PS 8 - Binomial and Poisson distributions

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1. For a whole number $N \geq 1$, recall the following identity:

$$(X+Y)^N = \sum_{j=0}^{N} \binom{N}{j} X^j Y^{N-j}$$

where X, Y are *variables*.

a. Explain why the identity $(X+Y)^N = (Y+X)^N$ implies that

$$\binom{N}{j} = \binom{N}{N-j}$$

for each $0 \leq j \leq N$.

b. Explain why the identity

$$\begin{aligned}(X+Y)^N &= (X+Y)(X+Y)^{N-1} \\ &= X(X+Y)^{N-1} + Y(X+Y)^{N-1}\end{aligned}$$

implies that

$$\binom{N}{j} = \binom{N-1}{j} + \binom{N-1}{j-1}$$

for each $0 \leq j \leq N-1$.

Hint: In each case, observe what the indicated identity says about the coefficient of $X^j Y^{N-j}$ in the given expression(s).

2. Using the identities in 1., one can argue inductively that

$$\binom{N}{j} = \frac{N!}{j! \cdot (N-j)!}$$

(in fact, this formula has been used in the "notebook"!)

Use this identity to compute the following limits:

$$\lim_{N \to \infty} \frac{1}{N^3} \binom{2N}{3} \qquad \text{and} \qquad \lim_{N \to \infty} \frac{1}{e^N} \binom{N}{N-4}.$$

3. Suppose that the probability that an automobile accident occurs during a 24 hour period in a certain stretch of freeway is given by the number p, $0 < p < 1$.

   Assume that the random variable X describing the *number of automobile accidents* is given by the Poisson distribution.

   Thus the probability that there are k accidents is given by

   $$P(X = k) = e^{-p} \cdot \frac{p^k}{k!}$$

   Give an expression for the probability that there no more than 3 accidents in a 24-hour period.

4. *Jane's Fish Tank Emporium* (*JFTE*) yet again

   Recall that in a class notebook last week, we discussed the operation of *JFTE* by considering the question: what is the optimal ordering strategy for fish tanks: ordering *on-demand*?, or putting in place a *standing order*?

   Our simulation used a `python` class `JFTE`; the *constructor* of the class `JFTE` (i.e. its member function `__init__`) creates the `customers` instance variable. For this, the version in the notebook invokes the function

```
def customer(prob=1./7):
   return rng.choice([1,0],p=[prob,1-prob])
```

   In this-week's-notebook, we created a function `arrival` which takes two arguments: `p` and `num_max`; this function returns the integer `k` with probability determined by the Poisson distribution and base probability `p`, where

```
0 <= k <= num_max.
```

   Edit the `JFTE` class so that is constructor uses the Poisson distribution to simulate arrival of customers.

   Recall that the constructor is the function `__init__`; it has the form:

```
def __init__(self,N,prob=1./7):
# ...
self.reset()
```

   You need to replace "`# ...`" with code to create the instance variable `self.customers` and assign it to be a list of integers containing `N` values returned by the function `arrival(prob,5)`.

   You may now apply these `strategy` functions to an instance of the `JFTE` class constructed using the Poisson customer-arrival function.

   Run the simulation 10 times with both strategy functions, as was done in the notebook. Discuss similarities/differences between the results obtained in the notebook.