Review for Quiz 1 - Solutions

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B. *Problems and questions.*

1. Suppose that the temperature last Wednesday at a point (x, y) in a national park is given by $T(x, y)$ degrees celcius. Here x represents the distance east from a certain fixed location $(0, 0)$, and y represents the distance north from this fixed location.

   The park is located in the mountains. The altitude above at the location (x, y) is $h(x, y)$ meters.

   Consider the problem of finding the maximum temperature (on Wednesday) in the park – and a position where it occurs – at a given altitude of a meters. Thus one needs to maximize the function $T(x, y)$ subject to the constraint $h(x, y) = a$.

   a. To use the method of Lagrange multipliers to find the point (x_0, y_0) at which $T(x, y)$ is maximal subject to $h(x, y) = a$, what equations must we solved? Choose one of the following:

   (i) $\dfrac{\partial T}{\partial x} = \dfrac{\partial g}{\partial x}, \dfrac{\partial T}{\partial y} = \dfrac{\partial g}{\partial y}$ and $g(x, y) = a$.

   (ii) $\dfrac{\partial T}{\partial x} = \lambda \dfrac{\partial g}{\partial x}, \dfrac{\partial T}{\partial y} = \lambda \dfrac{\partial g}{\partial y}$ for some real number λ.

   (iii) $\dfrac{\partial T}{\partial x} = \lambda \dfrac{\partial g}{\partial x}, \dfrac{\partial T}{\partial y} = \lambda \dfrac{\partial g}{\partial y}$ for some real number λ, and $g(x, y) = a$.

   Solution:
   The correct choice is (iii).

   b. If the position $(x, y) = (30-0.1a, 15-0.2a)$ maximizes the temperature $T(x, y)$ subject to the constraint $g(x, y) = a$, find the sensitivity of the x-coordinate of this solution to the quantity a when $a = 100$ meters above sea level.

   What are the *units* of the sensitivity figure?

   Recall that sensitivity is given by the formula $S(x, a) = \dfrac{dx}{da} \cdot \dfrac{a}{x(a)}$.

   Solution:
   We compute $\dfrac{dx}{da} = \dfrac{d}{da}[30 - 0.1a] = -0.1$
   Moreover, when $a = 100$ meters, $x(100) = 30 - 0.1 \cdot 100 = 30 - 10 = 20$ meters.
   Thus $S(x, a) = -0.1 \cdot \dfrac{100}{20} = -\dfrac{100}{200} = -0.5$
   Since x and a are both measured in meters, the units in the expression for the sensitivity *cancel* so the sensitivity is dimensionless.

c. Based on your answer to (b), finish the sentence:
"For the value $a = 100$ meters, a one percent change in a results in a ____________________ percent change in the x coordinate of the location (x, y) at the given altitude which had the highest temperature on Wednesday."

Solution:
"For the value $a = 100$ meters, a one percent change in a results in a -0.5 percent change in the x coordinate of the location (x, y) at the given altitude which had the highest temperature on Wednesday."

2. Consider a linear program $\mathscr{L}$ in *standard form*; say $\mathscr{L}$ is the linear program
`maximize` $\mathbf{c} \cdot \mathbf{x}$ where $\mathbf{c} = \begin{bmatrix} 10 & 20 & 1 & 1 \end{bmatrix}$
subject to costraints $A\mathbf{x} \le \mathbf{b}$ where $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$, and the non-negativity constraint $\mathbf{x} \ge \mathbf{0}$.

a. What are the inequalities that must be satisfied by the entries of the variable vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$?

Solution:
Computing the product $A\mathbf{x}$, we see that $A\mathbf{x} \le \mathbf{b}$ amounts to

$$\begin{bmatrix} x_1 + x_2 + 2x_4 \\ x_3 + x_4 \end{bmatrix} \le \begin{bmatrix} 10 \\ 15 \end{bmatrix}.$$

Thus the inequalities that the variables x_i must satisfy are as follows: $x_i \ge 0$ for each i, $x_1 + x_2 + 2x_4 \le 10$ and $x_3 + x_4 \le 15$.

b. What is the objective function of the *dual linear program* $\mathscr{L}'$?

Solution:
The objective function of the dual program is determined by the vector $\mathbf{b}^T = \begin{bmatrix} 10 & 15 \end{bmatrix}$.
If $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is a dual variable, the objective function for $\mathscr{L}'$ is given by

$$\mathbf{b}^T \mathbf{y} = 10y_1 + 15y_2$$

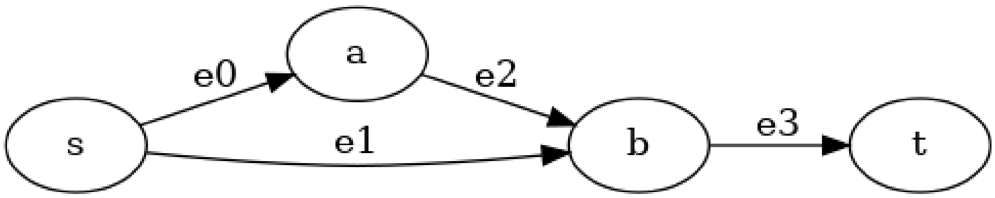
.

c. What are the inequality constraints of the dual linear program $\mathscr{L}'$?

Solution:

The inequality constraints of the dual linear program are given by $\mathbf{y} \geq 0$ together with $A^T\mathbf{y} \geq c^T$, which amounts to $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} 10 \\ 20 \\ 1 \\ 1 \end{bmatrix}$
Thus we need $y_1 \geq 0, y_2 \geq 0$, $y_1 \geq 10$, $y_1 \geq 20$, $y_2 \geq 1$ and $2y_1 + y_2 \geq 1$.

3. Consider the `max flow` linear program determined by the following *network flow diagram:*



The capacity `ci` through edge `ei` is given in the following table:

edge	`e0`	`e1`	`e2`	`e3`
capacity	10	15	20	5

We represent a *flow* as a vector $\mathbf{f} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$ where f_i denotes the flow through edge `ei`.

a. What vector $\mathbf{c}$ determines the *objective function* for the `max flow` linear program?

Solution:
To find the value $|\mathbf{f}$ for a flow, we just add the values of the flow on edges leaving the source node. Thus $|\mathbf{f}| = f_0 + f_1$.

b. What are the *inequality constraints* for the `max flow` linear program determined by this diagram?

Solution:
The inequality constraints for this linear program are determined by the capacities. Thus, we have $f_i \leq e_i$ and $0 \leq f_i$ for $i = 1, 2, 3, 4$. Written in matrix form this amounts to $A\mathbf{f} \leq \mathbf{e}$ where $A = \mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is the 4×4 identity matrix, and where

$$e = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 20 \\ 5 \end{bmatrix}$$

c. What is the *conservation law* at the interior vertex `b`? (Give an equation involving the quantities f_i).

Solution:
The vertex `b` has two incoming edges `a->b` and `s->b`, and one outgoing edge `b->t`. Thus, the conservation law for a flow $\mathbf{f}$ requires that `f1 + f2 = f3`.

d. The equality constraints for the `max flow` linear program are given by the vector equation $B\mathbf{f} = \mathbf{0}$ for a suitable matrix B. What is B? (Remember that B is determined by the conservation laws at the interior vertices).

Solution:
There are two interior vertices, `a` and `b`. In the previous problem, we worked out the conservation law at node `b`.
The conservation law for a flow $\mathbf{f}$ at node `a` requires that `f0 = f2`.
Thus, the matrix B is given by

$$B = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$