Assignment 1

sections of [Fitzpatrick] covered: $\S : 13.1, 13.2, 13.3$

Problem 1: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} xy/(x^2 + y^2) \text{ if } (x, y) \neq (0, 0) \\ 0 \text{ else} \end{cases}.$$

We saw in the lecture that $f_x = \partial f / \partial x$ and $f_y = \partial f / \partial y$ exist. Show that neither f_x nor f_y is continuous at the point $(0, 0)$.

Problem 2: Suppose that $g : \mathbb{R}^3 \to \mathbb{R}$ has the property that

$$|g(x, y, z)| \leq x^2 + y^2 + z^2 \text{ for all } (x, y, z) \in \mathbb{R}^3.$$

Prove that $\partial g / \partial x$, $\partial g / \partial y$ and $\partial g / \partial z$ all exist at $(0, 0, 0)$.

Problem 3: Let U be an open subset of $\mathbb{R}^3$ and let $g : U \to \mathbb{R}$ be a function which has first-order partial derivatives at each point $\vec{x} \in U$. Recall that the *gradient* of g is the function

$$\nabla g : U \to \mathbb{R}^3 \text{ given by } (\nabla g)(x, y, z) = \left(g_x(x, y, z), g_y(x, y, z), g_z(x, y, z)\right);$$

more succinctly, $\nabla g = \left(g_x, g_y, g_z\right)$.

Prove: if $(\nabla g)(\vec{x}) = 0$ for every $\vec{x} \in \mathbb{R}^3$ then g is *constant*; i.e. there is $c \in \mathbb{R}$ with $g(\vec{x}) = c$ for every $\vec{x} \in \mathbb{R}^3$.

Problem 4: **(Chain Rule)** Let U an open subset of $\mathbb{R}^3$ containing the point $\vec{x}$, and $f : U \to \mathbb{R}$ a function for which the partial derivative $f_x(\vec{x})$ exists.

Suppose that $g : \mathbb{R} \to \mathbb{R}$ is differentiable at the point $f(\vec{x})$.

Prove that the function $g \circ f : U \to \mathbb{R}$ has a partial derivative with respect to x and that

$$\frac{\partial}{\partial x}(g \circ f)(\vec{x}) = g'(f(\vec{x})) \cdot f_x(\vec{x}).$$

Problem 5: Find the gradient ∇f for each of the following functions.

a. $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\vec{x}) = e^{|\vec{x}|^2} = \exp(|\vec{x}|^2)$
b. $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(\vec{x}) = \sin(xy)/\sqrt{x^2 + y^2 + 1}$.

Problem 6: Assume that U is an open subset of $\mathbb{R}^3$ and that $f, g : U \to \mathbb{R}$ are continuously differentiable. For $\vec{x} \in U$, find a formula for $\nabla(fg)(\vec{x})$ in terms of $\nabla f(\vec{x})$ and $\nabla g(\vec{x})$.

Problem 7: Assume that U is an open subset of $\mathbb{R}^3 n$, that $f : U \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuously differentiable. For $\vec{x} \in U$ find a formula for $\nabla(g \circ f)(\vec{x})$ in terms of $\nabla f(\vec{x})$ and $g'(f(\vec{x}))$.

Problem 8: Let

$$f : \mathbb{R}^2 \to \mathbb{R} \text{ be given by } f(x, y) = \begin{cases} (x/|y|) \cdot \sqrt{x^2 + y^2} \text{ if } y \neq 0 \\ 0 \text{ if } y = 0 \end{cases}.$$

a. Prove that f is not continuous at $\vec{0}$.
b. Prove that $\nabla f(\vec{x})$ is defined for each $\vec{x} \in \mathbb{R}^2$.
c. Prove that for each $c \in \mathbb{R}$ there is a vector $\vec{p} \in \mathbb{R}^2$ with $|\vec{p}| = 1$ such that

$$\frac{\partial f}{\partial \vec{p}}(\vec{0}) = c.$$

Explain why this observation does not contradict Corollary 13.18 in [Fitzpatrick].

Notational remarks: Recall that

$$\frac{\partial f}{\partial \vec{p}} = \nabla f \cdot \vec{p}; \quad \text{i.e. } \frac{\partial f}{\partial \vec{p}}(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{p}.$$

In fact, Fitzpatrick writes

$$\frac{\partial f}{\partial \vec{p}} = \langle \nabla f, \vec{p} \rangle; \quad \text{i.e. } \frac{\partial f}{\partial \vec{p}}(\vec{x}) = \langle \nabla f(\vec{x}), \vec{p} \rangle$$

where $\langle \vec{v}, \vec{w} \rangle = \vec{v} \cdot \vec{w}$ represents the *dot product* of vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$.