

Math087 - Review for Quiz 1

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A. Summary

The modeling techniques discussed so far in our course all focus on *optimization*. These techniques include:

1. *Calculus-based methods* for optimization of functions of one or more variables.
 - An important related issue is the *sensitivity* of the optimal solution to parameters involved in the model.
 - Finding optimal values subject to *constraints* can be carried out using the method of *Lagrange multipliers*.
2. Optimization of linear objective functions of several variables subject to linear inequality constraints; this is known as *linear programming*.
 - From a practical point-of-view, an important task in solving linear programs is identifying the objective function and constraints in a manner that is easy to provide as input to a computer.
 - Reformulation the problem by consideration of the *dual linear program* can provide useful insight about the model; the coefficients of a dual solution can often be viewed as *unit prices* for the quantities involved in the original constraints.
 - Formulating the objective function and constraints for a linear program by using a *network flow model* can be useful.
 - Occasionally one needs to find a solution to linear programming problem where some or all of the coefficients of the solution vector are required to be *integers*. Solving such *integer programming* problems can be challenging; we discussed a *branch-and-bound* strategy for solving such problems which leverages solutions to related *relaxed* linear programs.

B. Problems and questions.

1. Suppose that the temperature last Wednesday at a point (x, y) in a national park is given by $T(x, y)$ degrees celcius. Here x represents the distance east from a certain fixed location $(0, 0)$, and y represents the distance north from this fixed location. The park is located in the mountains. The altitude above at the location (x, y) is $h(x, y)$ meters.
Consider the problem of finding the maximum temperature (on Wednesday) in the park – and a position where it occurs – at a given altitude of a meters. Thus one needs to maximize the function $T(x, y)$ subject to the constraint $h(x, y) = a$.

a. To use the method of Lagrange multipliers to find the point (x_0, y_0) at which $T(x, y)$ is maximal subject to $h(x, y) = a$, what equations must we solve? Choose one of the following:

(i) $\frac{\partial T}{\partial x} = \frac{\partial g}{\partial x}, \frac{\partial T}{\partial y} = \frac{\partial g}{\partial y}$ and $g(x, y) = a$.

(ii) $\frac{\partial T}{\partial x} = \lambda \frac{\partial g}{\partial x}, \frac{\partial T}{\partial y} = \lambda \frac{\partial g}{\partial y}$ for some real number λ .

(iii) $\frac{\partial T}{\partial x} = \lambda \frac{\partial g}{\partial x}, \frac{\partial T}{\partial y} = \lambda \frac{\partial g}{\partial y}$ for some real number λ , and $g(x, y) = a$.

b. If the position $(x, y) = (30 - 0.1a, 15 - 0.2a)$ maximizes the temperature $T(x, y)$ subject to the constraint $g(x, y) = a$, find the sensitivity of the x -coordinate of this solution to the quantity a when $a = 100$ meters above sea level.

What are the *units* of the sensitivity figure?

Recall that sensitivity is given by the formula $S(x, a) = \frac{dx}{da} \cdot \frac{a}{x(a)}$.

c. Based on your answer to (b), finish the sentence:

“For the value $a = 100$ meters, a one percent change in a results in a _____ percent change in the x coordinate of the location (x, y) at the given altitude which had the highest temperature on Wednesday.”

2. Consider a linear program \mathcal{L} in *standard form*; say \mathcal{L} is the linear program maximize $\mathbf{c} \cdot \mathbf{x}$ where $\mathbf{c} = [10 \ 20 \ 1 \ 1]$

subject to constraints $A\mathbf{x} \leq \mathbf{b}$ where $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$, and the non-negativity constraint $\mathbf{x} \geq \mathbf{0}$.

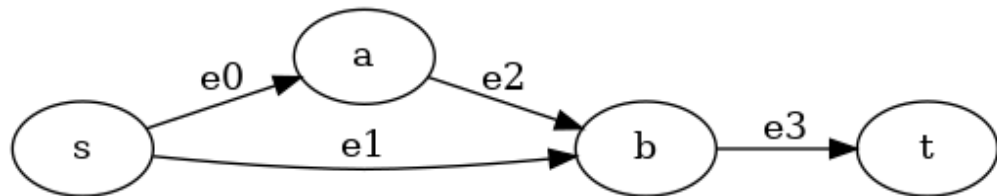
a. What are the inequalities that must be satisfied by the entries of the variable vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} ?$$

b. What is the objective function of the *dual linear program* \mathcal{L}' ?

c. What are the inequality constraints of the dual linear program \mathcal{L}' ?

3. Consider the **max flow** linear program determined by the following *network flow diagram*:



The capacity c_i through edge e_i is given in the following table:

edge	e0	e1	e2	e3
capacity	10	15	20	5

We represent a *flow* as a vector $\mathbf{f} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$ where f_i denotes the flow through edge e_i .

- a. What vector \mathbf{c} determines the *objective function* for the **max flow** linear program?
- b. What are the *inequality constraints* for the **max flow** linear program determined by this diagram?
- c. What is the *conservation law* at the interior vertex b ? (Give an equation involving the quantities f_i).
- d. The equality constraints for the **max flow** linear program are given by the vector equation $B\mathbf{f} = \mathbf{0}$ for a suitable matrix B . What is B ? (Remember that B is determined by the conservation laws at the interior vertices).