Problem Set week 13

Problem 1: Let A be a local ring with unique maximal ideal $\mathfrak{m}$, and write $k = A/\mathfrak{m}$ for the residue field of A.

Suppose that M is a free A-module of finite rank n. Let $\alpha_1, \ldots, \alpha_n \in M/\mathfrak{m}M$ be a $k = A/\mathfrak{m}$-basis for $M/\mathfrak{m}M$, and let $x_1, \ldots, x_n \in M$ be elements for which $\alpha_i = x_i + \mathfrak{m}M$ for $i = 1, \ldots n$. Prove that $x_1, \ldots, x_n$ forms an A-basis for M.

NB This problem needs an additional hypothesis: A is Noetherian. We need to know that a submodule of a finitely generated module is finitely generated.

Solution for 1:
Write $N = \sum_{i=1}^n A \cdot x_i$. We first argue $N = M$ or equivalently that $M/N = 0$.

Consider the short exact sequence

$$0 \to N \xrightarrow{\iota} M \to M/N \to 0.$$

We know that the tensor product functor $k \otimes_A -$ is right exact; thus

$$k \otimes_A N \xrightarrow{\mathrm{id} \otimes \iota} k \otimes_A M \to k \otimes_A (M/N) \to 0$$

is exact.

Since the x_i are a k-basis for M, the mapping $\mathrm{id} \otimes \iota : k \otimes_A N \to k \otimes_A M$ is an isomorphism. In particular it follows from exactness that $k \otimes_A (M/N) = 0$.

But for any A-module X, we have an A-module isomorphism $k \otimes_A X = X/ \mathfrak{m} \cdot X$. If X is finitely generated, the Nakayama Lemma thus shows that $k \otimes_A X = 0 \Rightarrow X = 0$. Since M – and hence M/N – is finitely generated, this shows that $M/N = 0$ so that $N = M$.

Now, let F be a free A-module on n generators $y_1, ..., y_n$ and consider the mapping $\varphi : F \to M$ given by $\varphi(y_i) = x_i$. Since the image of φ is N, the preceding argument shows that φ is surjective.

consider the kernel B of the mapping Φ; thus there is a short exact sequence

$$0 \to B \to F \xrightarrow{\varphi} M \to 0.$$

Again, we know that

$$k \otimes_A B \to k \otimes_A F \xrightarrow{\mathrm{id} \otimes \varphi} k \otimes_A M \to 0.$$

is exact.

By hypothesis, $\mathrm{id} \otimes \varphi$ is an isomorphism; thus $k \otimes_A B = 0$. Since A is Noetherian, B is finitely generated and the Nakayama Lemma now implies that $B = 0$.

Thus φ is an isomorphism. This shows in particular that M is free on the x_i – i.e. that the x_i are an A-basis for M.

Problem 2: Let A be a commutative ring and $P \subset A$ a prime ideal. Denote by A_P the *localization* of A at P; thus A_P is a local ring with unique maximal ideal $\mathfrak{m} = P^e = P \cdot A_P$.

a. Prove that $A_P / P \cdot A_P$ is isomorphic to the field of fractions of the integral domain A/P.

b. If P is a maximal ideal of A conclude that the fields A/P and $A_P / P \cdot A_P$ are isomorphic.

Solution for 2:
Set $S = A \setminus P$ and recall that $A_P = A[S^{-1}]$.

For a., write K for the field of fractions of A/P and consider the composite mapping

$$\pi : A \to A/P \to K.$$

Then π maps elements of S to non-zero elements of K. Since K is a field, there is a ring homomorphism $\iota : A_P \to K$ with $\iota(a/s) = \pi(a)\pi(s)^{-1}$.

Now, every element of K is a ratio of elements α/β of A/P. If $\beta \in A/P$ is non-zero, then $\alpha = s + P$ for $s \in S$. This shows that the mapping ι is surjective.

It only remains to show that the kernel of ι is the precisely the ideal PA_P. Certainly PA_P is contained in the kernel of ι since for $p \in P$, $\pi(p) = 0$ so that $\iota(p/1) = 0$.

On the other hand, suppose that $a/s \in \ker \iota$ for $a \in A$ and $s \in S$. Then evidently $\pi(a) = 0$ so that $a \in P$.

Now b. is an essentially immediate consequence of a. Indeed, if P is maximal then A/P is a field, and hence A/P coincides with its field of fractions.

Problem 3: Let A be a PID, $n \in \mathbb{N}$, let F be a free A-module of finite rank and let $\Phi \in \operatorname{End}_A(F)$ be an A-linear endomorphism.

For a prime $p \in A$, write A_{pA} for the localization of A at pA. Note that Φ determines an A_{pA}-homomorphism

$$\operatorname{id} \otimes \Phi : A_{pA} \otimes_A F \to A_{pA} \otimes_A F.$$

a. Fix an A-basis $\mathcal{B}$ for F. For $v \in M$, we write

$$v = \sum_{b \in \mathcal{B}} \alpha_b b \text{ for a function } \alpha : \mathcal{B} \to A$$

and we write $[v]_{\mathcal{B}} = \alpha \in A^{\mathcal{B}}$, where $A^{\mathcal{B}}$ is the module of all functions $\mathcal{B} \to A$ (recall that $|\mathcal{B}| = n < \infty$!)

Thus $v \mapsto [v]_{\mathcal{B}} : F \to A^{\mathcal{B}}$ is an isomorphism of A-modules where $A^{\mathcal{B}}$ is the module of

Show that there is a matrix $M \in \operatorname{Mat}_{\mathcal{B} \times \mathcal{B}}(A)$ for which

$$[\Phi v]_{\mathcal{B}} = M \cdot [v]_{\mathcal{B}} \text{ for } v \in F.$$

b. Let $d = \det(\Phi)$. Prove that $\operatorname{id} \otimes \Phi$ is an isomorphism – i.e. is an *automorphism* of $A_{pA} \otimes_A F$ – if and only if $\gcd(d, p) = 1$.

In particular if $d \neq 0$, then $\operatorname{id} \otimes \Phi$ is an automorphism for all but finitely many primes $p \in A$.

Solution for 3a:
We prove that when F is free of finite rank, an A-linear map $\Phi : F \to F$ is given by multiplication with a matrix in $\mathrm{Mat}_n(A)$.

If $\mathcal{B} = \{b_1, ..., b_n\}$ we write

$$\Phi(b_i) = \sum_j a_{ji} b_j \text{ for } a_{ji} \in A.$$

Write M for the matrix (a_{ji}). Let $v = \sum_i v_i b_i$ and let

$$[v]_{\mathcal{B}} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

Then a calculation shows that $\varphi(v) = w$ if and only if $M \cdot [v]_{\mathcal{B}} = [w]_{\mathcal{B}}$ for $v, w \in F$.

Solution for 3b:
Let $d = \det(\Phi)$ by which we mean $\det(M)$ for the matrix of Φ with respect to some basis $\mathcal{B}$ of F. Since a different choice of basis $\mathcal{B}'$ leads to a matrix $M' = PM'P^{-1}$ where P is an invertible matrix with A-coefficients.

Now, we prove that $\mathrm{id} \otimes \Phi$ is an automorphism of $A_{pA} \otimes_A F$ if and only if $\gcd(d, p) = 1$.

Observe that $A_{pA} \otimes_A F$ is a free A_{pA}-module, and that $\det(\mathrm{id} \otimes \Phi)$ coincides with the image in A_{pA} of $\det(\Phi) \in A$.

For any commutative ring B and a free B-module E of finite rank, an endomorphism $\Psi : E \to E$ is invertible if and only if $\det(\Psi) \in B^\times$.

Thus we see that $\mathrm{id} \otimes \Phi$ is invertible if and only if $d = \det(\Phi)$ is a unit in A_{pA}. Since $d \in A$, this holds if and only if $\gcd(d, p) = 1$.

Problem 4: Let A be a commutative ring and let M, N, P be A-modules. Prove that there is an isomorphism

$$M \otimes_A (N \oplus P) \simeq (M \otimes_A N) \oplus (M \otimes_A P).$$

Solution for 4:
Consider the mapping

$$F : M \times (N \oplus P) \to (M \otimes_A N) \oplus (M \otimes_A P)$$

given by the rule

$$F(m, (n, p)) = (m \otimes n, m \otimes p) \text{ for } m \in M, n \in N, p \in P.$$

It is straightforward to confirm that F is bilinear; thus F induces a homomorphism of A-modules

$$\Phi : M \otimes_A (N \oplus P) \to (M \otimes_A N) \oplus (M \otimes_A P)$$

with the property that

$$\Phi(m \otimes (n, p)) = (m \otimes n, m \otimes p).$$

On the other hand, there are homomorphisms of A-modules

$$\Psi_1 : M \otimes_A N \to M \otimes_A (N \oplus P) \text{ given by } m \otimes n \mapsto m \otimes (n, 0), \text{and}$$
$$\Psi_2 : M \otimes_A P \to M \otimes_A (N \oplus P) \text{ given by } m \otimes p \mapsto m \otimes (0, p);$$

using the mapping property of the direct sum, these homomorphisms induce an A-module mapping

$$\Psi : (M \otimes_A N) \oplus (M \otimes_A P) \to M \otimes_A (N \oplus P)$$

given by

$$\begin{aligned}\Psi(m_1 \otimes n, m_2 \otimes p) &= \Psi_1(m_1 \otimes n) + \Psi_2(m_2 \otimes p) \\ &= m_1 \otimes (n, 0) + m_2 \otimes (0, p).\end{aligned}$$

We claim that Φ and Ψ are inverses.

For $m \in M, n \in N, p \in P$ we note that

$$\Psi(\Phi(m \otimes (n, p))) = \Psi(m \otimes n, m \otimes p) = m \otimes (n, 0) + m \otimes (0, p) = m \otimes (n, p).$$

Since pure tensors generate $M \otimes_A (N \oplus P)$, it follows that $\Psi \circ \Phi = \mathrm{id}$.

On the other hand, for $m_1, m_2 \in M$, $n \in N$ and $p \in P$,

$$\begin{aligned}\Phi(\Psi(m_1 \otimes n, m_2 \otimes p)) &= \Phi(m_1 \otimes (n, 0) + m_2 \otimes (0, p)) \\ &= \Phi(m_1 \otimes (n, 0)) + \Phi(m_2 \otimes (0, p)) \\ &= (m_1 \otimes n, 0) + (0, m_2 \otimes p) = (m_1 \otimes n, m_2 \otimes p).\end{aligned}$$

Since $(M \otimes_A N) \oplus (M \otimes_A P)$ is generated by elements of the form $(m_1 \otimes n, m_2 \otimes p)$, it follows that $\Phi \circ \Psi = \mathrm{id}$. This proves that Φ and Ψ are isomorphisms, as required.

Problem 5: Let A be a PID and let $p, q \in A$. Write $d = \gcd(p, q) \in A$ for a greatest common divisor.

Prove that:

$$(A/pA) \otimes_A (A/q) \simeq A/dA.$$

Solution for 5:
Write $M = (A/pA) \otimes_A (A/qA)$ and notice that M is generated as A-module by the element $1 \otimes 1$.

Since M is generated by a single element, we know that $M \simeq A/fA$ for some $f \in A$; our task it to determine f.

Since d is a generator for the ideal $\langle p, q \rangle$, we may write

$$d = up + vq \text{ for } u, v \in A.$$

We claim that is annihilated by d. Well,

$$d.1 \otimes 1 = (up + vq)1 \otimes 1 = up \otimes 1 + vq \otimes 1 = up \otimes 1 + v1 \otimes q = 0 + 0 = 0;$$

this proves our claim. Since $M \simeq A/fA$ is annihilated by d, it follows that $f \mid d$.

On the other hand, $d \mid p \Rightarrow p \cdot A \subseteq d \cdot A$ and $d \mid q \Rightarrow q \cdot A \subseteq d \cdot A$, so there are surjective ring homomorphisms

$$\pi_1 : A/pA \to A/dA \text{ and } \pi_2 : A/qA \to A/dA$$

given by the rules $\pi_1(a + pA) = a + dA$ and $\pi_2(a + qA) = a + dA$.

The mapping

$$A/pA \times A/qA \to A/dA$$

given by $(\alpha, \beta) \mapsto \pi_1(\alpha) \cdot \pi_2(\beta)$ is bilinear and so induces an A-module mapping

$$\Phi : (A/pA) \otimes_A (A/qA) \to A/dA \text{ such that } \Phi(\alpha \otimes \beta) = \pi_1(\alpha) \cdot \pi_2(\beta).$$

Since π_1 is surjective and since $\Phi(\alpha \otimes 1) = \pi_1(\alpha)$, it is clear that Φ is surjective as well.

Thus, we have an surjective mapping $M \simeq A/fA \to A/dA$. This shows that f annihilates A/dA so that $f \in dA$; i.e. $d \mid f$.

Since $d \mid f$ and $f \mid d$, we conclude that d and f are associates and hence that $M \simeq A/dA$ as required.

Problem 6: Let F be a field and let V, W be a finite dimensional vector spaces over F.

Recall that the dual space V^* is the vector space $\operatorname{Hom}_F(V, F)$.

a. Show that $\dim_F V = \dim_F V^*$. **Hint:** exhibit a basis for V^*.

b. Show that there is an isomorphism

$$V^* \otimes_A W \xrightarrow{\sim} \operatorname{Hom}_F(V, W).$$

Hint: Use the mapping property of $\otimes$ to define the indicated map. Basis considerations show that this map is surjective. Now compare the dimension of the domain and co-domain.

Solution for 6a:

a. Suppose that $e_1, e_2, ..., e_d$ is an F-basis for V, and for $1 \leq j \leq d$ define a linear map $\varphi_j : V \to F$ by the rule $\varphi_j(e_i) = \delta_{i,j}$, the Kronecker delta.

Then $\varphi_j \in V^*$ and we claim that $\varphi_1, ..., \varphi_d$ is an F-basis for V^*.

For any $\varphi \in V^*$ and $1 \leq i \leq d$, write $\varphi(e_i) = a_i \in F$. We claim that

$$\varphi = \sum_{i=1}^{d} a_i \varphi_i;$$

to confirm this identity it suffices to check that for each $1 \leq j \leq d$ that

$$\varphi(e_j) = \left(\sum_{i=1}^{d} a_i \varphi_i\right)(e_j),$$

and in turn this identity is easily verified using the definition of the φ_i.

This proves that the φ_i span V^*.

To see that $\varphi_1, \varphi_2, ..., \varphi_d$ are linearly independent, suppose that

$$0 = \sum_{i=1}^{d} a_i \varphi_i \text{ for } a_i \in F.$$

For each j it follows that

$$0 = \left(\sum_{i=1}^{d} a_i \varphi_i\right)(e_j) = \sum_{i=1}^{d} a_i \varphi_i(e_j) = a_j$$

and this confirms the linear independence.

Since $\varphi_1, ..., \varphi_d$ is a basis for V^*, we see that $\dim_F V^* = d$ as required.

Remark THe basis $\varphi_1, ..., \varphi_d$ is called the *dual basis* of the basis $e_1, ..., e_d$.

Solution for 6b:
Consider the mapping

$$F : V^* \times W \to \mathrm{Hom}_F(V, W) \text{ given by } F(\varphi, w) = (v \mapsto \varphi(v) \cdot w).$$

It is is straightforward to confirm that F is bilinear; thus F induces a linear mapping

$$\Phi : V^* \otimes_F W \to \mathrm{Hom}_F(V, W) \text{ satisfying } \Phi(\varphi \otimes w) = (v \mapsto \varphi(v) \cdot w).$$

Now, we saw in part a that $\dim V^* = \dim V$. And we know that

$$\dim(V^* \otimes_F W) = (\dim V^*) \cdot (\dim W).$$

Write $d = \dim V$ and $e = \dim W$. On the other hand, we know that $\mathrm{Hom}_F(V, W)$ may be identified with the space $\mathrm{Mat}_{e\times d}(F)$ of $e \times d$ matrices; thus $\dim \mathrm{Hom}_F(V, W) = e \cdot d$.

Finally, we claim that Φ is surjective. Fix a basis

$$\mathcal{B} = \{e_1, ..., e_d\} \text{ of } V \text{ and } \mathcal{C} = \{f_1, ..., f_e\} \text{ of } W,$$

and let $\varphi_1, ..., \varphi_d \in V^*$ be the basis dual to the e_i. Then the (rank 1) linear maps $\Psi_{i,j} = \left(v \mapsto \varphi_i(v) \cdot f_j\right)$ form a basis for $\mathrm{Hom}_F(V, W)$.

(Note that the matrix $\left[\Psi_{i,j}\right]_{\mathcal{B},\mathcal{C}}$ is the *"matrix unit"* for which the (i, j)-entry is equal to 1 and all other entries are equal to 0).

Now note that $\Phi\left(\varphi_i \otimes f_j\right) = \Psi_{i,j}$ so that the image of Φ contains a basis of $\mathrm{Hom}_F \mathrm{Hom}(V, W)$; thus Φ is surjective.

Since Φ is a surjective linear map between finite dimensional vector spaces of the same dimension, Φ is an isomorphism.

Let A be a commutative ring. let M, M', N, N' be A-modules and let $\varphi : M \to N$ and $\varphi' : M \to N'$ be A-module homomorphisms. There is a unique homomorphism of A-modules

$$\varphi \otimes \varphi' : M \otimes_A N \to M' \otimes_A N'$$

such that

$$(\varphi \otimes \varphi')(m \otimes n) = \varphi(m) \otimes \varphi'(n).$$

Problem 7: Let F be a field and let $\varphi : V \to W$ be a homomorphism of F-vector spaces (a "linear transformation") and let X be an F-vector space.

If φ is injective, prove that $\mathrm{id}_X \otimes \varphi : X \otimes_F V \to X \otimes_F W$ is injective.

Remark: This shows that the functor $X \otimes_F -$ is *exact* for a field F; indeed, combine the preceding observation with the result proved in class that the functor $Y \otimes_A -$ is always right exact. In general, an A-module Y is said to be flat if $Y \otimes_A -$ is exact.

Solution for 7:
Identify V with its image under the injective mapping φ. After this identification, we view V as a vector subspace of W and φ is just the inclusion mapping $V \subseteq W$.

Let $\mathcal{B}$ be a basis of V. Important results of linear algebra imply that we may find a basis $\mathcal{C}$ of W containing $\mathcal{B}$.

Then $W = V \oplus V'$ is the internal direct sum, where V' is the span of $\mathcal{C} \setminus \mathcal{B}$. Let $\pi : W \to V$ be the linear mapping given by projection on the V factor; thus $\ker \pi = V'$ and $\pi \circ \varphi = \mathrm{id}_V$

Now consider

$$\mathrm{id} \otimes \varphi : X \otimes_F V \to X \otimes_F W.$$

We note that $\mathrm{id} \otimes \pi \circ \mathrm{id} \otimes \varphi = \mathrm{id} \otimes (\pi \circ \varphi) = \mathrm{id}_{X \otimes_A V}$. Thus $\mathrm{id} \otimes \varphi$ has a left inverse and in particular, $\mathrm{id} \otimes \varphi$ is injective, as required.

Problem 8: Let A be a commutative ring and let M be an A-module. If F is a free A-module on $\beta : \mathcal{B} \to F$, prove that $F \otimes_A M$ is isomorphic to $\bigoplus_{b \in \mathcal{B}} M$, a direct sum of copies of M index by $\mathcal{B}$.

Solution for 8:
For $b \in \mathcal{B}$, consider the linear mapping $\iota_b : M \to F \otimes_A M$ given by $m \mapsto b \otimes m$.

We claim that $(F \otimes_A M)$ together with the mappings ι_b for $b \in \mathcal{B}$ is direct sum (coproduct) of the family of modules $M = M_b$ for $b \in \mathcal{B}$ in the category $\operatorname{mod}(A)$.

To confirm this, let X be any A-module and let $\varphi_b : M \to X$ be an A-module homomorphism for each $b \in \mathcal{B}$. We must show that there is a unique A-module map $\varphi : F \otimes_A M \to X$ such that $\varphi \circ \iota_b = \varphi_b$ for each $b \in \mathcal{B}$.

We first note that if φ exists, it is unique. Indeed, suppose that such a mapping φ exists. We first observe that $F \otimes_A M$ is generated as an A-module by the elements $b \otimes m$ for $b \in \mathcal{B}$ and $m \in M$. Now notice that

$$\varphi(b \otimes m) = \varphi(\iota_b(m)) = \varphi_b(m)$$

so that $\varphi(b \otimes m)$ is completely determined by the assumption $\varphi \circ \iota_b = \varphi_b$.

It remains to prove existence of an A-module map φ with the required properties.

Since $\mathcal{B}$ is a basis for F, there is a unique bilinear mapping

$$\beta : F \times M \to X \text{ given by } \beta(b, m) = \varphi_{b(m)}.$$

By the universal mapping property of the tensor producct, this bilinear mapping induces an A -module homomorphism

$$\varphi : F \otimes_A M \to X \text{ such that } \varphi(b \otimes m) = \varphi_{b(m)},$$

as required.

Thus indeed $F \otimes_A M$ is isomorphic to $\bigoplus_{b \in \mathcal{B}} M$.