PS5 - Field extensions and degree

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1. Let $F = \mathbf{Q}(\alpha, \beta)$ where $\alpha = \sqrt{3}$ is a root of $T^2 - 3 \in \mathbf{Q}[T]$ and β is a root of $T^5 - 3T - 3 \in \mathbf{Q}[T]$.

   a. Compute $[F : \mathbf{Q}]$.
   b. Find a basis for F as a $\mathbf{Q}$-vector space.
   c. Find a basis for F as a $\mathbf{Q}(\beta)$-vector space.

2. Let $F \subset E$ be a field extension and let $\alpha, \beta \in E$ be algebraic over F, say with $n = \deg(\alpha)$ and $m = \deg(\beta)$.

   If $\gcd(n, m) = 1$ show that $[F(\alpha, \beta) : F] = nm$.

3. Let F be a field and let $f, g \in F[T]$ be irreducible polynomials with $n = \deg(f)$ and $m = \deg(g)$.

   Write $E = F[T]/\langle f \rangle$ so that E is a field extension of F with $[E : F] = n$.

   If $\gcd(n, m) = 1$ show that g remains irreducible in $E[T]$.

4. Consider the finite field $\mathbf{F}_5 = \mathbf{Z}/5\mathbf{Z}$ with 5 elements.

   a. Find a polynomial $f \in \mathbf{F}_5[T]$ of degree 2 which is irreducible.
   b. Find a polynomial $g \in \mathbf{F}_5[T]$ of degree 3 which is irreducible.
   c. Let $K = \mathbf{F}_5[T]/\langle f \rangle$ and let $\alpha = T + \langle f \rangle \in K$. so that $K = \mathbf{F}_5(\alpha)$ and $[K : \mathbf{F}_5] = 2$. Prove that g remains irreducible in the polynomial ring $K[T]$.
   d. Let $L = K[T]/\langle g \rangle$ and let $\beta = T + \langle g \rangle \in L$ so that $L = K(\beta) = \mathbf{F}_5(\alpha, \beta)$. Show that $[L : \mathbf{F}_5] = 6$.
   e. How many elements are in the field L?