Math 245 Final Exam

In this exam, all rings are assumed to have identity, and ring homomorphisms preserve the identity element.

Problem 1. (10 points)

Give careful answers to the following:

a. What is meant by the *annihilator* $\operatorname{Ann}_A(x)$ of an element $x \in M$ for an A-module M.

b. Give a careful statement for *Eisenstein's criteria* for the irreducibilty of a polynomial $f \in K[T]$ where K is the field of fractions of a PID A.

c. Let A be a commutative ring. If M is an A-module, what does it mean to say that M is a *torsion* module?

d. Let G be a finite group, and let p be a prime number. What does *Sylow's theorem* say about the number n_p of p-Sylow subgroups of G?

Problem 2. (10 points)

Let $\alpha, \beta, \gamma \in \mathbb{C}$ be 3 distinct complex numbers. Find representatives of the $\mathrm{GL}_5(\mathbb{C})$-orbits on 5×5 matrices M for which the characteristic polynomial M is $(T-\alpha)(T-\beta)(T-\gamma)^3 \in \mathbb{C}[T]$. Explain and justify your choices, briefly.

Solution for 2:
Let $V = \mathbb{C}^5$; a matrix M determines on V the structure of a finitely generated torsion module for the PID $A = \mathbb{C}[T]$. Thus

$$V = \bigoplus_p A_{p,\lambda_p}$$

where for each prime p, λ_p is a partition, where $\sum_p |\lambda_p| = 5$.

Now, we have seen (it is a consequence of the Cayley-Hamilton Theorem) that $|\lambda_p| > 0$ implies that p divides the characteristic polynomial.

Moreover, the matrix of the action of T on $A/(T-\delta)^r A$ is the $r \times r$ Jordan block matrix $J_{r,\delta}$.

Thus, it is a consequence of the structure theorem for finitely generated modules over a PID that the GL_5-orbits are represented by the following matrices:

- M with diagonal ("Jordan") blocks $J_{1,\alpha}$, $J_{1,\beta}$, $J_{3,\gamma}$
- M with diagonal ("Jordan") blocks $J_{1,\alpha}$, $J_{1,\beta}$, $J_{1,\gamma}$, $J_{2,\gamma}$
- M with diagonal ("Jordan") blocks $J_{1,\alpha}$, $J_{1,\beta}$, $J_{1,\gamma}$, $J_{1,\gamma}$, $J_{1,\gamma}$

Problem 3. (10 points)

Prove for a prime number p that

$$\mathbb{R} \otimes_{\mathbb{Q}} \left(\mathbb{Q}[T]/\langle T^2 - p\rangle\right) \simeq \mathbb{R} \times \mathbb{R}.$$

Solution for 3:
There is an isomorphism $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}[T] \simeq \mathbb{R}[T]$ and a resulting isomorphism

$$\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}[T]/\langle T^2 - p\rangle \simeq \mathbb{R}[T]/\langle T^2 - p\rangle.$$

Thus it will be enough to prove that

$$\mathbb{R}[T]/\langle T^2 - p\rangle \simeq \mathbb{R} \times \mathbb{R}.$$

Now, since p is a positive integer, there is a real square root of p - -i.e. $\sqrt{p} \in \mathbb{R}$.

In particular, we have a factorization

$$T^2 - p = (T - \sqrt{p}) \cdot (T + \sqrt{p}) \in \mathbb{R}[T].$$

Now, the polynomials $T \pm \sqrt{p} \in \mathbb{R}[T]$ are *irreducible* and non-associate, thus they are co-maximal. This can be checked directly by observing that

$$1 = \left(\frac{1}{2\sqrt{p}}\right)((T + \sqrt{p}) - (T - \sqrt{p})) \in \langle T + \sqrt{p}\rangle + \langle T - \sqrt{p}\rangle$$
$$\text{so that} \quad \mathbb{R}[T] = \langle T + \sqrt{p}\rangle + \langle T - \sqrt{p}\rangle.$$

Thus according to the Chinese Remainder Theorem, the mapping

$$\mathbb{R}[T] \to \mathbb{R}[T]/\langle T + \sqrt{p}\rangle \times \mathbb{R}[T]/\langle T - \sqrt{p}\rangle$$

given by the rule

$$f \mapsto (f + \langle T + \sqrt{p}\rangle, f + \langle T - \sqrt{p}\rangle)$$

is an isomorphism.

Finally, we note that $\mathbb{R}[T]/\langle T \pm \sqrt{p}\rangle \simeq \mathbb{R}$ since the polynomials $T \pm \sqrt{p}$ have degree 1. This completes the proof.

Problem 4. (10 points)

Let A be an integral domain and let K be its field of fractions. Suppose that F is a free A-module of finite rank and let $\varphi : F \to F$ be an A-endomorphism of F.

a. If φ is invertible, show that $\mathrm{id} \otimes \varphi : K \otimes_A F \to K \otimes_A F$ is invertible.

b. If

$$\mathrm{id} \otimes \varphi : K \otimes_A F \to K \otimes_A F$$

is invertible, is φ invertible? Give a proof or a counterexample.

Solution for 4:

a. Suppose that φ is invertible, say $\varphi^{-1} = \psi$. Thus $\varphi \circ \psi = \mathrm{id}_F$ and $\psi \circ \varphi = \mathrm{id}_F$.

   Since $K \otimes_A -$ is a functor, we see that

$$(\mathrm{id}_K \otimes \varphi) \circ (\mathrm{id}_K \otimes \psi) = \mathrm{id}_K \otimes (\varphi \circ \psi) = \mathrm{id}_K \otimes \mathrm{id}_F$$

   and

$$(\mathrm{id}_K \otimes \psi) \circ (\mathrm{id}_K \otimes \varphi) = \mathrm{id}_K \otimes (\psi \circ \varphi) = \mathrm{id}_K \otimes \mathrm{id}_F \,.$$

   Since $(\mathrm{id}_K \otimes \mathrm{id}_F)(a \otimes b) = a \otimes b$ for $a \in K$ and $b \in F$, it follows that

$$\mathrm{id}_K \otimes \mathrm{id}_F = \mathrm{id}_{K \otimes_A F} \,.$$

   Thus $\mathrm{id}_K \otimes \varphi$ and $\mathrm{id}_K \otimes \psi$ are inverse to one another, and in particular $\mathrm{id}_K \otimes \varphi$ is invertible.

b. In general, invertibility of $\mathrm{id}_K \otimes \varphi$ does not imply that φ is invertible.

   Here are two examples. Let $A = \mathbb{Z}$ so that $K = \mathbb{Q}$. Put $F = \mathbb{Z}/2\mathbb{Z}$ and and let $\varphi : F \to F$ be the 0-mapping.

   Then of course φ is not invertible. But $\mathbb{Q} \otimes_{\mathbb{Z}} F = 0$, and $\mathrm{id}_{\mathbb{Q}} \otimes \varphi : 0 \to 0$ is thus invertible.

   For the second example, again take $A = \mathbb{Z}$ so that $K = \mathbb{Q}$. Let $F = \mathbb{Z}$ and consider the mapping $\varphi : \mathbb{Z} \to \mathbb{Z}$ given by $\varphi(x) = 2x$. Since the maping φ is not surjective, φ is not invertible.

   Then $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Q}$, and $\mathrm{id}_{\mathbb{Q}} \otimes \varphi : \mathbb{Q} \to \mathbb{Q}$ is given by multiplication with 2. In particular, $\mathrm{id}_{\mathbb{Q}} \otimes \varphi$ *is* invertible.

Problem 5. (10 points)

Let A be a commutative local ring with unique maximal ideal $\mathfrak{m}$. Prove for $x \in A$ that x is a unit if and only if $x \notin \mathfrak{m}$.

Solution for 5:
First, suppose that x is a unit in A. Then $A \cdot x = \langle x \rangle = A$. Since $\mathfrak{m}$ is a maximal ideal, we know that $\mathfrak{m} \subsetneq A$. It follows that $x \notin \mathfrak{m}$.

On the other hand, suppose that $x \notin \mathfrak{m}$. We must find an inverse for x. Consider the ideal $I = \langle x \rangle$. First of all, suppose that I is a proper ideal of A. It follows that I is contained in some maximal ideal. Since A is local, $\mathfrak{m}$ is the unique maximal ideal of A. Then $I \subseteq \mathfrak{m}$ contradicting the assumption $x \notin \mathfrak{m}$.

This contradiction shows that $\langle x \rangle = I = A$. In particular, there is an element

$$y \in A \text{ with } x \cdot y = 1;$$

since A is commutative, $y = x^{-1}$ so that $x \in A^\times$ as required.

Problem 6. (10 points)

Let A be a commutative ring, let M be an A-module, and suppose that $X, Y \subseteq M$ are A-submodules such that $M = X + Y$ and $X \cap Y = \{0\}$.

Prove that $Y \simeq M/X$.

Solution for 6:

Let $\iota : Y \to M$ be the inclusion mapping, and let $\pi : M \to M/X$ be the quotient mapping. Set $\gamma = \pi \circ \iota$

We claim that $\gamma : Y \to M/X$ is an isomorphism.

To see that γ is onto, let $\alpha \in M/X$. We may write $\alpha = m + X$ for some $m \in M$. Since $M = X + Y$ by assumption, we may write $m = x + y$ for $x \in X$ and $y \in Y$. There is an equality of cosets

$$\alpha = m + X = y + X \text{ in } M/X.$$

Now we see that

$$\gamma(y) = \pi(y) = y + X = \alpha$$

which proves that γ is surjective.

To see that γ is injective, suppose that $y \in \ker \gamma$. Then

$$\pi(y) = y + X = 0 \text{ in } M/X.$$

Thus $y \in X$ so that $y \in X \cap Y$. The intersection $X \cap Y$ is 0 by assumption; thus we have proved that $y = 0$. It follows that $\ker \gamma = 0$, so that γ is injective.

Thus γ is an isomorphism as required.

Problem 7. (10 points)

Let K be a field and let L be a field extension of K (i.e. $K \subseteq L$ is a subfield). Let V be a finite dimensional K-vector space, and write $V^* = \mathrm{Hom}_K(V, K)$ for the *dual space* of V. Prove that

$$L \otimes_K V^* \simeq \mathrm{Hom}_L(L \otimes_K V, L).$$

Solution for 7:
We note first that

$$\dim_L L \otimes_K V^* = \dim_K V^* = \dim_K V$$

and that

$$\dim_L L \otimes_K V = \dim_K V \text{ so that } \dim_L \operatorname{Hom}_L(L \otimes_K V, L) = \dim_k V.$$

In particular,

$$L \otimes_K V^* \text{ and } \operatorname{Hom}_L(L \otimes_K V, L)$$

are L-vector spaces of the same dimension.

In fact, that observation solves the problem, since any two vector spaces of the same dimension are isomorphic.

To give an explicit (functorial!) map, proceed as follows:

There is a map of K-vector spaces

$$\Phi : V^* \to \operatorname{Hom}_L(L \otimes_K V, L)$$

such that

$$\Phi(\gamma) \text{ is the map determined by } \alpha \otimes v \mapsto \alpha \cdot \gamma(v).$$

Since $\operatorname{Hom}_L(L \otimes_K V, L)$ is an L-vector space, the mapping property of the scalar extension gives a unique mapping

$$\Psi : L \otimes_K V^* \to \operatorname{Hom}_L(L \otimes_K V, L)$$

such that for $\gamma \in V^*$,

$$\Psi(1 \otimes \gamma) = \Phi(\gamma).$$

We claim that Ψ is an isomorphism.

Since Ψ is an L-linear map between L-vector spaces of the same dimension, it suffices to prove that Ψ is surjective.

A choice of K-basis $b_1, \dots, b_n$ for V determines an L-basis $1 \otimes b_1, \dots, 1 \otimes b_n$ for $L \otimes_K V$. Write $\delta_1, \dots, \delta_n$ for the K-basis of V^* dual to $b_1, \dots, b_n$, so that $1 \otimes \delta_1, \dots, 1 \otimes \delta_n$ is an L-basis for $L \otimes_K V^*$.

On the other hand, let $\kappa_1, \dots, \kappa_n$ be the L-basis of $\operatorname{Hom}_L(L \otimes_K V, L) = (L \otimes_K V)^*$ dual to the basis $1 \otimes b_1, \dots, 1 \otimes b_n$.

A calculation shows that $\Psi(1 \otimes \delta_i) = \kappa_i$, which shows that Ψ is surjective and hence an isomorphism.