PS6 - Splitting fields

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The symbol F denotes a field.

1. Let $F(X)$ be the field of rational functions over F; thus $E = F(X)$ is the field of fractions of the polynomial ring $F[X]$.

   a. For $n \in \mathbf{Z}_{\geq 2}$, show that the polynomial $f(T) = T^n + XT + X$ is irreducible in $E[T] = F(X)[T]$.

   b. For distinct elements $a, b \in F$, find the degree $[E(\sqrt{X-a}, \sqrt{X-b}) : E]$.

2. Let E be a splitting field over $\mathbf{Q}$ of the polynomial $T^5 - 2$. Find $[E : \mathbf{Q}]$.

3. Let $F = \mathbf{F}_7 = \mathbf{Z}/7\mathbf{Z}$ be the finite field with 7 elements.

   a. Show that $F^\times$ is a cyclic group of order 6.

   b. Let E be a splitting field over F of the polynomial $T^3 - 2$. Compute $[E : F]$.

      Hint: If $\alpha = \sqrt[3]{2}$ is a chosen root of $T^3 - 2$, and if ω is a root of $T^2 + T + 1 = \dfrac{T^3 - 1}{T - 1}$ then $\alpha, \omega\alpha, \omega^2\alpha$ are the roots of $T^3 - 2$ i.e. $T^3 - 2 = (T - \alpha)(T - \omega\alpha)(T - \omega^2\alpha)$ in $F(\alpha, \omega)[T]$.

4. Let $c_1, c_2, \cdots, c_n \in F$ be distinct elements. Show that $\dfrac{1}{X - c_1}, \dfrac{1}{X - c_2}, \cdots, \dfrac{1}{X - c_n}$ are *linearly independent* over F in the field $F(X)$ of rational functions.

   i.e. show that if $\alpha_1, \cdots, \alpha_n \in F$ and if

   $$0 = \sum_{i=1}^{n} \frac{\alpha_i}{X - c_i}$$

   then $\alpha_i = 0$ for every i.