Review for Midterm Exam

1. Material

The final exam is comprehensive, but will weight material since the midterm exam more heavily. Recall that since the midterm, we have discussed the following:

- modules over a ring; direct sums and free modules (week 5)
- algebras, the construction of the polynomial ring (or polynomial algebra) (week 6)
- maximal ideals; existence of maximal ideals via Zorn's Lemma (week 6)
- principal ideal domains, Chinese remainder theorem (week 7)
- unique factorization domains, PID => UFD, rings of fractions (week 9)
- Gauss Lemma, Eisenstein criteria, R UFD $\Rightarrow R[T]$ UFD (week 9,10)
- extension of scalars of a module; tensor product of modules (week 10)
- Hom and $\otimes$ functors; Jordan-Holder theorem. (week 11)
- local rings; Nakayama Lemma (week 11)
- modules over a PID: torsion and torsion-free modules, p-primary modules, elementary divisors, invariant factors (week 12)
- Jordan blocks of a matrix; Cayley Hamilton Theorem (week 13)

You should know the important definitions and statements of important results.

2. Some problems

Review the homework assignments. We can talk in the Review on [2025-12-08 Mon] about which homework problems to emphasize in your exam preparation. (I'll bring printed copies of the assignments!)

Here are a few more problems to consider.

Problem 1: Let A be an commutative ring, and let F be a free module of finite rank $n \in \mathbb{N}$. Show that the choice of a basis $\mathcal{B} = \{b_1, b_2, ..., b_n\}$ of F determines an isomorphism

$$\mathrm{End}_A(F) \xrightarrow{\sim} \mathrm{Mat}_n(A)$$

of A-algebras given by $\varphi \mapsto [\varphi]_{\mathcal{B}}$ where $[\varphi]_{\mathcal{B}}$ denotes the matrix of φ in the basis $\mathcal{B}$.

Problem 2: Let F be an algebraically closed field, and let $\alpha, \beta \in F$ with $\alpha \neq \beta$. Find representatives for the $\mathrm{GL}_5(F)$ orbits of 5×5 matrices whose characteristic polynomial is given by $(T - \alpha)(T - \beta)^4$.

Problem 3: Let A be a commutative ring and let B be a commutative A-algebra.

a. Prove that $B \otimes_A A[T] \simeq B[T]$ where T is a polynomial variable.
b. Let $I = \langle f_1, f_2, ..., f_n \rangle \subset A[T]$ be the ideal generated by the elements $f_i \in A[T]$. Write $\overline{f_i} = 1 \otimes f_i \in B[T]$, and let $J = \langle \overline{f_1}, ..., \overline{f_n} \rangle$ be the corresponding ideal of
$$B[T] = B \otimes_A A[T].$$
Prove that $B[T]/J \simeq B \otimes_A (A[T]/I)$.

Problem 4: Use the results of the preceding problem to prove:
$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}.$$

Problem 5: Let $K \subset L$ be a field extension, let $f \in K[T]$ be a monic irreducible polynomial of degree d, and suppose that f has d distinct roots $\alpha_1, ..., \alpha_d \in E$, so that
$$f = \prod_{i=1}^{d} (T - \alpha_i).$$
Suppose that $E = F(\alpha_1, \alpha_2, ..., \alpha_d)$. Prove that
$$E \otimes_F E \simeq \prod_{i=1}^{n} E.$$