rightarrow F$  be the mapping  $\mathcal{F}(x) = x^p$ . Then  $\mathcal{F}$  is a ring homomorphism.*

*Proof.* Part a. of Lemma 12.4.2 shows that  $\mathcal{F}$  is a homomorphism of additive group. If  $x, y \in F$  then  $\mathcal{F}(xy) = (xy)^p = x^p y^p = \mathcal{F}(x)\mathcal{F}(y)$  which completes the proof.  $\square$

**Lemma 12.4.4.** *Let  $m, n$  be positive integers for which  $n = qm$ .*

*a.  $T^m - 1 \mid T^n - 1$  in the polynomial ring  $\mathbf{Z}[T]$ .*

*b. For any commutative ring  $R$  (with identity) and any  $y \in R$  we have  $y^m - 1 \mid y^n - 1$ .*

*Proof.* For a., first note that for a polynomial variable  $u$ , we have the identity

$$\frac{u^q - 1}{u - 1} = u^{q-1} + u^{q-2} + \cdots + u + 1 \quad (12.2)$$

in the field of fractions of  $\mathbf{Z}[u]$ . Substituting  $u = T^m$  in (12.2) gives

$$\begin{aligned} f(T) &= \frac{T^n - 1}{T^m - 1} = \frac{(T^m)^q - 1}{T^m - 1} \\ &= (T^m)^{q-1} + (T^m)^{q-2} + \cdots + T^m + 1 \\ &= T^{m(q-1)} + T^{m(q-2)} + \cdots + T^m + 1 \in \mathbf{Z}[T] \end{aligned}$$

Now b. follows from a. Indeed, if  $T^n - 1 = g(T) \cdot (T^m - 1)$  for  $g(T) \in \mathbf{Z}[T]$ , then for  $y \in R$  we see that  $y^n - 1 = g(y) \cdot (y^m - 1)$  since evaluation at  $y$  determines a ring homomorphism  $\mathbf{Z}[T] \rightarrow R$ .  $\square$

**Proposition 12.4.5.** *Let  $F$  be a field with  $p^n$  elements. Each subfield of  $F$  has  $p^m$  elements for some divisor  $m$  of  $n$ . Conversely, for each divisor  $m \mid n$ , there exists a unique subfield of  $F$  having  $p^m$  elements.*

*Proof.* Let  $F_0$  be the prime subfield of  $F$ . Any subfield  $E$  of  $F$  must contain  $F_0$  and must have  $p^m$  elements, where  $m = [E : F_0]$ . Since

$$n = [F : F_0] = [F : E][E : F_0] = [F : E] \cdot m$$

we conclude that  $m$  must be a divisor of  $n$ .

Conversely, let  $m$  be a divisor of  $n$ . Then  $p^m - 1$  is a divisor of  $p^n - 1$  by Lemma 12.4.4. Applying Lemma 12.4.4 a second time, we see that the polynomial  $g(T) = T^{(p^m-1)} - 1$  is a divisor of  $h(T) = T^{(p^n-1)} - 1$  in the polynomial ring  $F_0[T]$ .

Since  $F$  is the splitting field of  $T \cdot h(T)$  over  $F_0$ , it must contain all  $p^m$  distinct roots of  $T \cdot g(T)$ .

Now, part b. of Lemma 12.4.2 implies that the roots of  $T \cdot g(T) = T^{p^m} - T$  form a subfield  $E$  of  $F$ . Any other subfield having order  $p^m$  must be a splitting field of  $T \cdot g(T)$  and so it must coincide with  $E$ . This completes the proof.  $\square$

**Lemma 12.4.6.** *Let  $F$  be a field of char.  $p > 0$ . If  $n \in \mathbf{Z}_{>0}$  and  $n \not\equiv 0 \pmod{p}$  then  $T^n - 1$  has no repeated roots in any extension field of  $F$ . Put another way, if  $E$  denotes a splitting field of  $T^n - 1$  over  $F$ , then*

$$T^n - 1 = \prod_{i=1}^n (T - \alpha_i)$$

for  $n$  distinct elements  $\alpha_i \in E$ .

*Proof.* Let  $c$  be a root of  $T^n - 1$  in a splitting field  $E$ . The remainder theorem – Corollary 3.4.2 – shows that  $T^n - 1 = (T - c)g(T)$  for some polynomial

$$g(T) = \sum_{i=0}^{n-1} a_i T^i$$

with  $a_0, a_1, \dots, a_{n-1} \in F(c)$ . Now, we have

$$\begin{aligned} T^n - 1 &= (T - c)g(T) = (T - c) \left( \sum_{i=0}^{n-1} a_i T^i \right) = \left( \sum_{i=0}^{n-1} a_i T^{i+1} \right) - \left( \sum_{i=0}^{n-1} c a_i T^i \right) \\ &= \left( \sum_{i=1}^n a_{i-1} T^i \right) - \left( \sum_{i=0}^{n-1} c a_i T^i \right) \\ &= a_{n-1} T^n + \left( \sum_{i=1}^{n-1} a_{i-1} T^i - \sum_{i=1}^{n-1} c a_i T^i \right) - c a_0 \end{aligned}$$

Comparing coefficients, we find that  $a_{n-1} = 1$  and that  $a_{i-1} = c a_i$  for  $1 \leq i \leq n-1$ . Thus we find that  $a_i = c^{n-1-i}$  for  $1 \leq i \leq n-1$  and that  $a_0 = c^{n-1}$  since then  $c a_0 = c^n = 1$ . Thus

$$g(T) = T^{n-1} + c T^{n-2} + c^2 T^{n-3} + \dots + c^{n-2} T + c^{n-1}$$

To prove the Lemma, we must show that  $g = g(T)$  is not divisible by  $T - c$ . By the remainder theorem, it is sufficient to prove that  $g(c) \neq 0$ . But we have:

$$g(c) = c^{n-1} + c c^{n-2} + c^2 c^{n-3} + \dots + c^{n-2} c + c^{n-1} = n \cdot c^{n-1}$$

and the result follows since  $n 1_F \neq 0$  and  $c \neq 0$ . □

**Theorem 12.4.7.** *For every prime  $p$  and every positive integer  $n$ , there is a field  $\mathbf{F}_q$  with  $q = p^n$  elements, and any field of order  $q$  is isomorphic to  $\mathbf{F}_q$ .*

*Proof.* The uniqueness has already been proved; it remains to argue the *existence* of  $\mathbf{F}_q$  for  $q = p^n$ .

Let  $F$  be the splitting field of the polynomial  $T^{p^n} - T$  over  $\mathbf{Z}/p\mathbf{Z}$ . The previous Lemma shows that  $T^{p^n} - T$  has  $p^n$  distinct roots. By an earlier Lemma, these roots form a *subfield* of  $F$ , so we conclude that  $F$  consists exactly in these roots. Thus  $|F| = p^n$  as required. □

*Remark 12.4.8.* For a prime power  $q$ , some texts write  $\text{GF}(q)$  for the field we have denoted  $\mathbf{F}_q$ . The symbol GF stands for “Galois Field”.

## 12.5 Some examples of finite fields

We have seen in Theorem 12.4.7 that for each prime power  $q = p^n$ , there is a field of that order. The computer algebra system **sagemath** knows how to do some computations with finite fields. We are next going to demonstrate this facility with some calculations.

### 12.5.1 Extensions of $\mathbf{F}_{19}$

For example, we can ask to represent the field of  $19^2 = 361$  elements.

```
H.<a>=FiniteField(19^2)
a.minpoly()
```

```
x^2 + 18*x + 2
```

The output here tells us that

$$H = \mathbf{F}_{19}[T]/\langle T^2 + 18T + 2 \rangle.$$

We can construct larger finite fields and ask about subfields:

```
G.<z>=FiniteField(19^6)
z.minpoly()
```

```
x^6 + 17*x^3 + 17*x^2 + 6*x + 2
```

```
G.subfields()
```

```
[(Finite Field of size 19,
  Ring morphism:
    From: Finite Field of size 19
    To:   Finite Field in z of size 19^6
    Defn: 1 |--> 1),
 (Finite Field in z2 of size 19^2,
  Ring morphism:
    From: Finite Field in z2 of size 19^2
    To:   Finite Field in z of size 19^6
    Defn: z2 |--> 18*z^5 + 9*z^4 + 5*z^3 + 2*z^2 + 12*z + 7),
 (Finite Field in z3 of size 19^3,
  Ring morphism:
    From: Finite Field in z3 of size 19^3
    To:   Finite Field in z of size 19^6
    Defn: z3 |--> 13*z^5 + 10*z^4 + 2*z^3 + 15*z^2 + 7*z + 18),
 (Finite Field in z of size 19^6,
  Identity endomorphism of Finite Field in z of size 19^6)]
```

The output here tells us that the field  $G$  of order  $19^6 = 47045881$  – roughly forty seven million elements – has exactly 4 subfields:  $G = \mathbf{F}_{19}(z)$ , a subfield  $\mathbf{F}_{19}(z^3)$  of order  $19^3$ , a subfield  $\mathbf{F}_{19}(z^2)$  of order  $19^2$  and a subfield of order 19.

Here **sage** has found an element  $z$  for which

$$G = \mathbf{F}_{19}(z) \simeq \mathbf{F}_{19}[T]/\langle T^6 + 17 \cdot T^3 + 17 \cdot T^2 + 6 \cdot T + 2 \rangle,$$

The subfield

$$\mathbf{F}_{19}(z^3) = \mathbf{F}_{19}(13 \cdot z^5 + 10 \cdot z^4 + 2 \cdot z^3 + 15 \cdot z^2 + 7 \cdot z + 18)$$

has order  $19^3 = 6859$ .

The subfield

$$\mathbf{F}_{19}(z^2) = \mathbf{F}_{19}(18 \cdot z^5 + 9 \cdot z^4 + 5 \cdot z^3 + 2 \cdot z^2 + 12 \cdot z + 7)$$

has order  $19^2 = 361$ .

Let's pause and ask **sagemath** to compute the non-squares in  $\mathbf{F}_{19}$ :

```
F.<a>=FiniteField(19)
squares = [ x^2 for x in F]
nonSquares = [x for x in F if not(x in squares)]
len(nonSquares)
```

9

This output tells us that there are 9 elements  $a \in \mathbf{F}_{19}$  for which  $T^2 - a$  is **irreducible**. Those elements are:

```
nonSquares
```

[2, 3, 8, 10, 12, 13, 14, 15, 18]

According to Corollary 12.3.3, up to isomorphism there is a unique field of order  $19^2$ . It follows that

$$\mathbf{F}_{19}(\sqrt{2})$$

must contain a square root of each of these **nonSquares**. We can ask **sagemath** to describe these roots in terms of  $a = \sqrt{2}$  as follows:

We first describe solutions to  $T^2 - 2$ :

```
F= FiniteField(19)
R.<T>=PolynomialRing(F)
E.<a> = F.extension(T^2 - 2)
[x for x in E if x^2==2]
```

[a, 18\*a]

And here are solutions to  $T^2 - 13$ :

```
[x for x in E if x^2==13]
```

[4\*a, 15\*a]

Similarly we can find solutions to  $T^2 - 8$ :

```
[x for x in E if x^2==8]
```

[2\*a, 17\*a]

This makes clear for example that

$$\mathbf{F}_{19}(\sqrt{13}) = \mathbf{F}_{19}(4\sqrt{2}) = \mathbf{F}_{19}(\sqrt{2}).$$

In fact, we can get a full list of irreducible polynomials:

```
irred = [T^2 + a*T + b for a in F for b in F if (T^2+a*T+b).is_irreducible()]
len(irred)
```

171

The output tells us that there are 171 monic irreducible quadratic polynomials in  $\mathbf{F}_{19}[T]$ . Let's look at a few:

```
irred[0:11]
```

```
[T^2 + 1,
 T^2 + 4,
 T^2 + 5,
 T^2 + 6,
 T^2 + 7,
 T^2 + 9,
 T^2 + 11,
 T^2 + 16,
 T^2 + 17,
 T^2 + T + 2,
 T^2 + T + 3]
```

We can use the `sage` command `polroots` to find roots of a polynomial:

```
def polroots(p):
    return [x for x in E if p(x)==0]
[irred[10],
 polroots(irred[10])]
```

```
[T^2 + T + 3, [a + 9, 18*a + 9]]
```

The output shows that the two roots of  $T^2 + T + 3$  in  $\mathbf{F}_{19}(\sqrt{2})$  are

$$9 + \sqrt{2} \quad \text{and} \quad 9 + 18\sqrt{2} = 9 - \sqrt{2}.$$

(Of course, we have obtained those roots using the quadratic formula!)

This makes clear that  $\mathbf{F}_{19}(\sqrt{2})$  is a splitting field for  $T^2 + T + 3$ .

In fact, we know that  $\mathbf{F}_{19}(\sqrt{2})$  is a splitting field for all 171 polynomials  $p$  in the list `irred`.

### 12.5.2 Fields of order 4 and 8

There are 4 monic polynomials of degree 2 over the field  $\mathbf{F}_2$  of two elements. Of these, only one is irreducible, namely

$$T^2 + T + 1.$$

Thus

$$\mathbf{F}_4 \simeq \mathbf{F}_2(\alpha)$$

where  $\deg \alpha = 2$  and  $\alpha^2 = \alpha + 1$ . Notice that

$$T^2 + T + 1 = (T + \alpha)(T + \alpha + 1).$$

There are 8 monic polynomials of degree 3 over  $\mathbf{F}_2$ . Of these, only two are irreducible:

```
H = FiniteField(2)
R.<T>=PolynomialRing(H)
[T^3 + a*T^2 + b*T + c
 for a in H
 for b in H
 for c in H
 if (T^3+a*T^2+b*T + c).is_irreducible()]
```

```
[T^3 + T + 1, T^3 + T^2 + 1]
```

Thus  $\mathbf{F}_8 = \mathbf{F}_2(\beta)$  where  $\deg \beta = 3$  and  $\beta^3 = \beta + 1$ . And indeed we may confirm that  $\mathbf{F}_2(\beta)$  is a splitting field for *both* the irreducible polynomials of degree 3:

```
HH.<b>=FiniteField(8)
RR.<T>=PolynomialRing(HH)
[RR(T^3+T+1).factor(),
 RR(T^3+T^2+1).factor()]
```

```
[(T + b) * (T + b^2) * (T + b^2 + b),
 (T + b + 1) * (T + b^2 + 1) * (T + b^2 + b + 1)]
```

## 12.6 The multiplicative group of a finite field

Let  $F = \mathbf{F}_q$  be a finite field, where  $q = p^n$ . Then of course the multiplicative group  $F^\times = F \setminus \{0\}$  is a finite abelian group having  $q - 1$  elements.

In this section we are going to argue that the group  $F^\times$  is *cyclic*, so that

$$F^\times \simeq \mathbf{Z}/(q - 1)\mathbf{Z}.$$

We begin with a Lemma from group theory:

**Lemma 12.6.1.** *Let  $G$  be a finite abelian group (written multiplicatively). If  $a \in G$  is an element of maximal order in  $G$ , then the order of every element of  $G$  is a divisor of the order  $o(a)$  of  $a$ .*

*Proof.* Let  $x \in G$  be any element different from 1. If  $o(x) \nmid o(a)$  then in the prime factorizations of  $o(x)$  and  $o(a)$  we can find a prime  $p$  that occurs to a higher power in  $o(x)$  than in  $o(a)$ .

Write  $o(a) = p^\alpha n$  and  $o(x) = p^\beta m$  where  $\alpha < \beta$  and  $p \nmid n, p \nmid m$ .

Now  $o(a^{p^\alpha}) = n$  and  $o(x^m) = p^\beta$ , so the orders of  $a^{p^\alpha}$  and  $x^m$  are relatively prime. It follows that the order of the product  $a^{p^\alpha} \cdot x^m$  is equal to the product of the orders of the elements, i.e. to  $np^\beta$ . But this exceeds  $o(a)$  contrary to the hypothesis.  $\square$

**Theorem 12.6.2.** *Let  $F$  be any field. Any finite subgroup of the multiplicative group  $F^\times$  is cyclic.*

*Proof.* Let  $H$  be a finite subgroup of  $F^\times$  and let  $a \in H$  be an element with maximal order. Write  $N = o(a)$ . Now Lemma 12.6.1 shows that  $o(x) \mid N$  for all  $x \in H$ . Thus, every element of  $H$  is a root of the polynomial  $T^N - 1$ . Now, this polynomial has no more than  $N$  roots – see Corollary 3.4.3. It follows that  $|H| \leq N$ . Since the cyclic group  $\langle a \rangle$  has order  $N$ , conclude that  $H = \langle a \rangle$ .  $\square$

**Corollary 12.6.3.**  *$\mathbf{F}_q^\times$  is a cyclic group of order  $q - 1$  for any prime power  $q = p^n$ .*

**Corollary 12.6.4.** *For any prime power  $q = p^n$ , there is  $\alpha \in \mathbf{F}_q$  for which  $\mathbf{F}_q = \mathbf{F}_p(\alpha)$ . In words: each finite field is a primitive extension of its prime subfield.*

*Proof.* Let  $\beta$  be a generator for the cyclic group  $\mathbf{F}_q^\times$ . Then

$$\langle \beta \rangle \subseteq \mathbf{F}_p(\beta) \subseteq \mathbf{F}_q \implies q - 1 \leq |\mathbf{F}_p(\beta)| \leq q.$$

Since  $|\mathbf{F}_p(\beta)|$  must be a power of  $p$  – see Proposition 12.2.1 – it follows that  $\mathbf{F}_p(\beta) = \mathbf{F}_q$ .  $\square$

## 13 Perfect fields and separable polynomials

Let  $F$  be a field.

### 13.1 Common roots and root multiplicity

If  $f \in F[T]$  is a non-zero polynomial, recall that according to Theorem 5.2.1 we may write

$$f = u \prod_{i=1}^r p_i^{e_i}$$

where  $u \in F^\times$ , where the  $p_i \in F[T]$  are pairwise non-associate *irreducible* polynomials, and where  $e_i \geq 0$ . observe that a splitting field for  $f$  over  $F$  is the same as a splitting field for

$$g = \prod_{i=1}^r p_i.$$

**Lemma 13.1.1.** *Suppose that  $f, g \in F[T]$ .*

- a. *If  $\gcd(f, g) = 1$  then  $f$  and  $g$  have no common root in any extension of  $F$ .*
- b. *If  $f, g$  are irreducible and not associate, they have no common root in any extension of  $F$ .*

*Proof.* Assertion b. is of course an immediate consequence of assertion a.

As to a., note that  $\gcd(f, g) = 1 \implies$  that  $1 = uf + vg$  for polynomials  $u, v \in F[T]$  Proposition 4.3.4.

Let  $E$  be an extension field of  $F$  and suppose that  $\alpha \in E$  is a root of both  $f$  and  $g$ . Then  $0 = u(\alpha)f(\alpha) + v(\alpha)g(\alpha) = 1$  which is impossible. Thus there can be no such common root  $\alpha$ .  $\square$

Let  $f \in F[T]$  be monic and let  $E$  be a splitting field for  $f$  over  $F$ . Write

$$f = (T - \alpha_1)^{e_1} \cdots (T - \alpha_r)^{e_r}.$$

for **distinct** elements  $\alpha_i \in E$  and exponents  $e_i \in \mathbf{Z}_{\geq 1}$ . Since the linear polynomials  $T - \alpha_i$  are irreducible and pairwise relatively prime in  $E[T]$ , it follows from Theorem 5.2.1 that this representation is unique (up to re-ordering, of course).

*Definition 13.1.2.* We say that the root  $\alpha_i$  of  $f$  has *multiplicity*  $e_i$ . If  $e_i = 1$ , we say that  $\alpha_i$  is a *simple root* of  $f$ . If  $e_i > 1$ , we say that  $\alpha_i$  is a *repeated root* of  $f$ .

**Proposition 13.1.3.** *The polynomial  $f \in F[T]$  has no repeated roots if and only if  $\gcd(f, f') = 1$  where  $f'$  is the formal derivative of  $f$ .*

*Proof.* We are actually going to prove the (equivalent) assertion:  $f$  has a repeated root if and only if  $\gcd(f, f') \neq 1$ .

$\implies$ : We show that if  $f$  has a repeated root, then  $\gcd(f, f') \neq 1$ . Suppose that  $f$  has a repeated root  $\alpha$  in some extension field  $E$ .

In  $E[T]$  we may write

$$f = (T - \alpha)^2 \cdot g \quad \text{for some } g \in E[T].$$

One must check that the product rule holds for formal differentiation; using that rule, one then notes that

$$f' = (T - \alpha)^2 g' + 2(T - \alpha)g.$$

It is evident that  $\alpha$  is a root of both  $f$  and  $f'$  and thus Lemma 13.1.1 implies that  $\gcd(f, f') \neq 1$ .

$\Leftarrow$ : We suppose that  $\gcd(f, f') \neq 1$  and we must prove that  $f$  has a repeated root.

Our assumption implies that there is a polynomial  $g \in F[T]$  of positive degree which divides both  $f$  and  $f'$ . Let  $\alpha$  be a root of  $g$  in some extension field of  $F$ . Thus  $\alpha$  is a root of both  $f$  and  $f'$ . We now claim that  $\alpha$  is a repeated root of  $f$ .

Since  $\alpha$  is a root of  $f$ , we may write

$$f = (T - \alpha) \cdot h \quad \text{for some } h \in F[T].$$

In order to show that  $\alpha$  is a repeated root of  $f$ , we must argue that  $\alpha$  is a root of  $h$ .

Well, we find using the product rule that

$$f' = h + (T - \alpha) \cdot h'.$$

Since  $\alpha$  is a root of  $f'$  we find that

$$0 = f'(\alpha) = h(\alpha) + (\alpha - \alpha)h'(\alpha) = h(\alpha).$$

We have now argued that  $h(\alpha) = 0$ ; as already observed, this proves that  $\alpha$  is a repeated root of  $f$ .  $\square$

## 13.2 Repeated roots and the characteristic

**Lemma 13.2.1.** *Suppose that the field  $F$  has characteristic 0, and let  $g \in F[T]$  be a polynomial with  $\deg g \geq 1$ . Then the formal derivative  $g' \in F[T]$  is non-zero.*

*Proof.* Let  $d = \deg g \geq 1$  and write

$$g = \sum_{i=0}^d a_i T^i \in F[T]$$

with  $a_d \neq 0$ . Then

$$g' = \sum_{i=0}^d i \cdot a_i T^{i-1}$$

so that the coefficient of  $T^{d-1}$  in  $g'$  is equal to  $d \cdot a_d$ . Since  $F$  has characteristic 0,  $d1_F \neq 0$ . Since  $a_d \neq 0$  by assumption, we conclude that the coefficient of  $T^{d-1}$  in  $g'$  is non-zero, hence  $g'$  itself is indeed non-zero.  $\square$

**Proposition 13.2.2.** *Let  $f \in F[T]$  be an irreducible polynomial.*

*a. If  $F$  has characteristic 0, then  $f$  has no repeated roots.*

*b. If  $F$  has characteristic  $p > 0$  then  $f$  has no repeated roots unless  $f$  has the form*

$$f(T) = g(T^p)$$

*for some polynomial  $g \in F[T]$ .*

*Proof.* Suppose that  $f$  has a repeated root. It follows from Proposition 13.1.3 that  $\gcd(f, f') \neq 1$ . But  $\deg(f') < \deg(f)$ . Thus if  $f' \neq 0$ , the irreducibility of  $f$  guarantees that  $f$  and  $f'$  have no common factor. Hence, the assumption that  $f$  has a repeated root implies that  $(\clubsuit) \quad f' = 0$ .

Now a. follows since if  $F$  has characteristic 0, Lemma 13.2.1 shows that the polynomial  $f'$  is non-zero, contradicting  $(\clubsuit)$ .

Now suppose that the characteristic of  $F$  is  $p > 0$  and write

$$f = \sum_{i=0}^N a_i T^i \quad \text{for } a_i \in F.$$

Suppose that  $f' = 0$ . Then

$$f' = \sum_{i=1}^n a_i \cdot i \cdot T^{i-1}.$$

So  $f' = 0 \implies a_i \cdot i = 0$  for all  $i$ . This equation shows that  $a_i = 0$  whenever  $i \not\equiv 0 \pmod{p}$ .

Thus the polynomial  $f$  has the form

$$f = \sum_{j=0}^M a_{jp} T^{jp} = g(T^p)$$

where

$$g = \sum_{j=0}^M a_{jp} T^j.$$

□

### 13.3 Perfect fields

*Definition 13.3.1.* A polynomial  $f \in F[T]$  is said to be *separable* if each irreducible factor of  $f$  has no repeated roots in a splitting field.

Of course, since two splitting fields of  $f$  are isomorphic Theorem 11.3.5, the condition that an irreducible factor of  $f$  has no repeated roots in a splitting field is independent of the choice of a splitting field.

*Definition 13.3.2.* A field  $F$  is said to be *perfect* if each irreducible polynomial is separable.

*Remark 13.3.3.* a. Proposition 13.2.2 implies that any field of characteristic 0 is perfect.

b. Let  $F = \mathbf{F}_p(X)$  be the field of rational functions over  $\mathbf{F}_p$  in the variable  $X$ . Then  $F$  is not perfect.

Indeed, the polynomial  $T^p - X \in F[T]$  is irreducible by Eisenstein's criterion Theorem 7.4.1.

But this polynomial has only one root  $\alpha$  (with multiplicity  $p$ ) in a splitting field since

$$T^p - X = (T - \alpha)^p \text{ by (12.1).}$$

On the other hand, some fields of characteristic  $p$  are perfect. Here is a useful characterization:

**Proposition 13.3.4.** *Let  $F$  be a field of characteristic  $p > 0$ . Then  $F$  is perfect if and only if*

$$F = F^p = \{x^p \mid x \in F\}.$$

*Proof.*  $\Leftarrow$ : Suppose that  $F = F^p$  and let  $f \in F[T]$  be an irreducible polynomial. We must argue that  $f$  is separable.

If  $f$  has a repeated root, we argued above that  $f = g(T^p)$  for some polynomial

$$g = \sum_{i=0}^r a_i T^i.$$

For each  $i$ , choose  $b_i \in F$  with  $b_i^p = a_i$ . Then

$$f = g(T^p) = \sum_{i=0}^r a_i T^{pi} = \sum_{i=0}^r b_i^p T^{pi} = \left( \sum_{i=0}^r b_i T^i \right)^p.$$

But this equation contradicts the assumption that  $f$  is irreducible in  $F[T]$ .

$\Rightarrow$ : Suppose that  $F$  is perfect and let  $x \in F$ . Consider the polynomial

$$f = T^p - x$$

and let  $g$  denote a monic irreducible factor of  $f$  in  $F[T]$ . Find a root  $\alpha$  of  $g$  in some extension field of  $F$ .

Then  $\alpha$  is also a root of  $f$ , so that  $\alpha^p = x$ . In  $F(\alpha)[T]$  we have the identity

$$f = T^p - x = T^p - \alpha^p = (T - \alpha)^p.$$

By unique factorization in  $E[T]$  – see Theorem 5.2.1 –, we find that  $g = (T - \alpha)^m$  for some  $1 \leq m \leq p$ . But  $g$  is irreducible, so the assumption that  $F$  is perfect means  $g$  has no repeated roots in the extension field  $E$ . Thus  $m = 1$  so that  $g = (T - \alpha)$ . This implies that  $\alpha \in F$  so indeed  $x$  has a  $p$ -th root in  $F$ .  $\square$

We can now prove the following important fact:

**Proposition 13.3.5.** *A finite field is perfect.*

*Proof.* Let  $F$  be a finite field, and recall that the Frobenius mapping  $\mathcal{F}(x) = x^p$  is a ring homomorphism  $F \rightarrow F$  – see Lemma 12.4.3. Moreover,  $\ker \mathcal{F} = \{0\}$  since  $x^p = 0 \implies x = 0$ ; this shows that  $\mathcal{F}$  is *injective*.

Since  $F$  is finite and  $\mathcal{F}$  is injective, one knows that  $\mathcal{F}$  is also *surjective*. This proves that  $F = F^p$ ; thus the field  $F$  is perfect by Proposition 13.3.4.  $\square$

*Remark 13.3.6.* Observe that the proof shows that  $\mathcal{F}$  is always injective for a field of characteristic  $p$ . Moreover, the image  $\mathcal{F}(F)$  coincides with  $F^p$ , which is therefore a *subfield* of  $F$ .

We see that the following are equivalent:

- i)  $F$  is *perfect*,
- ii) the Frobenius mapping  $\mathcal{F}$  is *onto*,
- iii) the Frobenius mapping  $\mathcal{F}$  is bijective, i.e. an *automorphism* of  $F$ .

## 14 Automorphisms of algebraic objects

Consider an algebraic object  $\mathcal{X}$  – e.g. a *group*, or a *ring*, or a *field*, or a *field extension*, or a *vector space over a field*.

Within the family of algebraic objects of the same type, there is a notion of *isomorphism*.

For the above list, probably the only case that raises eyebrows is the question: “what is an isomorphism of a field extension?”

Though a related question is: what is the right notion for isomorphism of “vector spaces over fields”? We’ll have more to say on this in the examples, below.

Once one has agreed on a notion of isomorphism, then for a fixed object  $\mathcal{X}$  one can consider the collection of all isomorphisms

$$\mathcal{X} \rightarrow \mathcal{X}$$

This collection is a group

$$\text{Aut}(\mathcal{X}),$$

the group of *automorphisms* of  $\mathcal{X}$ .

### 14.1 Automorphism examples

- Vector spaces

For a field  $F$  and an  $n$ -dimensional vector space  $V$  over  $F$ , the automorphism group

$$\text{Aut}(V) = \text{GL}(V) \simeq \text{GL}_n(F)$$

identifies with the group of invertible  $n \times n$  matrices with coefficients in  $F$ .

- Automorphisms of some finite abelian groups

Let  $m \geq 1$  and consider the group

$$A = \mathbf{Z}_m \times \mathbf{Z}_m,$$

a group with  $|A| = m^2$ .

Let’s represent elements  $x$  of  $A$  as column vectors:

$$x = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{for } a, b \in \mathbf{Z}_m.$$

Any matrix

$$M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \quad \text{for } \alpha, \beta, \delta, \gamma \in \mathbf{Z}_m$$

determines a *group homomorphism*

$\phi_M : A \rightarrow A$  given by the rule

$$\phi_M(x) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} + b \begin{bmatrix} \beta \\ \delta \end{bmatrix}$$

and  $\phi_M$  is an *automorphism* if and only if the determinant of  $M$  is a unit in  $\mathbf{Z}_m$  – i.e.  $\det M \in (\mathbf{Z}_m)^\times$ .

Thus

$$\text{Aut}(A) \simeq \text{GL}_2(\mathbf{Z}_m).$$

So, for example the matrix

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

has determinant  $-2 \equiv m - 2 \pmod{m}$ , and so it defines an automorphism of  $A$  whenever

$$\gcd(m, m - 2) = 1$$

i.e. whenever  $m$  is odd.

## 14.2 Automorphisms of field extensions

Our real interest in this course is in automorphisms of a field extension  $F \subseteq E$ . Here, an automorphism of the field extension is an automorphism  $\phi : E \rightarrow E$  such that  $\phi(a) = a$  for all  $a \in F$ .

In this setting,  $F$  is sometimes called the *base field* or *ground field*. We write

$$\text{Aut}_F(E) \quad \text{or} \quad \text{Aut}(E/F)$$

for the automorphism group of this field extension, and we also write  $\text{Gal}(E/F)$  for this group and call it the Galois group of the extension.

**Proposition 14.2.1.** *Let  $f \in F[T]$  be a monic polynomial and let  $E$  be field extension of  $F$  such that  $f$  splits over  $E$ . Write  $f = \prod_{i=1}^d (T - \alpha_i)$  with  $\alpha_i \in E$ . The action of the Galois group  $\text{Gal}(E/F)$  on  $E$  determines an action of  $\text{Gal}(E/F)$  on the set  $S = \{\alpha_1, \dots, \alpha_d\} \subset E$ .*

*Proof.* Write  $f = \sum_{i=0}^d a_i T^i$  for  $a_i \in F$ . Note that  $\sigma(a_i) = a_i$  for each  $i$ , since  $a_i \in F$ . Now if  $\alpha \in S$  we must argue that  $\sigma(\alpha) \in S$ . For this, we must argue that  $f(\sigma(\alpha)) = 0$ .

We know that  $f(\alpha) = 0$  since  $\alpha \in S$ . Since  $\sigma$  is a ring homomorphism and  $\sigma(a_i) = a_i$  for each  $i$ , we find that

$$0 = \sigma(0) = \sigma(f(\alpha)) = \sigma\left(\sum_{i=0}^d a_i \alpha^i\right) = \sum_{i=0}^d \sigma(a_i) \sigma(\alpha)^i = \sum_{i=0}^d a_i \sigma(\alpha)^i = f(\sigma(\alpha)).$$

This shows that  $f(\sigma(\alpha)) = 0$  so indeed  $\sigma(\alpha) \in S$  as required.  $\square$

- Example: quadratic extensions when the characteristic is not 2

Suppose that  $[E : F] = 2$  and that the characteristic of  $F$  is not 2.

Then  $E = F(\beta)$  for some element  $\beta \in E$ ,  $\beta \notin F$ ,  $\beta^2 \in F$ .

Indeed, we may choose a basis of  $E$  as an  $F$  vector space of the form  $1, \gamma$ . Then linear independence implies that  $\gamma \notin F$ . Let

$$f(T) = T^2 + aT + b \in F[T]$$

be the monic minimal polynomial of  $\gamma$  over  $F$ .

For any  $s \in F$ , we claim that the minimal polynomial of the element  $\gamma - s \in E$  has the form

$$\begin{aligned} f(T + s) &= (T + s)^2 + a(T + s) + b \\ &= T^2 + (a + 2s)T + s^2 + as + b \end{aligned}$$

Taking  $s = \frac{-a}{2}$ , we find that  $g(T) = f(T + s)$  has the form

$$g(T) = T^2 - c$$

for some  $c \in F$ , so that  $\beta = \gamma - s$  satisfies  $\beta^2 = c$ . Now it just remains to observe that

$$F(\gamma) = F(\gamma + s) = F(\beta).$$

Now, notice that every element of  $E$  has the form

$$a + b\beta \quad \text{for } a, b \in F.$$

Now, the roots of  $T^2 - c$  in  $E$  are  $\pm\beta$ . Since  $T^2 - c$  is irreducible over  $F$ , it follows that there is an isomorphism

$$\phi : E = F(\beta) \rightarrow E = F(\beta)$$

for which  $\phi(\beta) = -\beta$  and  $\phi(s) = s$  for all  $s \in F$ .

Thus

$$\phi(a + b\beta) = a - b\beta \quad \text{for } a, b \in F.$$

**Proposition 14.2.2.** *Suppose that the characteristic of  $F$  not equal 2. For  $E = F(\gamma) = F(\beta)$  a quadratic extension as above,  $\text{Aut}_F(E) = \langle \phi \rangle$  and in particular  $|\text{Aut}_F(E)| = 2$ .*

- Example: quadratic extensions in characteristic 2.

Suppose that the characteristic of  $F$  is 2, and consider a polynomial of the form  $T^2 - c \in F[T]$ . If  $\beta$  is a root of this polynomial then

$$T^2 - c = T^2 - \beta^2 = (T - \beta)^2$$

since the characteristic is 2.

Just for emphasis, let's double check this:

$$(T - \beta)^2 = T^2 - 2\beta T + (-\beta)^2 = T^2 + c = T^2 - c.$$

Thus the polynomial  $T^2 - c$  has a single root  $\beta$  which is repeated twice. It is irreducible over  $F$  if and only if  $\beta \notin F$ .

However, in general at least, there are irreducible quadratic polynomials with distinct roots in characteristic 2.

Consider a polynomial of the form

$$f = T^2 + T + a \quad \text{for } a \in F$$

and suppose that  $\beta$  is a root of  $f$ ; thus

$$\beta^2 + \beta + a = 0.$$

We claim that also  $\beta + 1$  is a root of  $f$ . Indeed,

$$\begin{aligned}
 f(\beta + 1) &= (\beta + 1)^2 + (\beta + 1) + a \\
 &= \beta^2 + 1 + \beta + 1 + a \\
 &= \beta^2 + \beta + a + 2 \\
 &= \beta^2 + \beta + a \\
 &= f(\beta) = 0.
 \end{aligned}$$

It follows that

$$f = T^2 + T + a = (T + \beta)(T + \beta + 1)$$

i.e.  $\beta$  and  $\beta + 1$  are the distinct roots of  $f$ . Recall that  $\mathbf{F}_4 = \mathbf{F}_2(\beta)$  where  $\beta^2 + \beta = 1$ .

Note that for any  $F$  of char. 2, the polynomial  $f = T^2 + T + a$  is irreducible if and only if  $\beta \notin F$  – this follows from Proposition 7.1.4.

Suppose  $f$  is irreducible and let  $E = F(\beta)$ . Recall that an element of  $E$  has the form

$$a + b\beta \quad \text{for } a, b \in F.$$

Since  $\beta$  and  $\beta + 1$  are the roots of  $f$ , there is an automorphism

$$\phi : E = F(\beta) \rightarrow E = F(\beta + 1) = F(\beta)$$

for which  $\phi(\beta) = \beta + 1$  and  $\phi(s) = s$  for  $s \in F$ .

Thus

$$\phi(a + b\beta) = a + b + b\beta.$$

*Remark 14.2.3.* When  $F = \mathbf{F}_2$  and  $\beta^2 + \beta = 1$ , notice that

$$(a + b\beta)^2 = \phi(a + b\beta) = a + b + b\beta.$$

## 15 The Fundamental Theorem of Galois Theory

Let  $F$  be a field and let  $E$  be the splitting field over  $F$  of some *separable* polynomial  $g \in F[T]$ . Loosely speaking, the fundamental theorem of Galois Theory relates two things:

- *intermediate fields*  $L$ , where  $F \subseteq L \subseteq E$ , and
- *subgroups*  $H$ , where  $H \subseteq \text{Gal}(E/F)$ .

### 15.1 Subfields from subgroups

**Proposition 15.1.1.** *Let  $K$  be any field and let  $H$  be any subgroup of the group  $\text{Aut}(K)$  of automorphisms of  $K$ . Then*

$$K^H = \{x \in K \mid h \cdot x = x \quad \forall h \in H\}$$

*is a subfield of  $K$ .*

*Proof.* If  $x, y \in K^H$  with  $x \neq 0$ , we must argue that  $x - y \in K^H$ , that  $x \cdot y \in K^H$  and that  $\frac{1}{x} \in K^H$ . But for each  $h \in H$  we have:

$$h(x - y) = h(x) - h(y) = x - y \implies x - y \in K^H$$

$$h(x \cdot y) = h(x) \cdot h(y) = x \cdot y \implies x \cdot y \in K^H$$

and

$$h\left(\frac{1}{x}\right) = \frac{1}{h(x)} = \frac{1}{x} \implies \frac{1}{x} \in K^H.$$

□

### 15.2 Splitting fields and Galois groups

The following result follows the proof of Lemma 11.3.1

**Proposition 15.2.1.** *Let  $g \in F[T]$  be a separable polynomial and let  $E$  be a splitting for  $g$  over  $F$ . Suppose that  $\phi : F \rightarrow F_1$  is a field isomorphism and write  $g_1 = \phi(g) \in F_1[T]$  and write  $E_1$  for a splitting field for  $g_1$  over  $F_1$ . Then there are exactly  $[E : F]$  isomorphisms  $\theta : E \rightarrow E_1$  such that  $\theta_F = \phi$ .*

*Proof.* We are going to essentially repeat the proof of Lemma 11.3.1 with a little more book-keeping.

Proceed by induction on  $\deg g$ . If  $g$  has degree 0 or 1 then  $F = E$  and  $F_1 = E_1$  and there is nothing to prove.

So suppose that  $d = \deg g > 1$  and that the result is known for all fields  $F$  and all polynomials of degree less than  $d$ . Let  $p$  be an irreducible factor of  $g$  of degree  $e \leq d$  and write  $p_1 = \phi(p)$  which is thus an irreducible factor of  $g_1$ . Of course,  $p$  splits over  $E$  and we choose a root  $\alpha \in E$  of  $p$ .

Since  $g$  has no repeated roots, the same is true for  $g_1$  and  $p_1$ . Thus there are  $e$  roots of  $p_1$  in  $E_1$ , and for any root  $\beta \in E_1$  of  $p_1$  Lemma 11.3.1 gives an isomorphism  $\phi' : F(\alpha) \rightarrow F_1(\beta)$  such that  $\phi'_F = \phi$  and such that  $\phi'(\alpha) = \beta$ . Thus, there are exactly  $d = [F(\alpha) : F]$  isomorphisms  $F(\alpha) \rightarrow F_1(\beta)$  whose restriction to  $F$  is  $\phi$ .

Now, we may write  $g = (T - \alpha)s$  for  $s \in F(\alpha)[T]$ , and  $g_1 = (T - \beta)s_1$  for  $s_1 \in F_1(\beta)[T]$ . Since  $g_1 = \phi(g)$  we have  $s_1 = \phi'(s)$ . It is clear that  $E$  is a splitting field for  $s$  over  $F$  and that  $E_1$  is a splitting field for  $s_1$  over  $F_1$ .

Since  $\deg s = d - 1$ , for any isomorphism  $\theta'$  as in the preceding paragraph, the induction hypothesis guarantees that there are precisely  $[E : F(\alpha)]$  isomorphisms  $\theta : E \rightarrow E'$  for which  $\theta_{F(\alpha)} = \phi'$ .

It therefore follows that there are  $[E : F(\alpha)] \cdot [F(\alpha) : F] = [E : F]$  isomorphisms  $\theta : E \rightarrow E_1$  with  $\theta_F = \phi$ , as required.  $\square$

As an immediate consequence of Proposition 15.2.1 we obtain the following:

**Corollary 15.2.2.** *Let  $g \in F[T]$  be a separable polynomial, and let  $E$  denote a splitting field of  $g$  over  $F$ . Then  $|\text{Gal}(E/F)| = [E : F]$ .*

**Proposition 15.2.3.** *Suppose that  $E$  is the splitting field over  $F$  of a separable polynomial  $g \in F[T]$ . Let  $\Gamma = \text{Gal}(E/F)$ . Then  $F = E^\Gamma$ .*

*Proof.* Let  $L = E^\Gamma$ , so that  $L$  is an intermediate field:

$$F \subseteq L = E^\Gamma \subseteq E.$$

Viewing  $g$  as a polynomial in  $L[T]$ , it is clear that  $E$  is a splitting field of  $g$  over  $L$ .

Now Corollary 15.2.2 guarantees that

$$[E : L] = |\text{Gal}(E/L)| \quad \text{and} \quad [E : F] = |\text{Gal}(E/F)|.$$

Since  $F \subseteq L$ , we have  $\text{Gal}(E/L) \subseteq \Gamma = \text{Gal}(E/F)$ . The assumption  $L = E^\Gamma$  shows that any automorphism of  $E$  which is the identity on  $F$  is the identity on  $L$ ; this shows that

$$\Gamma = \text{Gal}(E/F) = \text{Gal}(E/L).$$

It now follows that  $[E : L] = [E : F]$  and hence that  $L = F$ .  $\square$

### 15.3 Fixed fields and some linear algebra

The correspondence between subgroups  $H \subset \text{Gal}(E/F)$  and intermediate fields  $F \subseteq L \subset E$  will be given by the assignment

$$H \mapsto E^H$$

(we'll formulate the statement more precisely later on).

We are ultimately going to argue that this assignment determines a one-to-one correspondence between the subgroups and the intermediate fields. For this, we require some numerical estimates relating the degrees  $[E : E^H]$  and the orders  $|H|$ . These estimates are obtained using a result of E. Artin:

**Proposition 15.3.1.** *Let  $G$  be a finite group of automorphisms of a field  $K$  and let  $L = K^G$ . Then  $[K : L] \leq |G|$ .*

*Proof.* If  $|G| = n$ , let us write

$$G = \{\theta_1, \theta_2, \dots, \theta_n\}$$

where  $\theta_1 = 1_G$ .

We must argue that  $[K : L] \leq n$ . Suppose the contrary, and choose  $n + 1$  elements  $u_1, u_2, \dots, u_{n+1} \in K$  which are linearly independent over  $L$ .

Now form the following  $n \times (n + 1)$  matrix with entries in  $K$ :

$$M = \begin{pmatrix} \theta_1(u_1) & \theta_1(u_2) & \cdots & \theta_1(u_{n+1}) \\ \theta_2(u_1) & \theta_2(u_2) & \cdots & \theta_2(u_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \theta_n(u_1) & \theta_n(u_2) & \cdots & \theta_n(u_{n+1}) \end{pmatrix} \in \text{Mat}_{n \times (n+1)}(K).$$

Since  $M$  has  $n$  rows, we know that the *rank* of  $M$  satisfies

$$\text{rk}(M) \leq n.$$

On the other hand, linear algebra tells us that

$$\dim_K \text{Null}(M) + \text{rk}(M) = n + 1 = \# \text{ of columns of } M.$$

Thus

$$\dim_K \text{Null}(M) = n + 1 - \text{rk}(M) \geq n + 1 - n = 1$$

and we conclude that there is a non-zero solution  $\mathbf{x} = \mathbf{a} \in K^{n+1}$  to the matrix equation

$$(\clubsuit) \quad M \cdot \mathbf{x} = \mathbf{0}.$$

Among all possible non-zero solutions

$$\mathbf{0} \neq \mathbf{a} = (a_1 \ a_2 \ \cdots \ a_{n+1})^T$$

to  $(\clubsuit)$ , choose one with the smallest number of non-zero coefficients  $a_i \in K$ .

After renumbering the indices on the  $u_i$  and the  $x_j$ , we may suppose that  $a_1 \neq 0$ . Since the vector  $\left(\frac{1}{a_1}\right) \cdot \mathbf{a}$ , remains a solution to  $(\clubsuit)$ , we may and will suppose that  $a_1 = 1$ .

Recall that  $\theta_1 = 1_G = \text{id}_K$ . The first coefficient in the vector equation

$$\mathbf{0} = M \cdot \mathbf{a}$$

gives

$$0 = \sum_{i=1}^{n+1} a_i \theta_1(u_i) = \sum_{i=1}^{n+1} a_i u_i.$$

Since the  $u_i$  are linearly independent over  $L$  by assumption, some  $a_j$  must be in  $K$  and not in  $L = K^G$ .

Renumbering again, we may and will suppose that  $a_2 \in K$ ,  $a_2 \notin L = K^G$ .

Of course,  $a_2 \notin K^G \implies ga_2 \neq a_2$  for some  $g \in G$ , and in turn we recall that  $g = \theta_i$  for some  $i > 1$  hence we have

$$\theta_i(a_2) \neq a_2.$$

Consider the matrix  $\theta_i(M) \in \text{Mat}_{n \times (n+1)}(K)$  given by

$$\theta_i(M) = \begin{pmatrix} \theta_i \cdot \theta_1(u_1) & \theta_i \cdot \theta_1(u_2) & \cdots & \theta_i \cdot \theta_1(u_{n+1}) \\ \theta_i \cdot \theta_2(u_1) & \theta_i \cdot \theta_2(u_2) & \cdots & \theta_i \cdot \theta_2(u_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \theta_i \cdot \theta_n(u_1) & \theta_i \cdot \theta_n(u_2) & \cdots & \theta_i \cdot \theta_n(u_{n+1}) \end{pmatrix}.$$

Since  $G$  is group, the rows of  $\theta_i(M)$  are the same as those of  $M$ , but in a different *order*. In particular,

$$\text{Null}(M) = \text{Null}(\theta_i(M)).$$

On the other hand, it is clear that

$$\mathbf{0} = \theta_i(\mathbf{0}) = \theta_i(M \cdot \mathbf{a}) = \theta_i(M) \cdot \theta_i(\mathbf{a}).$$

Thus, both

$$\mathbf{a} = (1 \quad a_2 \quad \cdots \quad a_{n+1})^T$$

and

$$\theta_i(\mathbf{a}) = (1 \quad \theta_i(a_2) \quad \cdots \quad \theta_i(a_{n+1}))^T.$$

are solutions to  $(\clubsuit)$ . Since  $a_2 \neq \theta_i(a_2)$ ,  $\mathbf{v} = \mathbf{a} - \theta_i(\mathbf{a})$  is a non-zero solution to  $(\clubsuit)$ , as well.

Now notice for any  $j$  that  $\mathbf{v}_j = \theta_i(a_j) - a_j \neq 0 \implies a_j \neq 0$ . Since the first coefficient of  $\mathbf{v}$  is 0 and the first coefficient of  $\mathbf{a}$  is non-zero, it is clear that  $\mathbf{v}$  has strictly fewer non-zero entries than does  $\mathbf{a}$ . This contradicts the choice of  $\mathbf{a}$ , and completes the proof.  $\square$

## 15.4 Normal and separable extensions

Let  $E$  be an algebraic extension of the field  $F$ .

*Definition 15.4.1.*  $E$  is a *normal extension* of  $F$  if every irreducible polynomial in  $F[T]$  that contains a root in  $E$  actually *splits* over  $E$ .

Recall Definition 13.3.1 that a polynomial is separable if it has no repeated roots in a splitting field.

*Definition 15.4.2.*  $E$  is a *separable extension* of  $F$  if the minimal polynomial  $m_\alpha \in F[T]$  over  $F$  of each element  $\alpha$  of  $E$  is a separable polynomial – i.e.  $m_\alpha$  has no repeated roots in a splitting field.

Of course,  $E$  is a separable extension of  $F$  just in case each irreducible polynomial in  $F[T]$  with a root in  $E$  is separable.

**Proposition 15.4.3.** *Let  $E$  be an extension field of  $F$ , and let  $\Gamma = \text{Gal}(E/F)$ .*

- a. If  $F = E^\Gamma$ , then  $E$  is a normal, separable extension of  $F$ .*
- b. Suppose that  $E$  is the splitting field over  $F$  of some separable polynomial  $g \in F[T]$ . Then  $E$  is a normal and separable extension of  $F$ .*

*Proof.* According to Proposition 15.2.3, if  $E$  is the splitting field over  $F$  of a separable polynomial, then  $F = E^\Gamma$ . Thus assertion b. is an immediate consequence of assertion a.

To prove a., let  $h \in F[T]$  be an *irreducible* polynomial, and suppose that  $\alpha \in E$  is a root of  $h$ . We must argue that  $h$  is separable and actually *splits* over  $E$ .

Consider the *orbit*  $\mathcal{O}$  of the root  $\alpha$  under the action of  $\Gamma$ :

$$\mathcal{O} = \{g\alpha \mid g \in G/H\}$$

where  $H = \text{Stab}_\Gamma(\alpha)$ .

If  $g_1, \dots, g_m$  is a system of coset representatives for  $H$  in  $\Gamma$ , there are  $m = [\Gamma : H]$  distinct elements of  $\mathcal{O}$ :

$$\mathcal{O} = \{g_1\alpha, g_2\alpha, \dots, g_m\alpha\}.$$

Form the polynomial

$$h_1 = \prod_{\beta \in \mathcal{O}} (T - \beta) = \prod_{g \in \Gamma/H} (T - g\alpha) \in E[T].$$

Note that by construction  $h_1$  has  $m$  distinct roots in  $E$ . We first claim that in fact  $h_1 \in F[T]$ .

Of course, for any polynomial  $\ell \in E[T]$ , we know that

$$\ell \in F[T] = E^\Gamma[T] \iff x\ell = \ell \text{ for all } x \in \Gamma.$$

Thus, we must argue for each  $x \in \Gamma$  that  $xh_1 = h_1$ .

Well, for  $x \in \Gamma$ , we have

$$xh_1 = x \cdot \left( \prod_{g \in \Gamma/H} (T - g\alpha) \right) = \prod_{g \in \Gamma/H} (T - xg\alpha) = (\diamond).$$

Now using the substitution  $h = xg$ , note that

$$(\diamond) = \prod_{h \in \Gamma/H} (T - h\alpha) = h_1.$$

This proves that  $h_1 \in F[T]$ .

Since  $h$  is the *minimal polynomial* of  $\alpha$  over  $F$ , since  $h_1 \in F[T]$ , and since  $h_1$  has  $\alpha$  as a root by construction, we conclude that  $h \mid h_1$ . Since  $h_1$  splits over  $E$ , unique factorization in  $E[T]$  shows that  $h$  splits over  $E$ . Since  $h_1$  is separable by construction (it has no repeated roots), also  $h$  is separable. This completes the proof that  $E$  is a normal, separable extension of  $F$ .  $\square$

**Proposition 15.4.4.** *Let  $E$  be a finite, normal, separable extension of  $F$ . Then  $E$  is the splitting field over  $F$  of a separable polynomial  $g \in F[T]$ .*

*Proof.* Let  $\alpha_1, \dots, \alpha_n \in E$  be elements such that  $E = F(\alpha_1, \dots, \alpha_n)$ . For  $1 \leq i \leq n$ , write  $f_i \in F[T]$  for the minimal polynomial over  $F$  of the element  $\alpha_i$ .

Since  $f_i$  has the root  $\alpha_i$  in  $E$  and since  $E$  is normal over  $F$ , the polynomial  $f_i$  splits over  $E$ . Since  $E$  is generated over  $F$  by the roots of the  $f_i$ , it follows that  $E$  is a splitting field of the separable polynomial

$$f = \prod_{i=1}^n f_i \in F[T].$$

$\square$

*Remark 15.4.5.* It is actually true that any finite separable extension  $F \subset E$  is *primitive*; namely, there is an element  $\alpha \in E$  such that  $E = F(\alpha)$  – this result is known as the *Primitive Element Theorem*. We don't require this fact, and so I haven't given a proof. The proof of the previous Proposition would be slightly more streamlined using the Primitive Element Theorem.

## 15.5 The Fundamental Theorem

Before stating the main theorem of Galois theory, observe that results so far enable us to recognize Galois groups in some useful situations:

**Proposition 15.5.1.** *Suppose that  $F \subseteq E$  is an extension field, that  $G \subset \text{Aut}(E)$  is a finite group of automorphisms, and that  $F = E^G$ . Then*

$$G = \text{Gal}(E/F).$$

*Proof.* According to Proposition 15.4.3,  $E$  is a separable and normal extension of  $F$ , and

$$[E : F] = |\text{Gal}(E/F)|.$$

Since  $F = E^G$ , note that  $G \subseteq \text{Gal}(E/F)$ . Artin's result – Proposition 15.3.1 – implies that  $[E : F] \leq |G|$ , and we see that

$$[E : F] \leq |G| \leq |\text{Gal}(E/F)| = [E : F].$$

Thus equality holds everywhere, and we conclude that  $G = \text{Gal}(E/F)$  as required.  $\square$

**Theorem 15.5.2.** *Let  $E$  be a splitting field over  $F$  of a separable polynomial  $g \in F[T]$ , and let  $\Gamma = \text{Gal}(E/F)$ .*

*a. There is a one-to-one correspondence between subgroups of  $\Gamma$  and intermediate fields of the extension  $F \subseteq E$  given by  $H \mapsto E^H$ .*

*i. If  $H$  is a subgroup of  $\Gamma$ , then*

$$H = \text{Gal}(E/E^H).$$

*ii. If  $F \subseteq K \subseteq E$  is an intermediate field, the corresponding subgroup is  $\text{Gal}(E/K) \subseteq \Gamma$ , and*

$$K = E^{\text{Gal}(E/K)}.$$

*b. For any subgroup  $H \subseteq \Gamma$ ,*

$$[E : E^H] = |H| \quad \text{and} \quad [E^H : F] = [\Gamma : H].$$

*c. Under the correspondence of a., the subgroup  $H$  is normal in  $\Gamma$  if and only if the subfield  $K = E^H$  is a normal extension of  $F$ . If this is the case, then*

$$\text{Gal}(E^H/F) = \Gamma/H \simeq \text{Gal}(E/F)/\text{Gal}(E/K).$$

*Proof.* For (a), write  $\mathcal{G}$  for the set of subgroups of  $\Gamma$  and write  $\mathcal{J}$  for the set of intermediate fields  $K$  (so  $F \subseteq K \subseteq E$ ).

We consider the mapping

$$\mathcal{G} \rightarrow \mathcal{J} \quad \text{given by } H \mapsto E^H$$

and the mapping

$$\mathcal{J} \rightarrow \mathcal{G} \quad \text{given by } K \mapsto \text{Gal}(E/K).$$

Let us pause to observe that if  $H_1, H_2 \subseteq G$  are subgroups with  $H_1 \subseteq H_2$ , then  $E^{H_2} \subseteq E^{H_1}$  – so the assignment  $H \mapsto E^H$  is *inclusion reversing*.

Similarly, if  $K_1 \subseteq K_2$  are intermediate fields, then  $\text{Gal}(E/K_2) \subseteq \text{Gal}(E/K_1)$ , so the assignment  $K \mapsto \text{Gal}(E/K)$  is inclusion reversing.

We observe that the statements of i. and ii. precisely confirm that these mappings are inverse to one another. So to prove a., we need to confirm that

i.  $\text{Gal}(E/E^H) = H$ , and

that

ii.  $K = E^{\text{Gal}(E/K)}$ .

Now, i. is an immediate consequence of Proposition 15.5.1.

On the other hand, suppose that  $K$  is an intermediate field:  $F \subseteq K \subseteq E$ . Since  $E$  is the splitting field of a separable polynomial over  $F$ , then  $E$  is the splitting field over  $K$  of this same polynomial. Thus Proposition 15.2.3 implies that  $E^{\text{Gal}(E/K)} = K$  as required.

This completes the proof of a. As to b., let  $H$  be a subgroup of  $\Gamma$ . Since  $E$  is a splitting field over  $E^H$  of a separable polynomial, and since we've already seen that  $H = \text{Gal}(E/E^H)$ , we know by Proposition 15.3.1 that

$$[E : E^H] = |H|.$$

Now, the same reasoning shows that  $F = E^\Gamma$  and

$$[E : F] = [E : E^\Gamma] = |\Gamma|.$$

The remaining statement of b. now follows from a calculation:

$$[E^H : F] = \frac{[E : F]}{[E : E^H]} = \frac{|\Gamma|}{|H|} = [\Gamma : H].$$

This completes the proof b.

Finally, consider c. Let  $F \subseteq K \subseteq E$  be an intermediate extension, and let  $H = \text{Gal}(E/K) \subseteq \Gamma$ . We must argue that  $K$  is a normal extension of  $F$  if and only if  $H$  is a normal subgroup of  $\Gamma$ , and in case  $H$  is normal, we will argue that  $\Gamma/H$  is isomorphic to  $\text{Gal}(K/F)$ .

$\Rightarrow$ : Suppose that  $K$  is a normal extension of  $F$ . To show that  $H$  is a normal subgroup of  $\Gamma$ , let  $\phi$  be an arbitrary element of  $\Gamma$ , and let  $\theta \in H = \text{Gal}(E/K)$ .

We must argue that  $\phi^{-1} \circ \theta \circ \phi \in H$ . For this, we must argue that  $\phi^{-1} \circ \theta \circ \phi$  is the identity on  $K$ .

Let  $u \in K$  and let  $p \in F[T]$  be the minimal polynomial of  $u$  over  $F$ . Since  $\phi \in \Gamma = \text{Gal}(E/F)$ , the element  $\phi(u)$  is again a root of  $p$ . Since  $K$  is a normal extension, it follows that  $\phi(u) \in K$ . Now,  $\theta|_K$  is the identity on  $K$ , so that

$$\theta(\phi(u)) = \phi(u) \implies \phi^{-1} \circ \theta \circ \phi(u) = u.$$

This proves that indeed  $H$  is normal in  $\Gamma$ .

$\Leftarrow$ : Suppose that  $H$  is a normal subgroup of  $\Gamma$ . We must argue that  $K$  is a normal extension of  $F$ .

We are first going to argue that  $\Gamma/H \simeq \text{Gal}(K/F)$ . To carry out this argument, we first contend that for any automorphism  $\phi$  in  $\Gamma$ , the restriction of  $\phi$  to  $K$  takes values in  $K$ . Let  $u \in K$ .

To argue that  $\phi(u) \in K = E^H$ , let  $\theta \in H$ . Since  $H$  is normal in  $\Gamma$ ,  $\theta_1 = \phi^{-1} \circ \theta \circ \phi \in H$ . Thus

$$\theta \circ \phi = \phi \circ \theta_1.$$

Now notice that

$$\theta(\phi(u)) = \phi(\theta_1(u)) = \phi(u)$$

since  $\theta_1$  is the identity on  $K$ . This shows that indeed  $\phi(u) \in E^H = K$ .

It now follows that the restriction of  $\phi$  to  $K$  takes values in  $K$ . Since  $\ker \phi = \{0\}$ ,  $\phi$  is a one-to-one mapping. Since  $\phi$  is an  $F$ -linear mapping and  $K$  is a finite dimensional vector space over  $F$ , conclude that  $\phi|_K$  is onto and thus determines an automorphism of  $K$ .

We have thus defined a *group homomorphism*

$$(\diamond) \quad \phi \mapsto \phi|_K : \text{Gal}(E/F) \rightarrow \text{Gal}(K/F).$$

The kernel of the group homomorphism  $(\diamond)$  consists in the automorphisms  $\phi$  whose restriction to  $K$  is the identity – i.e. the kernel is  $\text{Gal}(E/K) = H$ .

On the other hand, we claim that the homomorphism  $(\diamond)$  is *onto*. Indeed, since  $E$  is a splitting field over  $K$  of a (separable) polynomial, Section 11.3 shows that for any automorphism  $\theta : K \rightarrow K$ , we may find an automorphism  $\hat{\theta} : E \rightarrow E$  with  $\hat{\theta}|_K = \theta$ .

It now follows that  $\Gamma/H \simeq \text{Gal}(K/F)$ .

To complete the proof that  $K$  is normal, note first that  $[\Gamma : H] = [K : F]$  by b. This proves that  $|\text{Gal}(K/F)| = [K : F]$ . Since  $\text{Gal}(K/F)$  is a finite group, the [Proposition on normal extensions as fixed fields](#normal-as-fixed-field) implies that  $K$  is a normal separable extension of  $K^{\text{Gal}(K/F)}$ .

But then

$$[K : K^{\text{Gal}(K/F)}] = |\text{Gal}(K/F)| = [K : F]$$

which implies that  $F = K^{\text{Gal}(K/F)}$  and we conclude that  $K$  is a normal separable extension of  $F$ . This completes the proof of c, and of the Theorem.  $\square$

## 16 Examples of Galois groups

### 16.1 Galois groups for finite fields

Let  $K = \mathbf{F}_p$  be the field of  $p$  elements, let  $n \geq 1$  be an integer and let  $L = \mathbf{F}_q$  be the finite field having  $q = p^n$  elements.

We have seen that  $L$  is a splitting field over  $K$  of the polynomial  $T^q - T$  Proposition 12.3.1. And we have seen that  $K$  is *perfect*. Thus, Corollary 15.2.2 implies that

$$|\mathrm{Gal}(L/K)| = n.$$

Finally, we have seen that the mapping

$$x \mapsto x^p$$

is a ring homomorphism  $\mathcal{F} : L \rightarrow L$ ; see Lemma 12.4.3.

Since  $x^p = 0 \implies x = 0$ , we know that  $\ker \mathcal{F} = \{0\}$  so that  $\mathcal{F}$  is a one-to-one mapping  $L \rightarrow L$ . Since  $L$  is finite,  $\mathcal{F}$  is also onto. We call  $\mathcal{F}$  the *Frobenius automorphism* of  $L$ .

Fermat's Little Theorem tells us that  $m^p \equiv m \pmod{p}$  for  $m \in \mathbb{Z}$ ; this immediately shows that  $\mathcal{F}(t) = t$  for  $t \in K$  and thus

$$\mathcal{F} \in \mathrm{Gal}(L/K).$$

**Proposition 16.1.1.**

$$\mathrm{Gal}(L/K) = \mathrm{Gal}(\mathbf{F}_q/\mathbf{F}_p) = \langle \mathcal{F} \rangle;$$

*i.e. the Galois group of  $\mathbf{F}_{p^n}$  over  $\mathbf{F}_p$  is cyclic and is generated by the Frobenius*

*Proof.* We have already seen that  $\mathbf{F}_q$  is the splitting field over  $\mathbf{F}_p$  of the polynomial  $T^q - T$ , and we have seen that this polynomial has  $q$  distinct roots. Thus we may apply the previous Theorem to conclude that the order of the Galois group  $\mathrm{Gal}(L/K)$  is  $n$ . Thus, it suffices to argue that the order  $o(\mathcal{F})$  of the Frobenius automorphism in the Galois group is equal to  $n$ .

Well, notice for  $x \in L$  and  $m \geq 1$  that

$$\mathcal{F}^m(x) = \mathcal{F}(\mathcal{F}^{m-1}(x)) = \mathcal{F}(x^{p^{m-1}}) = \left(x^{p^{m-1}}\right)^p = x^{p^m}.$$

We have seen that each element of  $L$  is a root of  $T^q - T$ , and it follows that  $\mathcal{F}^n(x) = x^q = x$  so indeed  $\mathcal{F}^n = \mathrm{id}$ .

Now if  $\mathcal{F}^m = \mathrm{id}$  for  $m \leq n$  then every  $x \in L$  is a root of  $T^{p^m} - T$ . Since  $|L| = q = p^n$  it follows that  $p^n \leq p^m$  and hence  $n \leq m$ . We conclude that  $n = m$  so that indeed the order of the element  $\mathcal{F}$  is  $n$ . This completes the proof.  $\square$

*Remark 16.1.2.* In fact, for any  $p$ -power  $q = p^r$  and any  $m \geq 1$

$$\mathrm{Gal}(\mathbf{F}_{q^m}/\mathbf{F}_q) = \langle \mathcal{F}^r \rangle$$

*Example 16.1.3.*

$$\mathbf{F}_{p^p} \simeq \mathbf{F}_p(\alpha)$$

where  $\alpha$  is a root of the polynomial  $f = T^p - 1 - 1 \in \mathbf{F}_p[T]$ . More precisely we saw that

$$f = \prod_{i=0}^{p-1} (T - \alpha - i).$$

In particular, a typical element  $x$  of  $\mathbf{F}_{p^p}$  may be written uniquely in the form

$$x = t_0 + t_1\alpha + t_2\alpha^2 + \cdots + t_{p-1}\alpha^{p-1}$$

for scalars  $t_i \in \mathbb{F}_p = \mathbb{Z}_p$ .

Now, since  $\alpha$  is a root of  $f$ , we see that  $\alpha^p = \alpha + 1$ . Thus

$$\mathcal{F}(\alpha) = \alpha + 1$$

and in particular,

$$\begin{aligned} \mathcal{F}(x) &= \mathcal{F}(t_0 + t_1\alpha + t_2\alpha^2 + \cdots + t_{p-1}\alpha^{p-1}) \\ &= \mathcal{F}(t_0) + \mathcal{F}(t_1\alpha) + \mathcal{F}(t_2\alpha^2) + \cdots + \mathcal{F}(t_{p-1}\alpha^{p-1}) \\ &= t_0 + t_1\mathcal{F}(\alpha) + t_2\mathcal{F}(\alpha)^2 + \cdots + t_{p-1}\mathcal{F}(\alpha)^{p-1} \\ &= t_0 + t_1(\alpha + 1) + t_2(\alpha + 1)^2 + \cdots + t_{p-1}(\alpha + 1)^{p-1} \end{aligned}$$

For example, when  $p = 3$

$$\begin{aligned} \mathcal{F}(x) &= t_0 + t_1(\alpha + 1) + t_2(\alpha + 1)^2 \\ &= (t_0 + t_1 + t_2) + (t_1 + 2t_2)\alpha + t_2\alpha^2 \end{aligned}$$

## 16.2 Cyclic extensions

### 16.2.1 Some generalities about cyclic extensions

Suppose that  $\Gamma = \langle \sigma \rangle$  is a cyclic group of order  $n = |\Gamma|$ .

Recall that for each divisor  $d$  of  $n$ , there is a unique subgroup  $H$  of  $\langle \sigma \rangle$  of order  $d$ , namely  $H = \langle \sigma^{n/d} \rangle$ , and that these account for all subgroups of  $\Gamma$ .

Theorem 15.5.2 then implies that for each divisor  $d$  of  $n$ , there is a unique intermediate field  $K$  with  $[E : K] = d$ , and that these account for all intermediate fields of the extension. Moreover, the extension  $K$  with  $[E : K] = d$  is given by

$$K = E^{\langle \sigma^{n/d} \rangle}$$

Since there are exactly  $\phi(n)$  subgroups of  $\Gamma = \langle \sigma \rangle$ , there are exactly  $\phi(n)$  intermediate fields (where  $\phi$  is Euler's function).

$$\begin{array}{c} E \\ \Big|_d \\ K = E^{\langle \sigma^{n/d} \rangle} \\ \Big|_{n/d} \\ F \end{array}$$

Note that  $\text{Gal}(E/E^{\langle \sigma^{n/d} \rangle}) = \langle \sigma \rangle$  and  $\text{Gal}(E^{\langle \sigma^{n/d} \rangle}/F) = \langle \sigma \rangle / \langle \sigma^{n/d} \rangle$

### 16.2.2 Finite fields and cyclic extensions

Let  $p$  be a prime number and let  $F = \mathbf{F}_p$ . If  $q = p^n$  and  $E = \mathbf{F}_q$ , we showed in Proposition 16.1.1 that

$$\text{Gal}(E/F) = \text{Gal}(\mathbf{F}_q/\mathbf{F}_p) = \langle \mathcal{F} \rangle$$

is cyclic of order  $n$ , where  $\mathcal{F}$  is the *Frobenius automorphism*.

Thus the discussion in Section 16.2.1 shows that the intermediate extensions of  $\mathbf{F}_p \subset \mathbf{F}_q$  are labelled by divisors  $d$  of  $n$ . Of course, we already knew this – we proved in Theorem 12.4.7 that for each divisor  $e$  of  $n$ , there is a unique subfield of order  $\mathbf{F}_{p^e}$ .

### 16.2.3 Roots of unity of prime order

For a prime number  $p$ , the cyclotomic polynomial

$$f_p = \frac{T^p - 1}{T - 1}$$

is irreducible over  $\mathbf{Q}$ ; see Section 7.5.

Let  $\zeta = \zeta_p$  be a root of  $f_p$  in some extension of  $\mathbf{Q}$ . For each integer  $i$ , notice that  $\zeta^i$  is a root of  $T^p - 1$  since  $(\zeta^i)^p = (\zeta^p)^i = 1^i = 1$ . It follows that  $E = \mathbf{Q}(\zeta)$  is a splitting field for  $f_p$ . Moreover,  $[E : \mathbf{Q}] = p - 1$ .

You are going to show for homework that  $\text{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})$  is cyclic of order  $p - 1$ .

Let's study the case when  $p = 5$ .

Since  $\zeta$  and  $\zeta^2$  are roots of  $f_5$ , there is an automorphism  $\sigma : E \rightarrow E$  for which  $\sigma(\zeta) = \zeta^2$ .

**Proposition 16.2.1.** *a. The Galois group of  $E$  over  $\mathbf{Q}$  satisfies  $\text{Gal}(E/\mathbf{Q}) = \text{Gal}(\mathbf{Q}(\zeta_5)/\mathbf{Q}) = \langle \sigma \rangle$ , a cyclic group of order 4.*

*b. If  $H = \langle \sigma^2 \rangle \subset \text{Gal}(E/\mathbf{Q})$ , then  $E^H = E^{\langle \sigma^2 \rangle} = \mathbf{Q}(\zeta^2 + \zeta^3)$*

*Proof.* Note that  $1, \zeta, \zeta^2, \zeta^3$  is a  $\mathbf{Q}$ -basis for  $E$ . A typical element  $x \in E$  has the form

$$x = s_0 + s_1\zeta + s_2\zeta^2 + s_3\zeta^3$$

for  $s_i \in \mathbf{Q}$ .

Note that

$$\begin{aligned} \sigma(x) &= \sigma(s_0 + s_1\zeta + s_2\zeta^2 + s_3\zeta^3) \\ &= s_0 + s_1\sigma(\zeta) + s_2\sigma(\zeta)^2 + s_3\sigma(\zeta)^3 \\ &= s_0 + s_1\zeta^2 + s_2\zeta^4 + s_3\zeta^6 \end{aligned}$$

and using  $\zeta^6 = \zeta$  and  $\zeta^4 = -1 - \zeta - \zeta^2 - \zeta^3$  we see that

$$\sigma(x) = (s_0 - s_2) + (s_3 - s_2)\zeta + (s_1 - s_2)\zeta^2 - s_2\zeta^3.$$

Note that  $\sigma \in \Gamma$  and  $|\Gamma| = 4$ . Clearly  $\sigma \neq 1$ . We claim that  $\sigma^2 \neq 1$ ; this will show that  $\sigma$  has order 4.

Indeed, note that  $\sigma^2(\zeta) = \sigma(\zeta^2) = \zeta^4 = -1 - \zeta - \zeta^2 - \zeta^3 \neq \zeta$  which shows that  $\sigma^2 \neq 1$ .

We conclude that  $\Gamma = \langle \sigma \rangle$  is cyclic of order 4.

The Fundamental Theorem of Galois Theory now implies that  $\mathbf{Q} \subset E$  has a unique intermediate field  $K$  for which  $K \neq \mathbf{Q}$  and  $K \neq E$ . Moreover,  $[K : \mathbf{Q}] = 2$  and

$$K = E^{\langle \sigma^2 \rangle}.$$

□

*Remark 16.2.2.* In fact, we can identify the intermediate field  $\mathbf{Q}(\zeta^2 + \zeta^3)$  more precisely, as follows.

Using sage, we construct the field extension  $E = \mathbf{Q}(\zeta)$  and we find the minimal polynomial over  $\mathbf{Q}$  of  $\zeta^2 + \zeta^3$ :

```
R.<T>=PolynomialRing(QQ);
f = T^4 + T^3 + T^2 + T + 1

E.<z>=QQ.extension(f)
g=(z^2 + z^3).minpoly()
g
```

```
x^2 + x - 1
```

Now, using the quadratic formula, the roots of  $T^2 + T - 1$  are  $\frac{-1 \pm \sqrt{5}}{2}$ . Since we know  $[\mathbf{Q}(\zeta^2 + \zeta^3) : \mathbf{Q}] = 2$  it follows that  $\mathbf{Q}(\zeta^2 + \zeta^3) = \mathbf{Q}(\sqrt{5})$ .

Now let's study the case  $p = 13$ . Thus  $E = \mathbf{Q}(\zeta)$  where  $\zeta$  is a root of  $g = \frac{T^{13} - 1}{T - 1} \in \mathbf{Q}[T]$ . Recall that we know  $g$  to be irreducible (since 13 is prime) and hence  $[E : \mathbf{Q}] = 12$ .

Now, the roots of  $g$  are precisely the elements  $\zeta^i$  for  $1 \leq i \leq 12$ , which are all contained in  $E$ . Thus  $E$  is a splitting field for the separable polynomial  $g$ . Let  $\Gamma = \text{Gal}(E/\mathbf{Q})$ ; then  $|\Gamma| = [E : \mathbf{Q}] = 12$ .

Since  $E = \mathbf{Q}(\zeta) = \mathbf{Q}(\zeta^i)$  for  $1 \leq i \leq 12$  and since the roots of  $g$  are precisely the  $\zeta^i$ , there is an automorphism

$$\sigma : E \rightarrow E$$

with the property that

$$\sigma(\zeta) = \zeta^2.$$

Let us note that

$$\begin{aligned} \sigma(\zeta) &= \zeta^2 \\ \sigma^2(\zeta) &= \sigma(\zeta^2) = \sigma(\zeta)^2 = (\zeta^2)^2 = \zeta^4 \\ \sigma^3(\zeta) &= \sigma(\zeta^4) = \sigma(\zeta)^4 = (\zeta^2)^4 = \zeta^8 \\ &\vdots \\ \sigma^i(\zeta) &= \sigma(\zeta)^i = (\zeta^{2^{i-1}})^i = \zeta^{2^i} \quad (\clubsuit) \end{aligned}$$

Now, you can check that the element 2 is an element of order 12 in the multiplicative group  $\mathbb{Z}_{13}^\times$

```
[ (i, 2^i % 13) for i in range(1,13) ]
```

```
[(1, 2),
 (2, 4),
 (3, 8),
 (4, 3),
 (5, 6),
 (6, 12),
 (7, 11),
 (8, 9),
 (9, 5),
 (10, 10),
 (11, 7),
 (12, 1)]
```

Thus  $\mathbb{Z}_{13}^\times = \langle 2 \rangle$ . Together with ( $\clubsuit$ ) it now follows that  $\sigma$  has order 12 in  $\Gamma$ , and since  $\Gamma$  has order 12, we see that

$$\Gamma = \langle \sigma \rangle.$$

Now, let's describe the intermediate extension  $K$  of  $\mathbf{Q} \subset E$  which has degree 3 over  $\mathbf{Q}$ .

So we need to find the subgroup  $H \subset \Gamma$  with  $K = E^H$ . If we want  $[K : \mathbf{Q}] = 3$  then we need  $[E : K] = 4$ , and so using the fundamental theorem of galois theory, we know that  $H$  must be the subgroup of  $\Gamma = \langle \sigma \rangle$  of order 4. Thus  $H = \langle \sigma^3 \rangle$ .

Now,  $K = E^{\langle \sigma^3 \rangle}$  is an extension of degree 3 of  $\mathbf{Q}$ , so if we find  $a \in K$ ,  $a \notin \mathbf{Q}$ , then automatically  $[K : \mathbf{Q}(a)] = 1$  so that  $K = \mathbf{Q}(a)$ .

Now, notice that if we take any element  $x$  of  $E$ , we can look at the **orbit sum**

$$s = \sum_{h \in H} h.x$$

Then  $s$  is always contained in  $E^H$ , since for any  $g \in H$ ,

$$gs = g\left(\sum_{h \in H} h.x\right) = \sum_{h \in H} gh.x = \sum_{h' \in H} h'.x = s$$

where the reindexing for the last term corresponds to  $h' = gh$  or  $h = g^{-1}h'$ .

We now compute the orbit sum for the element  $\zeta \in E$  using ( $\clubsuit$ ):

$$\begin{aligned} \alpha &= \sum_{h \in H} h.\zeta = \sum_{i=0}^3 (\sigma^3)^i(\zeta) \\ &= \zeta + \sigma^3(\zeta) + \sigma^6(\zeta) + \sigma^9(\zeta) \\ &= \zeta + \zeta^{2^3} + \zeta^{2^6} + \zeta^{2^9} \\ &= \zeta + \zeta^8 + \zeta^{12} + \zeta^5 \end{aligned}$$

Thus we find that  $\alpha = \zeta + \zeta^8 + \zeta^{12} + \zeta^5 \in E^{\langle \sigma^3 \rangle} = K$ . Since  $\alpha \notin \mathbf{Q}$  we conclude that

$$K = E^{\langle \sigma^3 \rangle} = \mathbf{Q}(\alpha) = \mathbf{Q}(\zeta + \zeta^8 + \zeta^{12} + \zeta^5).$$

### 16.2.4 Cyclic extension of the field of rational functions over a finite field

Let  $F = \mathbf{F}_{11}(X)$  be the field of rational functions over the field with 11 elements.

Recall that  $\mathbf{F}_{11}^\times$  is a cyclic group of order 10 Corollary 12.6.3; let's write

$$\mathbf{F}_{11}^\times = \langle \gamma \rangle$$

for a suitable generator  $\gamma$ .

Consider the polynomial

$$g = T^{10} - (X + 1) \in F[T];$$

$g$  is irreducible over  $F$  by an application of Eisenstein's criteria Theorem 7.4.1.

Let  $E = F(\alpha)$  where  $\alpha$  is a root of  $g$ . Since  $g$  is the minimal polynomial of  $\alpha$ , we know that  $[E : F] = 10$ .

We now observe that  $E$  is a *splitting field* for  $g$ . Since  $E$  is generated over  $F$  by a root of  $g$ , it suffices to check that  $g$  splits over  $E$ .

Note that  $\gamma^i \alpha$  is a root of  $g$  for  $0 \leq i \leq 9$ . Since  $o(\gamma) = 10$ , these elements are distinct, and thus

$$\{\gamma^i \alpha\} = \{\alpha, \gamma\alpha, \dots, \gamma^9 \alpha\}$$

are 10 distinct roots of  $g$  in  $E$ . In particular,

$$g = \prod_{i=0}^9 (T - \gamma^i \alpha) \in E[T] = F(\alpha)[T] = \mathbf{F}_{11}(X, \alpha)[T].$$

Since  $E$  is a splitting field over  $F$  of the separable polynomial  $g$ , it follows that the galois group

$$\Gamma = \text{Gal}(E/F)$$

has order  $|\Gamma| = [E : F] = 10$ .

We now construct an element of  $\Gamma$  – i.e. an automorphism of  $E$ . Since  $\alpha$  and  $\gamma\alpha$  are two roots of the irreducible polynomial  $g$ , we know that there is an automorphism  $\sigma : E \rightarrow E$  for which  $\sigma|_F = \text{id}$  and  $\sigma(\alpha) = \gamma\alpha$ .

Since  $\gamma \in \mathbf{F}_{11}$ , we know that  $\sigma(\gamma) = \gamma$ . We now observe:

$$\begin{aligned} \sigma(\alpha) &= \gamma\alpha \\ \sigma^2(\alpha) &= \sigma(\gamma\alpha) = \gamma\sigma(\alpha) = \gamma^2\alpha \\ \sigma^3(\alpha) &= \sigma(\gamma^2\alpha) = \gamma^2\sigma(\alpha) = \gamma^3\alpha \\ &\vdots \\ \sigma^j(\alpha) &= \gamma^j\alpha \end{aligned}$$

We now see that  $\sigma^j(\alpha) = \alpha \iff j \equiv 0 \pmod{10}$ . Since  $\sigma^j = \text{id} \iff \sigma^j(\alpha) = \alpha$ , it follows that  $\sigma$  has order 10 in the group  $\Gamma$ . This proves that

$$\Gamma = \langle \sigma \rangle$$

is a cyclic group.

Recall that a cyclic group of order  $n$  has exactly one subgroup of order  $d$  for each divisor  $d$  of  $n$ . Thus, the proper non-trivial subgroups of  $\Gamma$  are  $\langle \sigma^2 \rangle$  which has order 5, and  $\langle \sigma^5 \rangle$  which has order 2.

Let  $K = E^{\langle \sigma^2 \rangle}$  has  $[E : K] = 5$  and thus  $[K : F] = 2$ .

Similarly,  $L = E^{\langle \sigma^5 \rangle}$  has  $[E : L] = 2$  and thus  $[L : F] = 5$ .

To describe  $K$  as an extension of  $F$ , we just need to find an element  $a$  in  $K$  not in  $F$  (indeed: since  $[K : F]$  is prime, the field  $F \neq F(a)$  must coincide with  $K$ ).

Now, we have an  $F$ -basis for  $E$  of the form

$$\{1, \alpha, \alpha^2, \dots, \alpha^9\}.$$

So a typical element of  $E$  has the form

$$x = s_0 + s_1\alpha + \dots + s_9\alpha^9$$

for elements  $s_i \in F$ .

Now, we notice that

$$\sigma(\alpha^i) = \gamma^i \alpha^i$$

and more generally that

$$\sigma^j(\alpha^i) = (\gamma^j)^i \alpha^i = \gamma^{ij} \alpha^i.$$

Thus we can check that

$$\sigma^2(\alpha^5) = \gamma^{10} \alpha^5 = \alpha^5 \implies \alpha^5 \in E^{\langle \sigma^2 \rangle} = K \implies K = F(\alpha^5).$$

And similarly

$$\sigma^5(\alpha^2) = \gamma^{10} \alpha^2 = \alpha^2 \implies \alpha^2 \in E^{\langle \sigma^5 \rangle} = L \implies L = F(\alpha^2).$$

So the intermediate fields (other than  $E, F$ ) of the extension

$$F = \mathbf{F}_{11}(X) \subset E = \mathbf{F}_{11}(X, \alpha) = \mathbf{F}_{11}(X, \sqrt[10]{X+1})$$

are:

$$K = F(\alpha^5) = \mathbf{F}_{11}(X, \alpha^5) = \mathbf{F}_{11}(X, \sqrt{X+1})$$

and

$$L = F(\alpha^2) = \mathbf{F}_{11}(X, \alpha^2) = \mathbf{F}_{11}(X, \sqrt[5]{X+1})$$

### 16.3 An example with a dihedral galois group

Consider  $g = T^3 - 2 \in \mathbf{Q}[T]$ , and let  $\alpha, \omega$  in some extension of  $\mathbf{Q}$  where  $\alpha = \sqrt[3]{2}$  is a root of  $g$ , and  $\omega$  is a root of  $T^2 + T + 1$ .

An application of Eisenstein's criteria shows that  $g$  is irreducible over  $\mathbf{Q}$ . We know that  $h = T^2 + T + 1$  is irreducible over  $\mathbf{Q}$  as well (e.g. because neither root of the degree 2 polynomial  $h$  is a real number, and hence  $h$  has no root in  $\mathbf{Q}$ ).

Since  $\deg g$  and  $\deg h$  are relatively prime, an earlier homework exercise shows that  $h$  remains irreducible over  $\mathbf{Q}(\alpha)$  and  $g$  remains irreducible over  $\mathbf{Q}(\omega)$ .

In particular,  $[E : \mathbf{Q}] = 6$ . Since  $E$  is a normal extension of  $\mathbf{Q}$ , we conclude that the galois group  $\Gamma = \text{Gal}(E/\mathbf{Q})$  has  $|\Gamma| = 6$ .

Since  $E$  generated by a root of the irreducible polynomial  $g = T^3 + 2$  over the field  $\mathbf{Q}(\omega)$ , there is an automorphism

$$\sigma : E \rightarrow E$$

for which  $\sigma|_{\mathbf{Q}(\omega)} = \text{id}$  and  $\sigma(\alpha) = \omega\alpha$ .

Let's observe that

$$\sigma^2(\alpha) = \sigma(\omega\alpha) = \omega\sigma(\alpha) = \omega^2\alpha$$

and that

$$\sigma^3(\alpha) = \sigma(\omega^2\alpha) = \omega^2\sigma(\alpha) = \omega^3\alpha = \alpha.$$

This shows that  $\sigma^3 = \text{id}$  so that  $\sigma$  has order 3 in the group  $\Gamma$ .

Now,  $E$  is generated by a root of the irreducible polynomial  $h = T^2 + T + 1$  over the field  $\mathbf{Q}(\alpha)$ . Thus, there is an automorphism

$$\tau : E \rightarrow E$$

for which  $\tau|_{\mathbf{Q}(\alpha)} = \text{id}$  and for which  $\tau(\omega) = \omega^2 = \bar{\omega}$ .

Note that  $\tau^2(\omega) = \tau(\omega^2) = \omega^4 = \omega$  so that  $\tau^2 = \text{id}$ ; thus  $\tau$  has order 2 in the group  $\Gamma$ .

We now claim that  $(\clubsuit) \quad \tau\sigma\tau = \sigma^2$ . Well, since  $E$  is generated by  $\alpha$  and  $\omega$ , it suffices to check that the automorphisms  $\tau\sigma\tau$  and  $\sigma^2$  agree on these two elements:

$$\begin{aligned} (\tau\sigma\tau)(\alpha) &= (\tau\sigma)(\alpha) = \tau(\omega\alpha) = \omega^2\alpha \\ \sigma^2(\alpha) &= \sigma(\omega\alpha) = \omega\sigma(\alpha) = \omega^2\alpha \\ (\tau\sigma\tau)(\omega) &= (\tau\sigma(\omega^2)) = \tau(\omega^2) = \omega \\ \sigma^2(\omega) &= \sigma(\omega) = \omega \end{aligned}$$

Using the relation  $(\clubsuit)$ , one sees that the subgroup  $\langle \sigma, \tau \rangle$  has 6 elements, namely

$$\{\text{id}, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\}.$$

Thus

$$\Gamma = \langle \sigma, \tau \rangle.$$

The group  $\Gamma$  has exactly one subgroup of order 3, namely  $\langle \sigma \rangle$ . The corresponding intermediate field  $K = E^{\langle \sigma \rangle}$  has  $[E : K] = 3$  and  $[K : \mathbf{Q}] = 2$ . Since  $\sigma(\omega) = \omega$ , we know that  $\omega \in K$  and thus  $K = \mathbf{Q}(\omega)$ .

Note that  $\langle \sigma \rangle$  is a *normal* subgroup  $\Gamma$ ; this corresponds to the fact that  $K$  is a normal extension of  $\mathbf{Q}$ , which one can also see because  $K$  is a splitting field for  $h = T^2 + T + 1$ .

The group  $\Gamma$  has three subgroups of order 2, namely  $H_1 = \langle \tau \rangle$ ,  $H_2 = \langle \sigma\tau \rangle$  and  $H_3 = \langle \sigma^2\tau \rangle$ . The corresponding intermediate fields  $L_i = E^{H_i}$  all satisfy  $[E : L_i] = 2$  and thus  $[L_i : \mathbf{Q}] = 3$ . Again, since 3 is prime, in order to find a generator for  $L_i$  as an extension of  $\mathbf{Q}$  it suffices to find an element of  $L_i$  which is not in  $\mathbf{Q}$ .

- $i = 1$ :  $H_1 = \langle \tau \rangle$ .

$$\tau(\alpha) = \alpha \implies \alpha \in L_1 \implies L_1 = \mathbf{Q}(\alpha) = \mathbf{Q}(\sqrt[3]{2})$$

- $i = 2$ :  $H_2 = \langle \sigma\tau \rangle$ .

$$\begin{aligned} \sigma\tau(\omega^2\alpha) &= \sigma(\omega\alpha) = \omega\sigma\alpha = \omega^2\alpha \\ \implies \omega^2\alpha &\in L_2 \\ \implies L_2 &= \mathbf{Q}(\omega^2\alpha) = \mathbf{Q}(\omega^2\sqrt[3]{2}) \end{aligned}$$

- $i = 3$ :  $H_3 = \langle \sigma^2 \tau \rangle$ .

$$\begin{aligned}
\sigma^2 \tau(\omega \alpha) &= \sigma^2(\omega^2 \alpha) = \omega^2 \sigma^2(\alpha) = \omega^4 \alpha = \omega \alpha \\
&\implies \omega \alpha \in L_3 \\
&\implies L_3 = \mathbf{Q}(\omega \alpha) = \mathbf{Q}(\omega \sqrt[3]{2})
\end{aligned}$$

Thus  $E, F, K, L_1, L_2, L_3$  is the complete list of intermediate fields.

The subgroups  $H_i$  are not normal (in fact,  $\sigma H_1 \sigma^{-1} = H_3$  and  $\sigma^2 H_1 \sigma^{-2} = H_2$ ). This non-normality corresponds to the fact that the  $L_i$  are not normal extensions of  $\mathbf{Q}$ . In fact, the polynomial  $g$  has a root in each of the fields  $L_i$ , but  $g$  does not split over  $L_i$  (for any  $i$ ).

## 17 The insolubility of the quintic

We want to make sense formally of what it would mean to *solve* a polynomial equation. Thus if  $g \in F[T]$ , we'd like to know how to describe the roots of  $g$  – in some extension of  $F$  – using some formula in the coefficients of  $g$ .

We know this to be possible for quadratic polynomials (at least when the characteristic of  $F$  is not 2): the roots of  $aT^2 + bT + c$  are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

*Example 17.0.1.* There is a formula for the roots of a cubic polynomial (at least when the characteristic of  $F$  is not 2 or 3); one of the roots of

$$g = aT^3 + bT^2 + cT + d$$

has the form

$$x = \frac{-1}{3a} \left( b + C + \frac{\Delta_0}{C} \right)$$

where

$$C = \sqrt[3]{\frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}$$

and

$$\begin{aligned}\Delta_0 &= b^2 - 3ac \\ \Delta_1 &= 2b^3 - 9abc + 27a^2d.\end{aligned}$$

To find all three roots of the cubic, one must make all possible choices of the cube root  $C$ . So in our formulas, we allow ourselves to “extract  $n$ -th roots” of algebraic expressions in  $F$ .

### 17.1 Some group theory

Let us recall the following definition for a group  $G$ .

*Definition 17.1.1.* The group  $G$  is *solvable* provided there is a finite chain of subgroups

$$1 = N_k \subseteq N_{k-1} \subseteq N_{k-2} \subseteq \cdots \subseteq N_0 = G$$

such that:

- $N_i$  is a normal subgroup of  $N_{i+1}$  for  $i = 1, 2, \dots, k$ ,
- and  $N_{i-1}/N_i$  is abelian for  $i = 1, 2, \dots, k$

**Proposition 17.1.2.**  $G$  is solvable if and only if there is a normal subgroup  $N \subseteq G$  such that  $N$  and  $G/N$  are solvable.

*Proof.* Left as an exercise! □

## 17.2 Radical extensions and the main result

*Definition 17.2.1.* A field extension  $F \subseteq E$  is a *radical extension* of  $F$  if there are elements  $u_1, \dots, u_d \in E$  and positive integers  $n_1, \dots, n_d$  such that:

- $E = F(u_1, \dots, u_d)$ , and
- $u_1^{n_1} \in F$  and  $u_i^{n_i} \in F(u_1, \dots, u_{i-1})$  for  $i = 2, \dots, d$ .

*Definition 17.2.2.* A polynomial  $g \in F[T]$  is *solvable in radicals* if there is a radical extension  $E$  of  $F$  which contains a splitting field for  $g$ .

*Example 17.2.3.* The formulas given in Example 17.0.1 show (at least in characteristic  $\neq 2, 3$ ) that any degree 3 polynomial  $g$  is solvable in radicals:

Indeed, the polynomial  $g$  splits over  $F(u_1, u_2, u_3)$  where

$$u_1^2 = \Delta_1^2 - 4\Delta_0^3, \quad u_2^3 = \frac{\Delta_1 + u_1}{2}, \quad u_3 = \omega;$$

here  $\omega$  is a root of  $\frac{T^3 - 1}{T - 1} = T^2 + T + 1$ .

*Remark 17.2.4.* If  $E = F(u)$  and  $u^d = a \in F$  for some  $d$ , then of course  $E$  has an  $F$ -basis consisting of the powers  $1, u, u^2, \dots, u^{d-1}$  of the *radical*  $u$ . Similarly, if  $E$  is a radical extension of  $F$ , then  $E$  has a basis consisting of products of powers of *radicals*.

If the polynomial  $g \in F[T]$  splits over  $E$ , then each root  $\alpha$  of  $g$  may be written as an  $F$ -linear combination of products of powers of radicals, which may be viewed as some sort of “formula” for the root  $\alpha$ .

In this section, we are going to prove the following result.

**Theorem 17.2.5.** *Suppose that  $F$  has characteristic zero – i.e. that  $\mathbf{Q} \subseteq F$ . Let  $g \in F[T]$  be a polynomial. Then  $g$  is solvable in radicals if and only if  $\text{Gal}(g)$  is a solvable group.*

*Remark 17.2.6.* In fact, the Theorem remains valid without the assumption that  $F$  has characteristic, provided that  $g$  is *separable*. However, the proof is slightly more complicated, so we focus here on characteristic zero.

## 17.3 Cyclotomic extensions

Recall that we suppose  $F$  to have characteristic zero.

**Proposition 17.3.1.** *For any  $n \geq 1$ , let  $E$  denote a splitting field over  $F$  of  $T^n - 1$ . Then  $\Gamma = \text{Gal}(E/F)$  is an Abelian group.*

*Proof.* Since the characteristic of  $F$  is 0,  $\frac{d}{dT}(T^n - 1) = nT^{n-1}$  is non-zero and has the unique root 0. Since 0 is not a root of  $T^n - 1$ , it follows from Proposition 13.1.3 that  $T^n - 1$  has  $n$  distinct roots in the splitting field  $E$ .

Now, the set  $A$  of roots of  $T^n - 1$  form a subgroup of the multiplicative group  $E^\times$ , and Theorem 12.6.2 shows that this subgroup must be *cyclic*; let us write  $A = \langle a \rangle$  so that the order  $o(a)$  is  $n$ .

Any element  $\sigma \in \Gamma$  defines an automorphism of the group  $A$ . In particular,  $\sigma(a)$  is another generator of the group  $A$ ; thus  $\sigma(a) = a^s$  where  $s = s(\sigma)$  is an integer with  $\gcd(s, n) = 1$ .

Now the assignment  $\sigma \mapsto s(\sigma)$  defines a group homomorphism

$$\Gamma \rightarrow (\mathbf{Z}/n\mathbf{Z})^\times.$$

Indeed, if  $\sigma, \tau \in \Gamma$  where  $\sigma(a) = a^s$  and  $\tau(a) = a^t$  then

$$(\sigma \circ \tau)(a) = \sigma(a^t) = \sigma(a)^t = (a^s)^t = a^{st}.$$

Finally, this group homomorphism is one-to-one. Indeed, we must show that the kernel is trivial. But if  $\sigma$  is in the kernel, then  $\sigma(a) = a$ . Since  $E$  is a splitting field of  $T^n - 1$ , we know that  $E = F(A) = F(a)$  and since  $\sigma|_F = \text{id}$ , it follows at once that  $\sigma = \text{id} = 1_\Gamma$  so indeed the kernel is trivial.

We now conclude that  $\Gamma$  is isomorphic to a subgroup of  $(\mathbf{Z}/n\mathbf{Z})^\times$ ; since  $(\mathbf{Z}/n\mathbf{Z})^\times$  is Abelian, the Proposition has been proved.  $\square$

## 17.4 Van Der Monde matrices

Let  $a_1, \dots, a_m$  be  $m$  elements in some field, write  $\mathcal{P}_m$  for the space of polynomials of degree  $< m$  and consider the linear transformation

$$\Phi : \mathcal{P}_m \rightarrow F^m$$

given by  $f \mapsto [f(a_1) \ f(a_2) \ \cdots \ f(a_m)]^T$ .

One observes that the matrix of  $\Phi$  with respect to the standard monomial basis of  $\mathcal{P}_m$  and the standard basis of  $F^m$  is the *van der Monde* matrix

$$V = \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{m-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_m & a_m^2 & \cdots & a_m^{m-1} \end{bmatrix}$$

**Proposition 17.4.1.** *With  $A$  as above, we have*

$$\det A = \pm \prod_{i < j} (a_i - a_j)$$

*Proof.* One way of seeing this formula for  $\det V$  is to use the following basis for  $\mathcal{P}_m$ :

$$\begin{aligned} b_1 &= 1 \\ b_2 &= (T - a_1) \\ b_3 &= (T - a_1)(T - a_2) \\ &\vdots \\ b_m &= (T - a_1)(T - a_2) \cdots (T - a_{m-1}) \end{aligned}$$

Note that since  $\deg b_i = i - 1$  the  $b_i$  are linearly independent and hence form a basis for  $\mathcal{P}_m$ .

Now, the matrix for  $\Phi$  with respect to the basis  $\{b_i\}$  of  $\mathcal{P}_m$  and the standard basis for  $F^m$  has the form

$$W = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & a_2 - a_1 & 0 & \cdots & 0 \\ 1 & a_3 - a_1 & (a_3 - a_1)(a_3 - a_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_m - a_1 & \cdots & \cdots & \prod_{i=1}^{m-1} (a_m - a_i) \end{bmatrix}$$

For a matrix of this form, the determinant is the product of the diagonal entries – i.e.

$$\det W = \prod_{i < j} (a_j - a_i)$$

To show that  $\det W = \det V$  it remains to note that  $W = VU$  where  $U$  is an upper triangular  $m \times m$  matrix with ones along the diagonal.  $\square$

If the  $a_i$  are all distinct, one knows that  $\ker \Phi = \{0\}$  since a non-zero polynomial of degree  $\leq m-1$  can not have all of  $a_1, \dots, a_m$  as roots. This implies at once that  $V$  has  $\text{Null}(V) = \{0\}$  and thus  $V$  is an invertible matrix (by the *Invertible Matrix Theorem* of linear algebra).

## 17.5 Main result on solvable extensions

**Proposition 17.5.1.** *Let  $K$  denote a splitting field over  $F$  of  $T^n - 1$ . Let  $a \in K$ , consider the polynomial  $g = T^n - a \in K[T]$ , and let  $E$  be a splitting field for  $g$  over  $K$ . Then  $\Gamma = \text{Gal}(E/K)$  is a cyclic group whose order is a divisor of  $n$ .*

*Proof.* Fix a generator  $\zeta \in K$  for the group  $A \subseteq K$  of solutions to  $T^n - 1$  (one says that  $\zeta$  is a *primitive  $n$ -th root of unity*).

If  $u \in E$  is a root of  $g = T^n - a$ , then  $E = K(u)$  is a splitting field for  $g$  over  $K$  since for any integer  $i$ , we know that  $\zeta^i u$  is a root of  $g$ . We find in this way  $n$  distinct roots, and in particular

$$g = \prod_{i=0}^{n-1} (T - \zeta^i u).$$

Now, any element  $\sigma \in \Gamma$  is completely determined by the value  $\sigma(u)$ , which must be a root of  $g$  and hence of the form

$$\sigma(u) = \zeta^i u$$

for some integer  $i$  which is well-defined (mod  $n$ ) – thus we view  $i = i(\sigma)$  as an element of  $\mathbf{Z}/n\mathbf{Z}$ .

In this way, we obtain an assignment

$$(\sigma \mapsto i(\sigma)) : \Gamma \rightarrow \mathbf{Z}/n\mathbf{Z}.$$

We now claim that this assignment is a group homomorphism. Well, suppose  $\sigma, \tau \in \Gamma$  and that  $\sigma \mapsto i = i(\sigma)$  and  $\tau \mapsto j = i(\tau)$  for  $i = i(\sigma), j = i(\tau) \in \mathbf{Z}/n\mathbf{Z}$ , then

$$(\sigma \circ \tau)(u) = \sigma(\zeta^j u) = \zeta^j \sigma(u) = \zeta^j \zeta^i \sigma(u) = \zeta^{i+j} u.$$

This shows that  $i(\sigma\tau) = i + j = i(\sigma) + i(\tau)$ , so indeed we have found a group homomorphism from the (multiplicatively written) group  $\Gamma$  to the (additively written) group  $\mathbf{Z}/n\mathbf{Z}$ .

Finally, we argue that this group homomorphism is one-to-one. For this, we argue that the kernel is trivial. Well, if  $\sigma$  is in the kernel then  $i(\sigma) = 0 \implies \sigma(u) = \zeta^0 u = u$ . Since  $E = K(u)$  and since  $\sigma$  is the identity on  $K$ , conclude that  $\sigma = \text{id}$  so indeed the kernel is trivial.

Since the indicated group homomorphism is one-to-one, we may now conclude that  $\Gamma$  is isomorphic to a subgroup of  $\mathbf{Z}/n\mathbf{Z}$ . In particular,  $\Gamma$  is cyclic and its order  $|\Gamma|$  must divide  $n$ .  $\square$

**Theorem 17.5.2.** *Let  $p$  be a prime number and let  $K$  denote a splitting field over  $F$  of  $T^p - 1$ . Let  $E$  be an extension of  $K$ . Assume that  $[E : K] = |\Gamma| = p$  where  $\Gamma = \text{Gal}(E/K)$ . Then  $E = K(u)$  for some element  $u \in E$  with  $u^p \in K$ .*

*Proof.* Choose any element  $w \in E$  with  $w \notin K$ . Since  $[E : K] = p$  is prime, conclude that  $E = K(w)$ .

Let  $A = \{\zeta_1, \zeta_2, \dots, \zeta_p\}$  be the roots of  $T^p - 1$  in  $K$ .

Finally, since  $\Gamma$  has prime order, it is a cyclic group; let us write  $\theta$  for a generator; thus  $\Gamma = \langle \theta \rangle$ .

Now write

$$\begin{aligned} w_1 &= w \\ w_2 &= \theta(w_1) = \theta(w) \\ w_3 &= \theta(w_2) = \theta^2(w) \\ &\vdots \\ w_i &= \theta(w_{i-1}) = \theta^{i-1}(w) \end{aligned}$$

Since  $\zeta^p = 1$ , we have  $\theta(w_p) = w_1$ .

For each  $i$  consider the element

$$u_i = w_1 + \zeta_i w_2 + \zeta_i^2 w_3 + \dots + \zeta_i^{p-1} w_p.$$

Observe that

$$\begin{aligned} \theta(u_i) &= \theta(w_1) + \zeta_i \theta(w_2) + \zeta_i^2 \theta(w_3) + \dots + \zeta_i^{p-1} \theta(w_p) \\ &= w_2 + \zeta_i w_3 + \zeta_i^2 w_4 + \dots + \zeta_i^{p-1} w_1 \\ &= \zeta_i^{-1} (\zeta_i w_2 + \zeta_i^2 w_3 + \dots + \zeta_i^p w_1) \\ &= \zeta_i^{-1} u_i. \end{aligned}$$

We conclude in particular that  $\theta(u_i^p) = \theta(u_i)^p = (\zeta_i^{-1} u_i)^p = u_i^p$ . This proves for each  $i$  that  $u_i^p \in E^{\langle \theta \rangle} = E^\Gamma = K$  since the extension  $K \subseteq E$  is normal.

We are going to complete the proof by arguing for at least one index  $i_0$  that  $u_{i_0} \notin K$ ; thus  $E = K(u_{i_0})$  and  $u_{i_0}^p \in K$  as required.

To accomplish this, we first write the relation between the  $u_i$  and the  $w_i$  in matrix form. Let

$$\mathbf{u} = [u_1 \quad u_2 \quad \dots \quad u_p]^T \quad \text{and} \quad \mathbf{w} = [w_1 \quad w_2 \quad \dots \quad w_p]^T$$

and

$$A = \begin{bmatrix} 1 & \zeta_1 & \dots & \zeta_1^{p-1} \\ 1 & \zeta_2 & \dots & \zeta_2^{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_p & \dots & \zeta_p^{p-1} \end{bmatrix} \in \text{Mat}_{p \times p}(K).$$

Then

$$A\mathbf{w} = \mathbf{u}.$$

The matrix  $A$  is a *van der Monde matrix* Section 17.4. For such a matrix, we have

$$\det A = \pm \prod_{i < j} (\zeta_i - \zeta_j)$$

by Proposition 17.4.1. Since the  $\zeta_i$  are distinct,  $\det A \neq 0$  and in particular  $A$  is invertible.

Now, the invertibility of  $A$  implies in particular that we may write  $w = w_1$  as a  $K$  linear combination of  $u_1, \dots, u_p$ . Now, if for each  $i$  we have  $u_i \in K$  then for every  $j$  we have  $w_j \in K$ , contrary to our choice of  $w$ . This proves for some  $i_0$  that  $u_{i_0} \notin K$  as required.  $\square$

## 17.6 Normal radical extensions

**Lemma 17.6.1.** *Let  $E$  be a radical extension of  $F$ . Then there is an extension  $L$  of  $E$  such that  $L$  is a normal and radical extension of  $F$ .*

*Proof.* Write  $E = F(u_1, \dots, u_m)$  as in the definition of radical extension; thus there are whole numbers  $n_1, n_2, \dots, n_m$  such that  $u_1^{n_1} \in F$  and  $u_i^{n_i} \in F(u_1, \dots, u_{i-1})$  for  $i \geq 2$ .

Denote by  $p_i$  the minimal polynomial of  $u_i$  over  $F$  for  $1 \leq i \leq m$ , let  $g = p_1 \dots p_m \in F[T]$  and let  $L$  be a splitting field over  $E$  of the polynomial  $g$ ; since  $E$  is generated over  $F$  by roots of  $g$ , it follows that  $L$  is a splitting field over  $F$  of  $g$  as well, hence  $L$  is a normal extension of  $F$ .

It remains to argue that  $L$  is a radical extension of  $F$ . Let  $\Gamma = \text{Gal}(L/F)$  be the galois group.

Since the galois group acts transitively on the roots of each irreducible factor  $p_i$  of  $g$ , it follows that each root of  $g$  has the form  $\theta(u_i)$  for some  $\theta \in \Gamma$  and some  $1 \leq i \leq m$ .

For each  $i$  observe that

$$\theta(u_i)^{n_i} \in F(\theta(u_1), \dots, \theta(u_{i-1})).$$

Now write

$$\Gamma = \{\theta_1, \dots, \theta_\ell\},$$

consider the elements

$$v_{i,j} = \theta_j(u_i) \in L$$

and consider any total ordering on the pairs  $(i, j)$  satisfying

$$(i, j) \leq (i', j') \iff i \leq i'.$$

Then  $L = F(v_{i,j})$  is a radical extension for the whole numbers  $m_{i,j} = n_i$  with respect to the chosen total ordering.  $\square$

**Lemma 17.6.2.** *Let  $E$  be a normal radical extension of  $F$ . Then the galois group  $\Gamma = \text{Gal}(E/F)$  is a solvable group.*

*Proof.* Consider  $\zeta$  a primitive  $M$ -th root of unity for some  $M \geq 1$ . Then  $E(\zeta)$  is evidently a radical extension of  $F(\zeta)$  and of  $F$ .

Moreover, the fundamental theorem of Galois Theory shows that  $\text{Gal}(E/F)$  is a quotient of  $\text{Gal}(E(\zeta)/F)$ . Since a quotient of a solvable group is solvable, it suffices to show that  $\Gamma = \text{Gal}(E(\zeta)/F)$  is a solvable group.

Now recall that  $E = F(u_1, \dots, u_m)$  where  $u_1^{n_1} \in F$  and  $u_i^{n_i} \in F(u_1, \dots, u_{i-1})$  for  $i \geq 2$ . Take  $M$  to be least common multiple of the  $n_i$ .

Write  $L_i = F(\zeta, u_1, \dots, u_i)$ . Then  $E(\zeta) = L_m$ . Now set  $N = \text{Gal}(E(\zeta)/F(\zeta)) \subseteq \Gamma$  and  $N_i = \text{Gal}(E(\zeta)/L_i) \subseteq \Gamma$  for  $1 \leq i \leq m-1$ .

By our choice of  $M$ ,  $L_i$  is a splitting field over  $L_{i-1}$  of the polynomial  $T^{n_i} - a_i$  for some  $a_i \in L_i$ . Thus  $N_i$  is a normal subgroup of  $N_{i-1}$ , and the fundamental theorem of Galois Theory – Theorem 15.5.2 – implies that

$$N_{i-1}/N_i \simeq \text{Gal}(L_i/L_{i-1}).$$

Moreover, Proposition 17.5.1 implies that  $N_{i-1}/N_i$  is an cyclic – hence Abelian – group.

Finally,  $N$  is normal in  $\Gamma$  since  $F(\zeta)$  is a normal extension of  $F$ , and  $\Gamma/N \simeq \text{Gal}(F(\zeta)/F)$  is abelian by Proposition 17.3.1.

Thus the descending chain of subgroups

$$\Gamma \supseteq N \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_m = \{1\}$$

shows that  $\Gamma$  is solvable, as required. □

We now prove give the proof of Theorem 17.2.5:

*Proof.* Recall we must prove the following:

Let  $g \in F[T]$  be a polynomial. Then  $g$  is solvable in radicals if and only if  $\Gamma = \text{Gal}(g)$  is a solvable group.

( $\Rightarrow$ ): First suppose that  $g$  is solvable in radicals. According to Lemma 17.6.1, we may find a radical extension  $E$  of  $F$  which is normal over  $F$  over which  $g$  splits.

Now,  $E$  contains a splitting field  $E_1$  of  $g$  over  $F$  and the galois group  $\Gamma = \text{Gal}(g) = \text{Gal}(E_1/F)$  is a quotient of  $\text{Gal}(E/F)$  by the fundamental theorem of Galois Theory (Theorem 15.5.2).

Now the solvability of  $\Gamma$  follows from Lemma 17.6.2.

( $\Leftarrow$ ): Suppose that  $\Gamma$  is solvable, and let  $E$  be a splitting field for  $g$  over  $F$ .

Write  $|\Gamma| = n$ , let  $\zeta$  be a primitive  $n$ -th root of unity, and let  $L = E(\zeta)$ .

We claim that  $\phi(E) = E$  for any  $\phi \in \text{Gal}(E(\zeta)/F(\zeta))$ . For this claim, let  $p$  be the minimal polynomial over  $F$  of an element  $a \in E$ . Then  $\phi(a)$  is a root of  $p$  hence  $\phi(a) \in E$  since  $E$  is normal over  $F$ .

We see that the mapping  $\phi \mapsto \phi|_E$  obtained by restriction of  $\phi$  to  $E$  defines a group homomorphism

$$\text{Gal}(E(\zeta)/F(\zeta)) \rightarrow \Gamma$$

which is one-to-one. This shows that  $\Gamma_1 = \text{Gal}(E(\zeta)/F(\zeta))$  is isomorphic to a subgroup of  $\Gamma$  and is thus *solvable* – see Proposition 17.1.2.

Thus we may find a finite chain of subgroups

$$\Gamma_1 \supseteq N_1 \supseteq \cdots \supseteq N_r = \{1\}$$

where each subgroup is normal in the next and the quotients are cyclic of prime order  $p_i$ .

In turn, according to the fundamental theorem Theorem 15.5.2, these subgroups determine subfields

$$F(\zeta) \subseteq F_1 \subseteq \cdots \subseteq F_r = E$$

with  $N_i = \text{Gal}(E/F_i)$  and  $\text{Gal}(F_{i+1}/F_i) \simeq N_i/N_{i+1}$ .

Now since  $\text{Gal}(F_{i+1}/F_i) \simeq N_i/N_{i+1}$  has prime order  $p_i$  and since  $F_i$  contains the primitive  $n$ -th root of unity  $\zeta$ , we can apply the Theorem 17.5.2 to conclude that  $g$  is solvable in radicals over  $F(\zeta)$  and hence over  $F$ . □

## 17.7 Insolvability of higher degree equations

We've seen that cubic equations are solvable in radicals – see Example 17.2.3. In fact, quartic equations – i.e. polynomial equations of degree 4 – are also solvable in radicals. But we are going to show using the fundamental theorem of Galois theory that for any degree  $d \geq 5$ , there are polynomials of degree  $d$  over  $\mathbf{Q}$  which are not solvable in radicals.

For simplicity, we are going to work over a field  $F$  of characteristic 0; thus we can ignore questions of separability.

Now let  $g \in F[T]$  be a polynomial, and let  $E$  be a splitting field for  $g$  over  $F$ . Recall that we write  $\text{Gal}(g) = \text{Gal}(E/F)$ .

If  $g$  is **solvable in radicals**, We will argue that  $\text{Gal}(g)$  is a **solvable group**.

Thus if  $h \in F[T]$  is a polynomial for which  $\text{Gal}(h)$  is not a solvable group, then we may conclude that  $h$  is **not** solvable in radicals.

Our main example of a non-solvable group will be the symmetric group  $S_n$ ; for  $n \geq 5$ , we have seen that  $S_n$  is never solvable (since the derived group  $A_n = (S_n)'$  is equal to the alternating group and is a simple group).

## 18 The symmetric group as Galois group

Let  $E$  be a splitting field over  $F$  for the separable polynomial  $h \in F[T]$ .

Recall that if  $h \in F[T]$  has degree  $d$ , then  $\Gamma = \text{Gal}(h)$  may be identified with a subgroup of  $S_d$ . Indeed, write  $a_1, \dots, a_d$  for the roots of  $h$  in a splitting field  $E = F(a_1, \dots, a_d)$ . Any element  $\sigma \in \Gamma$  must map the set  $\Delta = \{a_1, \dots, a_d\} \subset E$  to itself. Since  $E$  is generated by  $\Delta$ , the natural group homomorphism

$$\Gamma \rightarrow \text{Sym}(\Delta)$$

given by  $\sigma \mapsto \sigma|_{\Delta}$  is one-to-one.

In these notes, we argue that for suitable fields  $F$  and polynomials  $h \in F[T]$ , this homomorphism is onto as well.

We are going to give full details when  $d = 5$

The following group-theoretic result provides a condition which guarantees that a subgroup of the symmetric group  $S_5$  is equal to  $S_5$ .

**Lemma 18.0.1.** *Let  $H \subset S_5$  be a subgroup such that (i)  $H$  contains a 5-cycle, and (ii)  $H$  contains a transposition. Then  $H = S_5$ .*

*Proof.* After renaming elements of  $S_5$ , we may and will suppose that  $(1, 2) \in H$ . Now, replacing the element of  $H$  of order 5 by some power, we may and will suppose that  $(1, 2, a, b, c) \in H$ ; renumbering again we may and will suppose that  $(1, 2, 3, 4, 5) \in H$ .

Now we must argue that

$$S_5 = \langle (1, 2), (1, 2, 3, 4, 5) \rangle.$$

To do so, write  $H_0 = \langle (1, 2), (1, 2, 3, 4, 5) \rangle \subset H$ . We will argue that  $H_0$  contains all transpositions.

Well,  $\gamma = (1, 2)(1, 2, 3, 4, 5) = (2, 3, 4, 5) \in H_0$ . Thus

$$\begin{aligned} \gamma(1, 2)\gamma^{-1} &= (1, 3) \in H_0 \\ \gamma^2(1, 2)\gamma^{-2} &= (1, 4) \in H_0 \\ \gamma^3(1, 2)\gamma^{-2} &= (1, 5) \in H_0 \end{aligned}$$

Combining these observations with the formula

$$(1, n)(1, m)(1, n) = (m, n),$$

we see that  $(m, n) \in H_0$  for all  $1 \leq m < n \leq 5$  as required.  $\square$

**Lemma 18.0.2.** *Let  $h \in F[T]$  be an irreducible, separable polynomial of degree 5, and let  $E$  be a splitting field of  $h$  over  $F$ . Then  $\Gamma = \text{Gal}(h) = \text{Gal}(E/F)$  contains a 5-cycle.*

*Proof.* Indeed, let  $\alpha$  be a root of  $h$  in  $E$ . Since  $h$  is irreducible  $[F(\alpha) : F] = 5$ . The tower law shows that

$$5 = [F(\alpha) : F] \text{ divides } [E : F].$$

On the other hand, we know that  $|\Gamma| = [E : F]$ . Since the prime number 5 divides  $|\Gamma|$  we know that  $\Gamma$  contains an element of order 5. But we have identified  $\Gamma$  with a subgroup of  $S_5$ , and the only elements of  $S_5$  which have order 5 are the 5-cycles.  $\square$

## 18.1 How to find a suitable polynomial

## 18.2 Using calculus

In this section, we consider the case  $F = \mathbf{Q}$ , so that  $h \in \mathbf{Q}[T]$  is irreducible of degree 5. We are going to use [Lemma on  $S_5$  generators](#s5-generators) and [Lemma on 5-cycles in Galois](#five-cycle-in-galois). Thus we must produce an irreducible polynomial  $h$  for which the galois group contains a transposition.

**Proposition 18.2.1.** *Let  $h \in \mathbf{Q}[T]$  be an irreducible polynomial of degree 5. If  $h$  has exactly three roots  $a_1, a_2, a_3$  in the field  $\mathbb{R}$  of real numbers, then  $\text{Gal}(h) = S_5$ .*

*Proof.* Write  $\Gamma = \text{Gal}(h)$  and identify  $\Gamma$  as a subgroup of  $S_5$  using the action of  $\Gamma$  on the roots of  $h$ . According to [Lemma on  $S_5$  generators](#s5-generators), to show that  $\Gamma = S_5$ , it suffices to argue that  $\Gamma$  contains a 5-cycle and a transposition.

According to [Lemma on 5-cycles in Galois](#five-cycle-in-galois),  $\Gamma$  contains a 5-cycle whenever  $h$  is irreducible (since  $F = \mathbf{Q}$  has characteristic 0, any irreducible polynomial is separable).

So it remains to argue that  $\Gamma$  contains a transposition. Let  $F = E \cap \mathbb{R} = \mathbf{Q}(a_1, a_2, a_3)$ . Then

$$h = (T - a_1)(T - a_2)(T - a_3)h_1 \quad \text{for } h_1 \in F[T]$$

where  $\deg h_1 = 2$ . The field  $E$  is a splitting field over  $F$  of the quadratic polynomial  $h_1$  which is irreducible over  $F$ , and  $h_1$  has two roots  $z_1, z_2$  in  $E$ .

It follows that there is an automorphism  $\tau : E \rightarrow E$  for which  $\tau|_F = \text{id}$  and for which  $\tau(z_1) = z_2$ .

Viewing  $\tau$  as an element of  $\Gamma$ , observe that  $\tau(a_i) = a_i$  for  $i = 1, 2, 3$  and thus  $\tau$  determines a transposition in  $S_5$ .  $\square$

**Theorem 18.2.2.** *There are polynomials  $h \in \mathbf{Q}[T]$  with  $\deg = 5$  and  $\text{Gal}(h) = S_5$ .*

*Proof.* Using the previous result, we see that the Theorem will follow if we exhibit a polynomial which is irreducible of degree 5 having exactly 3 roots in the field  $\mathbb{R}$  of real numbers.

Suppose that  $h$  is monic. View  $h$  as a function  $\mathbb{R} \rightarrow \mathbb{R}$ ; since the leading term has odd degree, we have

$$\lim_{x \rightarrow \infty} h(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow -\infty} h(x) = -\infty.$$

*Claim:* ( $\clubsuit$ ) if the derivative  $h'$  has exactly two real roots  $a_0 < a_1$ , and if  $h(a_0) > 0$  and  $h(a_1) < 0$  then  $h$  has exactly three real roots.

Indeed,  $h$  is increasing on  $(-\infty, a_0)$ , decreasing on  $(a_0, a_1)$  and increasing on  $(a_1, \infty)$  and under the specified conditions,  $h$  must change sign exactly once on each of these intervals.

Now let  $p > 0$  be a prime number and consider the monic polynomial

$$h = T^5 - pT^3 - pT - p$$

Observe that  $h$  is irreducible over  $\mathbf{Q}$  by Eisenstein's criteria.

Now,  $h' = 5T^4 - 3pT^2 - p$ . Using the quadratic formula and the substitution  $U = T^2$ , we see that any root  $a$  of  $h'$  must satisfy

$$a^2 = \frac{3p \pm \sqrt{9p^2 + 20p}}{10}$$

Since  $\sqrt{9p^2 + 20p} > 3p$ , it follows that for any real root  $a$  of  $h'$  we have

$$a^2 = \frac{3p + \sqrt{9p^2 + 20p}}{10}.$$

Thus  $h'$  has exactly two real roots  $a$  and  $-a$ .

Now we confirm ( $\clubsuit$ ) for all primes  $p < 50$  by a computer computation.

```
h(t,p) = t^5 - p*t^3 - p*t - p
s(p) = (3.*p + sqrt(9.*p^2 + 20.*p))/10
r(p) = sqrt(s(p))
[(p,h(-r(p),p),h(r(p),p)) for p in primes(50)]
```

```
[(2, 1.36948536859665, -5.36948536859665),
 (3, 4.08660815330037, -10.0866081533004),
 (5, 14.2740412556404, -24.2740412556404),
 (7, 31.7144026134529, -45.7144026134529),
 (11, 92.2075014863997, -114.207501486400),
 (13, 136.959876953231, -162.959876953231),
 (17, 259.243800671028, -293.243800671028),
 (19, 338.139705228473, -376.139705228473),
 (23, 534.582615744191, -580.582615744191),
 (29, 934.480355273911, -992.480355273911),
 (31, 1097.98388970545, -1159.98388970545),
 (37, 1686.05091826109, -1760.05091826109),
 (41, 2164.03196447407, -2246.03196447407),
 (43, 2430.13218369729, -2516.13218369729),
 (47, 3018.66196208224, -3112.66196208224)]
```

□

### 18.3 An algebraic tool

We pause to introduce a result that is useful for our present goals.

For this, let  $R$  be a principal ideal domain with field of fractions  $F$ , let  $P = \pi R \subset R$  be a prime ideal, and write  $k = R/P = R/\pi R$ . Our goal here is to relate galois groups over  $F$  with galois groups over  $k$  in a suitable sense.

More precisely, let  $g \in R[T]$  be a polynomial of degree  $d \geq 1$ . Since  $R \subset F$ , we may view  $g$  as a polynomial with coefficients in  $F[T]$ ; let  $E$  be a splitting field for  $g$  over  $F$ .

On the other hand, note that there is a ring homomorphism  $R[T] \rightarrow (R/\pi R)[T] = k[T]$  for which  $T \mapsto T$ . Write  $\bar{g} \in k[T]$  for the image of  $g$  under this homomorphism, and let  $\ell$  denote a splitting field over  $k$  of the polynomial  $\bar{g} \in k[T]$ .

**Proposition 18.3.1.** *Suppose that  $\bar{g}$  has  $d$  distinct roots in  $\ell$ .*

1. *There is a subgroup  $D \subset \text{Gal}(E/F)$  and a group homomorphism*

$$\pi : D \rightarrow \text{Gal}(\ell/k)$$

*such that  $\pi$  maps **onto**  $\text{Gal}(\ell/k)$ .*

2. Write

$$\bar{g} = p_1 p_2 \cdots p_t$$

for irreducible polynomials  $p_i$  in  $k[T]$  of degree  $e_i$  for  $1 \leq i \leq t$ . Suppose that the galois group  $\text{Gal}(p_i) \subset S_{e_i}$  contains an  $e_i$ -cycle for each  $i$ . Then  $\Gamma \subset S_n$  contains a element  $\sigma$  which can be written as a product of disjoint cycles of lengths  $e_1, e_2, \dots, e_t$ .

*Remark 18.3.2.* For a polynomial  $f \in F[T]$ , there is a quantity  $\text{disc}(f) \in F$  which may be described by a formula involving the coefficients of  $f$  with the property that  $f$  has repeated roots if and only if  $\text{disc}(f) = 0$ .

*Remark 18.3.3.* The condition in part 2. – namely, that each galois group  $\text{Gal}(p_i) \subset S_{e_i}$  contains an  $e_i$ -cycle – is automatic when  $k$  is a finite field (since then  $\text{Gal}(p_i)$  is a cyclic group of order  $e_i$ ).

The condition on  $\text{Gal}(p_i)$  is also immediate when  $e_i = 1$  or when  $e_i$  is a prime number.

## 18.4 Example for rational field

In this section, we give an alternate way of constructing polynomials  $g \in \mathbf{Q}[T]$  of degree 5 for which either  $\text{Gal}(g) = A_5$  or  $S_5$ .

We replace the use of “calculus” with the use of Proposition 18.3.1.

Let  $p, q$  be prime numbers, and notice that

$$g = T^5 + pT + (q+1)p \in \mathbf{Q}[T]$$

is irreducible by Eisenstein’s criterion (Theorem 7.4.1) applied to the prime  $p$ .

A *partition*  $\lambda$  of a whole number  $n$  is a non-decreasing list  $0 < \lambda_1 \leq \lambda_2 \leq \cdots$  of whole numbers with  $\sum_i \lambda_i = n$ .

We say that  $\bar{g}$  has *shape* equal to the partition  $\lambda$  of 5 modulo a prime  $q$  if the irreducible factors of  $\bar{g}$  have degrees  $\lambda_1, \lambda_2, \dots$ .

Use `sage math` we can compute the shape of  $\bar{g}$  modulo various primes  $q$ :

```
def pred(p,q):
    R.<T>=PolynomialRing(FiniteField(q))
    g = T^5 + p*T + p*(q+1)

    shape = [x[0].degree() for x in factor(g)]
    return discriminant(g) != 0 and shape == [1,1,1,2]

bound = 100
myprimes = list(primes(bound))
[[p,q] for q in myprimes if pred(p,q)] for p in myprimes]
```

```
[[2, 19],
 [3, 47], [3, 97],
 [5, 37], [5, 67],
 [7, 47], [7, 73],
 [11, 23], [11, 53],
 [13, 73], [13, 83],
```

```

[],
[(19, 41), (19, 43), (19, 47), (19, 53)],
[(23, 17)],
[(29, 59)],
[],
[(37, 89)],
[(41, 23), (41, 31), (41, 37), (41, 73)],
[(43, 29), (43, 59), (43, 73), (43, 97)],
[],
[(53, 41)],
[(59, 19), (59, 71)],
[],
[(67, 89)],
[(71, 47), (71, 53)],
[(73, 83)],
[(79, 37)],
[(83, 67), (83, 89)],
[],
[(97, 19), (97, 47), (97, 53)]

```

**Proposition 18.4.1.** *Let*

$$g = T^5 + pT + p(q + 1) \in \mathbf{Q}[T]$$

*where  $p, q$  are prime numbers. For each pair  $(p, q)$  in the above list, the polynomial  $g$  has galois group  $\Gamma = \text{Gal}(g)$  equal to  $S_5$*

*Sketch of proof.* Again using Lemma 18.0.2, the irreducibility of  $g$  allows us to conclude that  $\Gamma$  contains a 5-cycle.

Since  $p, q$  is in the preceding list, we know that  $\bar{g} \in \mathbf{F}_q[T]$  has shape  $\lambda = (1, 1, 1, 2)$ .

Now Proposition 18.3.1 implies that  $\Gamma$  contains a transposition. □

## 18.5 Example for rational field

Here we give a version of the number-theoretic argument found in Section 18.3 which is valid for the field  $F = k(X)$  over certain fields of constants  $k$ , instead of  $\mathbf{Q}$ . We give in particular a construction when  $k = \mathbb{R}$ ; i.e. when  $F = \mathbb{R}(X)$ .

Let us fix an element  $a \in k$  and consider the irreducible polynomials  $p = X - a \in k[X]$ . Form the polynomial

$$g = T^5 + (X - a)T^3 + (X - a) \in F[T] = k(X)[T].$$

Observe that  $g \in k[X][T]$ ; i.e. viewing  $g$  as a polynomial in the variable  $T$ , the coefficients of  $g$  are themselves polynomials in the variable  $X$ . Moreover,  $g$  is irreducible by Eisenstein's criteria.

Consider the second irreducible polynomial  $q = X - b$ . The quotient ring  $k[X]/\langle X - b \rangle$  may be identified with  $k$  and the mapping

$$(h \mapsto \bar{h}) : k[X] \rightarrow k = k[X]/\langle X - b \rangle$$

is just given by evaluation at  $X = b$ .

Notice that if  $q = X - b$  then

$$\bar{g} = T^5 + (b - a)T^3 + (b - a) \in k[T].$$

**Proposition 18.5.1.** *Let  $a \in k$  and let*

$$g = T^5 - (X - a)T^3 + (X - a) \in F[T] = k(X)[T],$$

*and let  $E$  be a splitting field of  $g$  over  $F = k(X)$ . If  $\bar{g} \in k[T]$  has shape  $(1, 1, 1, 2)$ , then the galois group  $\Gamma = \text{Gal}(g) = \text{Gal}(E/F)$  satisfies*

$$\Gamma = S_5.$$

*Proof.* The proof is essentially the same as the argument given in [the previous section](#example-over-Q).  $\square$

## 18.6 Rational function field with real coefficients

Consider  $k = \mathbb{R}$ , and let

$$g = T^5 + (X - a)T^3 + (X - a) \in \mathbb{R}(X)[T],$$

as before.

We claim that  $\text{Gal}(g) = S_5$  for all real numbers  $a$ .

Well, it will be enough to argue that  $\bar{g}$  has shape  $(1, 1, 1, 2)$  for  $q = X - b$  for some  $b \in \mathbb{R}$ .

We first use `sage math` to compute the discriminant and the **shape** of the real polynomial  $T^5 - aT^3 - a \in \mathbb{R}[T]$  (this amounts to  $\bar{g}$  when  $q = X$ ). We get:

```
def pred(a):
    R.<T>=PolynomialRing(RealField())
    g = T^5 - a*T^3 - a
    shape = [x[0].degree() for x in factor(g)]
    if discriminant(g) != 0 and shape == [1,1,1,2]:
        return a

[a for a in range(1,10) if pred(a)]
```

```
[4, 5, 6, 7, 8, 9]
```

This shows that  $\bar{g} = T^5 - aT^3 - a$  has shape  $(1, 1, 1, 2)$  e.g. when  $a = 4$ .

Now, for general  $a$  take  $q = X - (4 - a)$  and find that  $\bar{g} = T^5 - 4T^3 - 4$  which we have just seen has shape  $(1, 1, 1, 2)$ .

## 19 Elementary symmetric functions

Let  $k$  be any field, let  $n \geq 1$  and consider the field of rational functions in  $n$  variables

$$E = k(T_1, T_2, \dots, T_n).$$

From a permutation  $\sigma \in S_n$ , we obtain a field automorphism

$$\Pi_\sigma : E \rightarrow E$$

given by the rule  $\Pi_\sigma(T_i) = T_{\sigma(i)}$ .

e.g. if  $\sigma = (1, 2, 3)$  and  $n \geq 4$ , we have

$$\Pi_{(1,2,3)} \left( \frac{T_1 + T_4}{T_2^2 T_3 + 1} \right) = \frac{T_2 + T_4}{T_3^2 T_1 + 1}.$$

It is immediate to check that the group homomorphism

$$\Pi : S_n \rightarrow \text{Aut}(E)$$

given by  $\sigma \mapsto \Pi_\sigma$  is one-to-one.

Let  $F = E^{S_n}$  be the fixed field for this action. Then we have proved before that  $E$  is a normal extension of  $F$ , and that in fact  $E$  is the splitting field for some separable polynomial in  $F[T]$ .

Moreover, we have proved that  $\text{Gal}(E/F) = S_n$ .

This gives a straightforward construction of extensions which are not solvable in radicals, though it is perhaps unsatisfying because the field  $F$  was not specified in advance.

The field  $F$  and the field extension  $F \subset E$  are quite important, though I'm only going to sketch some brief comments.

Note that the polynomial

$$s_1 = T_1 + T_2 + \dots + T_n$$

is actually contained in  $F = E^{S_n} = k(T_1, \dots, T_n)^{S_n}$ ; indeed, for any  $\sigma \in S_n$ , we see that

$$\Pi_\sigma(s_1) = T_{\sigma(1)} + T_{\sigma(2)} + \dots + T_{\sigma(n)} = s_1.$$

Similarly, the polynomial

$$s_2 = T_1 T_2 + T_1 T_3 + \dots + T_1 T_n + T_2 T_1 + T_2 T_3 + \dots = \sum_{1 \leq i < j \leq n} T_i T_j$$

is in  $F = E^{S_n}$  since

$$\Pi_\sigma(s_2) = \sum_{1 \leq i < j \leq n} T_{\sigma(i)} T_{\sigma(j)} = s_2.$$

In fact, for  $1 \leq \ell \leq n$  define the  $\ell$ -th elementary symmetric polynomials as follows:

$$s_\ell = \sum_{1 \leq i_1 < i_2 < \dots < i_\ell \leq n} T_{i_1} T_{i_2} \dots T_{i_\ell}.$$

Note e.g. that  $s_1$  and  $s_2$  are the polynomials described before, and

$$s_n = T_1 T_2 \dots T_n.$$

Then  $s_\ell \in F = E^{S_n}$  for each  $1 \leq \ell \leq n$ , and in fact it is a classical Theorem that

$$F = k(s_1, s_2, \dots, s_n) = k(T_1, \dots, T_n)^{s_n}$$

i.e.  $F$  is generated over  $k$  by the elementary symmetric functions. Each  $s_i$  is transcendental over  $k$ , and in fact for each  $\ell$  the element  $s_\ell$  is transcendental over  $k(s_1, \dots, s_{\ell-1})$ .

A slightly stronger statement is true:

$$k[s_1, s_2, \dots, s_n] = k[T_1, \dots, T_n]^{s_n}$$

i.e. every polynomial in  $T_1, \dots, T_n$  which is invariant under the action of  $s_n$  can be written as a polynomial in the elementary symmetric polynomials  $s_\ell$ .

For example, if  $n = 3$ , then by inspection we see that

$$T_1^2 + T_2^2 + T_3^2 \in k[T_1, T_2, T_3]^{s_3}.$$

Moreover, one checks that

$$T_1^2 + T_2^2 + T_3^2 = (T_1 + T_2 + T_3)^2 - 2(T_1T_2 + T_1T_3 + T_2T_3) = s_1^2 - 2s_2$$

As a final remark, if we grant the *Primitive Element Theorem* (though we didn't prove it) we conclude that  $E = F(\alpha)$  for some rational function  $\alpha \in E = k(T_1, \dots, T_n)$  having degree  $n!$  over  $F$ ; i.e. that

$$k(T_1, \dots, T_n) = k(s_1, \dots, s_n, \alpha).$$

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