Math146 - Review for midterm 1

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2025-02-19

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I. The exam will cover what is (currently) in sections 1 - 7 of the course lecture notes.

II. You should be able to give careful statements for the definitions of the following terms:

- a a *commutative ring* R, a *field* F, an *integral domain* R, a *ring homomorphism* $f : R \to S$, an *ideal* I of a commutative ring R,a *principal ideal* of a commutative ring R, a *principal ideal domain*, the *quotient ring* R/I where I is an ideal of a commutative ring,
- b an *irreducible element* of a commutative ring R, the greatest common divisor $\gcd(a, b)$ for elements $a, b \in R$ of a principal ideal domain R, a *unit* of a commutative ring, an *associate* of an element of a commutative ring, a 0-divisor of a commutative ring R
- c the *field of fractions* F of an integral domain R, the *polynomial ring* $R[T]$ for a commutative ring R

III. You should know the statements of the following results.

a. The *first isomorphism theorem for rings*
b. the result that the *unique factorization property* holds in a PID
c. The *division algorithm* for the polynomial ring $F[T]$ where F is a field.
d. Eisenstein's irreducibility criterion
e. the Gauss Lemma and consequences

IV. Be able to give examples of the following:

a. an *integral domain* that is not a *principal ideal domain*
b. a field F and a polynomial $f \in F[T]$ such that f has no root in F but f is *reducible.*
c. A finite field F with exactly 9 elements. (**Hint:** Consider the field $\mathbf{F}_3 = \mathbf{Z}/3\mathbf{Z}$ of order 3, and find a polynomial of the form $p = T^2 - a \in \mathbf{F}_3[T]$ that is *irreducible.* How many elements are in the quotient $F[T]/\langle p \rangle$?)

V. You should be able to write careful solutions to problems similar to the following:

1. Let F be a field and let $f, g \in F[T]$ be polynomials for which $\gcd(f, g) = 1$. Consider the mapping
$$\phi : F[T] \to F[T]/\langle f\rangle \times F[T]/\langle g\rangle$$
given by the rule $\phi(h) = (h + \langle f\rangle, h + \langle g\rangle)$.

   a. Show that $\ker \phi = \langle fg\rangle$ and that ϕ induces an isomorphism
$$\overline{\phi} : F[T]/\langle fg\rangle \xrightarrow{\sim} F[T]/\langle f\rangle \times F[T]/\langle g\rangle$$

   b. As a consequence, show that $\mathbf{Q}[T]/\langle T^7 - 1\rangle$ is isomorphic to the direct product of two fields.

2. Let R be a PID, let $a_1, a_2, \cdots, a_n \in R$ not all 0, and let $d = \gcd(a_1, a_2, \cdots, a_n)$. Note that $\dfrac{a_i}{d} \in R$ for each i. Prove that $\gcd\left(\dfrac{a_1}{d}, \dfrac{a_2}{d}, \cdots, \dfrac{a_n}{d}\right) = 1$

3. Show that $u = 2 + T + \langle T^3\rangle$ is a unit in the quotient ring $\mathbf{Q}[T]/\langle T^3\rangle$.

4. Let F be a field. Prove that $\sqrt{T}$ is not in $F(T)$. (**Hint:** Suppose the contrary, namely that $\sqrt{T} = \dfrac{f}{g}$ for $f, g \in F[T]$. Explain why we find an equation $g^2T = F^2$ in $F[T]$. Now apply unique factorization in the PID to deduce a contradiction).

5. If R is a PID and $p, q \in R$ are non-associate irreducible elements, compute $\gcd(p^2q, pq^2)$?

6. Consider the field $\mathbf{F}_5$ with 5 elements.
   - Prove that $T^2 - 3 \in \mathbf{F}_5[T]$ is irreducible.
   - Let $\gamma = T + \langle T^2 - 3\rangle \in \mathbf{F}_5[T]/\langle T^2 - 3\rangle$ so that $\mathbf{F}_5(\gamma) = \mathbf{F}_5[T]/\langle T^2 - 3\rangle$. Find $s, t \in \mathbf{F}_5$ so that $(s + t\gamma) \cdot (1 + \gamma) = 1$.

7. Be able to give the proof of the following results (taken from the notes).

   Let R be a PID and let $p \in R$ irreducible.
   - If $p \mid ab$ for $a, b \in R$ then $p \mid a$ or $p \mid b$.
   - The quotient ring $R/\langle p\rangle$ is a field.