Math146 - PS4 due 2025-02-23

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1. Find the minimal polynomial over $\mathbf{Q}$ of the element $\omega = \exp(2\pi i/6) \in \mathbf{C}$.

   Hint: First note that ω is a root of $\dfrac{T^6-1}{T-1} = T^5 + T^4 + T^3 + T^2 + T + 1$. Does $T^3 - 1$ divide $T^6 - 1$?

2. Consider the field extension $E = \mathbf{Q}(\alpha)$ of $\mathbf{Q}$ where the minimal polynomial of α is equal to $f(T) = T^2 - 3T - 3 \in \mathbf{Q}[T]$.

   a. Show that f is irreducible over $\mathbf{Q}$.

   b. Viewing E as a vector space over $\mathbf{Q}$, we know that $\{1, \alpha\}$ is a *basis* of E as a $\mathbf{Q}$-vector space. Let's write $\Phi : E \to \mathbf{Q}^2$ for the vector space isomorphism

   $$s + t\alpha \mapsto \begin{bmatrix} s \\ t \end{bmatrix}.$$

   Fix $\beta = a + b\alpha \in \mathbf{Q}(\alpha)$ for $a, b \in \mathbf{Q}$ and consider the $\mathbf{Q}$-linear transformation $\lambda_\beta : E \to E$ given by "multiplication by β". Thus

   $$\lambda_\beta(\gamma) = \beta \cdot \gamma \quad \text{for} \quad \gamma \in E = \mathbf{Q}(\alpha).$$

   Find the 2×2 *matrix* $M = M_\beta \in \operatorname{Mat}_{2\times 2}(\mathbf{Q})$ with the property that

   $$M \cdot \Phi(\gamma) = \Phi(\lambda_\beta(\gamma)).$$

   Otherwise stated, the matrix M is determined by the condition

   $$M \cdot \begin{bmatrix} s \\ t \end{bmatrix} = \Phi((a + b\alpha)(s + t\alpha)) \quad \text{for every } s, t \in \mathbf{Q}.$$

   c. Show that the assignment $\beta \mapsto M_\beta$ determines a homomorphism of rings

   $$E \to \operatorname{Mat}_{2\times 2}(\mathbf{Q})$$

   (For this, you need to show that $M_{\beta_1+\beta_2} = M_{\beta_1} + M_{\beta_2}$ and that $M_{\beta_1\cdot\beta_2} = M_{\beta_1} \cdot M_{\beta_2}$ for $\beta_i \in E$).

d. Using the quadratic formula, we see that the roots of f have the form

$$\frac{3 \pm \sqrt{21}}{2}.$$

Choosing $\alpha = \dfrac{3+\sqrt{21}}{2}$ let us write $\overline{\alpha} = \dfrac{3-\sqrt{21}}{2} = 3 - \alpha$.
For $\beta = s + t\alpha \in E$ with $s, t \in \mathbf{Q}$ put $\overline{\beta} = s + t\overline{\alpha} = (s+3t) - t\alpha$.
Show that the assignment $\beta \mapsto \overline{\beta}$ determines an isomorphism of rings $E \to E$.

e. Show that $\beta \cdot \overline{\beta} \in \mathbf{Q}$ for every $\beta \in E$.

f. Verify that

$$\det M_\beta = \beta \cdot \overline{\beta}.$$

for every $\beta \in E$.

g. Write $\dfrac{1}{2+\alpha} \in E$ in the form

$$\frac{1}{2+\alpha} = s + t\alpha$$

for $s, t \in \mathbf{Q}$.