Extra Problems

I was going to assign these problems, but decided against it. Thought I'd post them anyhow!

Problem 1: Let A be a PID, let F be a free A-module, and let $\varphi : F \to F$ be an A-module homomorphism.

a. Use Proposition 11.1.7 from the notes to prove that there is a basis $\mathcal{B}$ of F together with A -module homomorphisms $\psi, \gamma : F \to F$ such that
   - $\varphi = \psi \circ \gamma$
   - γ is invertible, and
   - there are elements $a_1, a_2, ..., a_n \in A$ such that

   $$[\psi]_{\mathcal{B}} = \operatorname{diag}(a_1, a_2, ..., a_n) = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix},$$

   is the diagonal matrix with the indicated entries.
b. Conclude that for any A-basis $\mathcal{C}$ of F, $\det([\psi]_{\mathcal{C}}) = u a_1 a_2 ... a_n$ for some unit $u \in A^{\times}$.

   Hint: The determinant of an invertible matrix in $\operatorname{Mat}_n(A)$ is a unit in A.

Problem 2: Let $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ be a $\mathbb{Z}$-linear map and suppose that $\operatorname{id} \otimes \varphi : \mathbb{Q}^n \to \mathbb{Q}^n$ is invertible. Prove that

$$|\operatorname{coker} \varphi| = |\det \varphi|.$$

Hint: Use problem 1.

Problem 3: Let k be a field, let $k[T]$ be the polynomial ring over k in one variable T and let $F = k[T]^n$ be the free $k[T]$-module of rank n consisting of column vectors with entries in $k[T]$.

Let $M \in \operatorname{Mat}_{n \times n}(k[T])$ be an $n \times n$ matrix with entries in $k[T]$. Notice that $\det M \in k[T]$ is a polynomial. Suppose that $\det M \neq 0$.

Consider the $k[T]$-module homomorphism $\varphi : F \to F$ given by multiplication with M; thus $\varphi(v) = M \cdot v$ for $v \in F = k[T]^n$.

Prove that the dimension as k-vector space of the $k[T]$-module $\operatorname{coker} \varphi = F / \operatorname{im} \varphi$ is given by the degree $\deg(\det M)$ of $\det M$; i.e.

$$\dim_k(\operatorname{coker} \varphi) = \deg(\det M).$$

Hint: Use problem 1.