PS 5 - Matching & Finite state machines

Math087 - George McNinch

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1. A mining company needs to set up supply line between some quarries –labeled by the set U – and some processing plants –labeled by the set W. The company is interested in finding *perfect matchings* of quarries to plants, i.e. each quarry should have a supply line to a unique processing plant, such that no quarry or plant is on two supply lines. Conveniently, there are as many quarries as there are processing plants – i.e. $|U| = |W|$. Less conveniently, supply lines need to travel along roads. Let E be the set of roads. We assume that each road in E starts at a quarry and travels directly to a processing plant, and that the roads do not intersect.

   a. Suppose that |U| = |W| = 4 and that there are seven roads total. Construct an example where no perfect matching is possible.

      Suggestion: for this problem and the rest, we suggest you model this set up as a bipartite graph.

   b. If no perfect matching exists, the company would like to know what is the maximal number of quarry/plant pairs such that no quarry or plant is on two supply chains (call this a maximal matching). Compute this for your example in a). How many additional roads must be made in order for a perfect matching to exist?

   c. If $|U| = |W| = n$, what is the largest possible size of E? Give your answer as an expression in n.

   d. Suppose that for each $x \in U$, there is exactly one road involving x, and write this edge as $x \to w(x)$ for some $w(x) \in W$. Explain why the size of a maximum matching is equal to the number of distinct nodes $\{w(x)\}$.

   e. (Optional) Suppose that |U | = |W | = n and let m = |E|. Can you always find a configuration of roads such that a perfect matching is impossible? Put differently, is there a minimal m such that a perfect matching will exist no matter how the roads are placed?

2. *I've removed problem 2; it will appear in next week's homework assignment*