Review for Midterm Exam

1. Some problems

Problem 1: Let A be an commutative ring, and let F be a free module of finite rank $n \in \mathbb{N}$. Show that the choice of a basis $\mathcal{B} = \{b_1, b_2, ..., b_n\}$ of F determines an isomorphism

$$\mathrm{End}_A(F) \xrightarrow{\sim} \mathrm{Mat}_n(A)$$

of A-algebras given by $\varphi \mapsto [\varphi]_{\mathcal{B}}$ where $[\varphi]_{\mathcal{B}}$ denotes the matrix of φ in the basis $\mathcal{B}$.

Solution for 1:
Sorry; actually this is redundant with problem 3a on the problem set from week 13.

Problem 2: Let F be an algebraically closed field, and let $\alpha, \beta \in F$ with $\alpha \neq \beta$. Find representatives for the $\mathrm{GL}_5(F)$ orbits of 5×5 matrices whose characteristic polynomial is given by $(T - \alpha)(T - \beta)^4$.

Solution for 2:
Since F is algebraically close, every irreducible polynomial in $F[T]$ is associate with a monic linear polynomial $T - a$ for $a \in F$.

Any 5×5 matrix M makes $V = F^5$ into an $F[T]$-module, and our assumption means for an irreducible polynomial p that the p-primary part of V is 0 unless p is associate with $T - \alpha$ or $T - \beta$.

According to Prop. 12.3.2, the $\mathrm{GL}(V)$-conjugacy class of M is determined by the isomorphism class of the $F[T]$-module V_M.

For $s \in F$ and a partition λ, we write $V_{s,\lambda}$ for the $F[T]$-module given

$$V_{s,\lambda} = \bigoplus_{i=1}^{e} F[T]/(T-s)^{\lambda_i} F[T].$$

Using Theorem 12.1.1, we can describe all torsion $F[T]$-modules of length 5 for which the only p-primary parts that are non-zero are when p is associate to $T - \alpha$ or $T - \beta$.

The possibilities are $V = V_{a,\lambda} \bigoplus V_{b,\mu}$ where $|\lambda| = 1$ and $|\mu| = 4$.

Thus the full list is:

- $V_{\alpha,(1)} \bigoplus V_{\beta,(4)}$.
- $V_{\alpha,(1)} \bigoplus V_{\beta,(1,3)}$.
- $V_{\alpha,(1)} \bigoplus V_{\beta,(1,1,2)}$.
- $V_{\alpha,(1)} \bigoplus V_{\beta,(2,2)}$.

For $s \in F$ and $d \in \mathbb{N}$, a Jordan block is a $d \times d$ matrix of the form

$$J_{s,d} = \begin{pmatrix} s & 0 & 0 & \dots & 0 & \\ 1 & s & 0 & \dots & 0 & 0 \\ 0 & 1 & s & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & s & 0 \\ 0 & 0 & 0 & \dots & 1 & s \end{pmatrix}$$

With respect to a suitable basis, the action of T on the $F[T]$-module $V_{s,\lambda}$ is a block-diagonal matrix where the diagonal blocks - the "Jordan blocks" - are precisely $J_{s,\lambda_1}, J_{s,\lambda_2}, \dots, J_{s(\lambda_e)}$.

Thus, a full list of $\mathrm{GL}(V)$-conjugacy classes of 5×5 matrices with the indicated properties are precisely the following:

- matrix with Jordan blocks $J_{\alpha,1}, J_{\beta,4}$
- matrix with Jordan blocks $J_{\alpha,1}, J_{\beta,1}, J_{\beta,3}$.
- matrix with Jordan blocks $J_{\alpha,1}, J_{\beta,1}, J_{\beta,1}, J_{\beta,2}$.
- matrix with Jordan blocks $J_{\alpha,1}, J_{\beta,2}, J_{\beta,2}$.

Problem 3: Let A be a commutative ring and let B be a commutative A-algebra.

a. Prove that $B \otimes_A A[T] \simeq B[T]$ where T is a polynomial variable.
b. Let $I = \langle f_1, f_2, ..., f_n \rangle \subset A[T]$ be the ideal generated by the elements $f_i \in A[T]$. Write $\overline{f_i} = 1 \otimes f_i \in B[T]$, and let $J = \langle \overline{f_1}, ..., \overline{f_n} \rangle$ be the corresponding ideal of

$$B[T] = B \otimes_A A[T].$$

Prove that $B[T]/J \simeq B \otimes_A (A[T]/I)$.

Solution for 3:

a. We note that $B \otimes_A A[T]$ is a B-algebra via

the mapping $b \mapsto b \otimes 1$. (The product in $B \otimes_A A[T]$ is given on pure tensors by the formula $(b \otimes f) \cdot (b' \otimes f') = b \cdot b' \otimes f \cdot f'$.)

We check that $B \otimes_A A[T]$ satisfies the mapping property of the polynomial B-algebra $B[T]$. Thus, let C be any B-algebra and let $x \in C$ be any element. We must argue that there is a unique B-algebra homomorphism $B \otimes_A A[T] \to C$

Well, viewing C as an A-algebra and using the mapping property of the polynomial A-algebra $A[T]$, there is a unique A-algebra homomorphism $\varphi : A[T] \to C$ for which $\varphi(T) = c$.

Now we get a unique B-module homomorphism

$$\Phi : B \otimes_A A[T] \to C \text{ by extension of scalars}$$

which satisfies $\Phi(b \otimes f) = b \cdot \varphi(f)$ for $b \in B$ and $f \in A[T]$.

One has to check that Φ is actually a homomorphism of B-algebras; this is straightforward. Now Φ is the required unique homomorphism of B-algebras, and this proves that $B \otimes_A A[T]$ satisfies the mapping property of the polynomial ring. Thus $B \otimes_A A[T] \simeq B[T]$.

a. Consider the short exact sequence of A-modules

$$0 \to I \xrightarrow{\iota} A[T] \to A[T]/I \to 0.$$

We know that the extension of scalars functor $B \otimes_A -$ is right exact; thus applying the result from a. we see that the sequence

$$B \otimes_A I \xrightarrow{\mathrm{id} \otimes \iota} B[T] \to B \otimes_A (A[T]/I) \to 0.$$

is exact.

Now the result follows from exactness by observing that the image of $\mathrm{id} \otimes \iota$ is precisely the ideal J of $B[T]$ generated by the elements $\overline{f_i} \in B[T]$.

Problem 4: Use the results of the preceding problem to prove:

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}.$$

Solution for 4:
There is an isomorphism of $\mathbb{R}$-algebras $\mathbb{C} \simeq \mathbb{R}[T]/(T^2+1)\mathbb{R}[T]$. Now the result from the previous problem shows that

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[T]/(T^2+1)\mathbb{R}[T] \simeq \mathbb{C}[T]/(T^2+1) \quad (\clubsuit)$$

Now the Chinese Remainder Theorem shows that

$$\mathbb{C}[T]/(T^2+1)\mathbb{C}[T] \simeq \mathbb{C}[T]/(T+i)\mathbb{C}[T] \times \mathbb{C}[T]/(T-i)\mathbb{C}[T] \simeq \mathbb{C} \times \mathbb{C}.$$

Problem 5: Let $K \subset L$ be a field extension, let $f \in K[T]$ be a monic irreducible polynomial of degree d, and suppose that f has d distinct roots $\alpha_1, ..., \alpha_d \in E$, so that

$$f = \prod_{i=1}^{d} (T - \alpha_i).$$

Suppose that $E = F(\alpha_1, \alpha_2, ..., \alpha_d)$. Prove that

$$E \otimes_F E \simeq \prod_{i=1}^{n} E.$$

Solution for 5:
The statement isn't quite correct (as I pointed out in class). I need to suppose that $E = F(\alpha_1)$.

Then $E \simeq F[T]/f \cdot F[T]$ and by problem 3 we find

$$E \otimes_F E \simeq E \otimes_F F[T]/f \cdot F[T] \simeq E[T]/f \cdot E[T].$$

Now by the Chinese Remainder Theorem we see that

$$E[T]/f \cdot E[T] \simeq \prod_{i=1}^{d} E[T]/(T - \alpha_i)E[T] \simeq \prod_{i=1}^{d} E.$$