

Math 65 Final Exam

Name: _____

Problem 1. (12 points)

Give careful answers to the following:

- (a). Let X and Y be sets, and suppose that $f : X \rightarrow Y$ is a function. What does it mean to say that f is *injective*?
- (b). If X is a set and $\sim$ a relation on X , what does it mean to say that $\sim$ is an *equivalence relation*?
- (c). Consider a sample space (E, P) and events A and B . Give a formula for the *conditional probability* $P(A|B)$. Your formula could involve $P(A)$, $P(B)$, and $P(A \cap B)$.
- (d). What does it mean to say that a set X is *uncountably infinite*?

Solution for 1:

- (a). f is injective if whenever $x, x' \in X$ and $f(x) = f(x')$ then $x = x'$.
- (b). $\sim$ is an equivalence relation if it is reflexive, symmetric, and transitive.
- (c). $P(A|B) = P(A \cap B)/P(B)$.
- (d). X is uncountably infinite if X is not finite – i.e. is not in bijection with $I_n = \{1, 2, \dots, n\}$ for any $n \in \mathbb{N}$ – and is not countable – i.e. is not in bijection with the set $\mathbb{N}$ of natural numbers)

Problem 2. (4 points)

Let $\Gamma = (V, E, \iota)$ be an undirected graph. If $a, b \in V$ are vertices of Γ , a *path* in Γ from a to b is a sequence

$$v_0 = a, e_1, v_1, e_2, \dots, v_{n-1}, e_n, v_n = b$$

where for each i , $v_i \in V$, $e_i \in E$ and the endpoints of e_i are v_{i-1} and v_i – in symbols $\iota(e_i) = [v_{i-1}, v_i]$. Given this definition of a *path*, what does it mean to say that the graph Γ is *connected*?

Solution for 2:

Γ is connected if for all $a, b \in V$, there is a path in Γ from a to b .

Problem 3. (8 points)

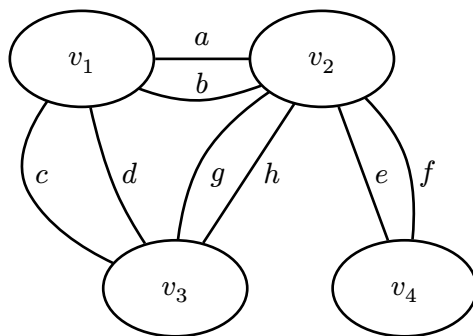
Indicate whether each of the following statements is True or False by circling the correct choice. (Here you do not need to write an argument justifying your choice).

- (a). **True / False** : If $\Gamma = (V, E, \iota)$ is an *undirected graph* and if $v \in V$ is a vertex of Γ , the *degree of v* is the number of edges incident to v , where a loop – i.e. an edge $e \in E$ with $\iota(e) = [v, v]$ – is counted twice.

- (b). **True / False** : If (E, P) is a sample space, and if A, B are events with $P(A) \neq 0$ and $P(B) \neq 0$, then *Bayes Theorem* states that $P(A|B) = P(B|A)$.
- (c). **True / False** : In contract bridge, a hand consists of 13 playing cards dealt from a deck of 52 playing cards. The number of possible bridge hands containing at least 2 Queens and at least 2 Jacks is given by the binomial coefficient

$$\binom{52-4}{13-4} = \binom{48}{9}.$$

- (d). **True / False** : The following graph has no Euler circuit:



Solution for 3:

- (a). True
(b). False
(c). False
(d). False

Problem 4. (9 points)

Let $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ be given by the formula $f(x, y) = (x \cdot y, y)$.

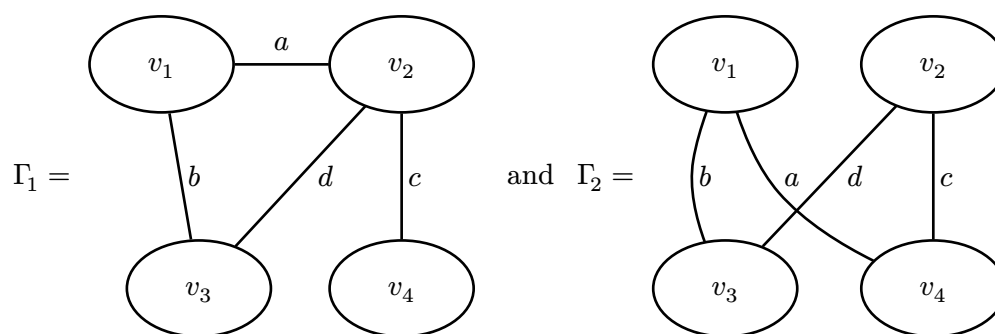
Decide whether f is *injective*. Justify your decision with a proof.

Solution for 4:

The function f is not injective. Indeed, $f(1, 0) = (0, 0) = f(0, 0)$, while $(1, 0) \neq (0, 0)$.

Problem 5. (8 points)

Consider the graphs Γ_1 and Γ_2 given as follows:



Prove or disprove: Γ_1 and Γ_2 are *isomorphic*.

Solution for 5:

The graphs are not isomorphic.

Indeed, the list of degrees in non-decreasing order of the vertices of Γ_1 is 1, 2, 2, 3 while that for the vertices of Γ_2 is 2, 2, 2, 2. If Γ_1 and Γ_2 were isomorphic, these lists would coincide, so there can be no isomorphism.

Problem 6. (8 points)

A gardener has seedlings of three different flowering plants: roses, tulips, and daisies. They want to plant some of the seedlings in a small garden bed, but there is only enough room for 10 seedlings.

If they insist on planting at least 4 tulips and at most 2 daisies, how many configurations are possible? You may suppose that the gardener has at least 10 of each type of seedling. And you may leave your answer expressed in terms of binomial coefficients.

Solution for 6:

Using the method of stars-and-bars, we see that the number of possible configurations of seedlings with at least 4 tulips is given by

$$\binom{6+2}{2} = \binom{8}{2} = 28.$$

The number of possible configurations of seedlings with at least 4 tulips and at least 3 daisies is given by

$$\binom{3+2}{2} = \binom{5}{2} = 10.$$

Thus, the number of configurations of seedlings with at least 4 tulips and at most 2 daisies is given by

$$\binom{8}{2} - \binom{5}{2} = 28 - 10 = 18.$$

Alternatively, you might instead count the following numbers of configurations:

- number of configurations with at least 4 tulips and no daisies: $\binom{7}{1} = 7$
- number of configurations with at least 4 tulips and 1 daisy: $\binom{6}{1} = 6$
- number of configurations with at least 4 tulips and 2 daisies: $\binom{5}{1} = 5$

Thus the number of configurations of seedlings with at least 4 tulips and at most 2 daisies is given by

$$7 + 6 + 5 = 18.$$

Problem 7. (9 points)

Prove or disprove: the function

$$f : \mathbb{Z}_5 \times \mathbb{Z}_5 \rightarrow \mathbb{Z}_5 \text{ given by } f([x], [y]) = [x \cdot y]$$

is well-defined.

Solution for 7:

Suppose that $([x], [y]), ([x'], [y']) \in \mathbb{Z}_5 \times \mathbb{Z}_5$ and that $([x], [y]) = ([x'], [y'])$. To see that f is well-defined, we must argue that

$$(\clubsuit) \quad f([x], [y]) = f([x'], [y']).$$

The condition $([x], [y]) = ([x'], [y'])$ means that $x \equiv x' \pmod{5}$ and $y \equiv y' \pmod{5}$; i.e. there are integers $k, \ell \in \mathbb{Z}$ for which $x' = x + 5k$ and $y' = y + 5\ell$.

Now we see that

$$f([x'], [y']) = [x' \cdot y'] = [(x + 5k) \cdot (y + 5\ell)]$$

while

$$f([x], [y]) = [x \cdot y].$$

Thus, to prove $(\clubsuit)$ we must argue that

$$[(x + 5k) \cdot (y + 5\ell)] = [x \cdot y];$$

i.e. we must argue that

$$(x + 5k) \cdot (y + 5\ell) \equiv x \cdot y \pmod{5}.$$

But

$$\begin{aligned} (x + 5k) \cdot (y + 5\ell) - x \cdot y &= (x \cdot y + 5\ell x + 5ky + 5^2 k\ell) - x \cdot y \\ &= 5\ell x + 5ky + 5^2 k\ell \\ &= 5(\ell x + ky + 5k\ell) \end{aligned}$$

which proves that $5 \mid (x + 5k) \cdot (y + 5\ell) - x \cdot y$. Thus, indeed

$$(x + 5k) \cdot (y + 5\ell) \equiv x \cdot y \pmod{5}.$$

This completes the proof.

Problem 8. (9 points)

Suppose that a_n is a sequence of real numbers and that $\lim_{n \rightarrow \infty} a_n = 1$. Using the *definition* of convergence and the *definition* of a *Cauchy sequence*, prove that a_n is a Cauchy sequence.

Solution for 8:

To prove that a_n is Cauchy, let $\varepsilon > 0$. We must find M so that

$$(*) \quad n_1, n_2 \geq M \Rightarrow |a_{n_1} - a_{n_2}| < \varepsilon.$$

To prove this, we use the convergence assumption. Since a_n converges to 1, we may find $N \in \mathbb{N}$ so that

$$n \geq N \Rightarrow |a_n - 1| < \varepsilon/2.$$

We now take $M = N$ and confirm (*).

So, let $n_1, n_2 \geq M$. Since $n_1, n_2 \geq N = M$, we have

$$|a_{n_1} - 1| < \varepsilon/2 \text{ and } |a_{n_2} - 1| < \varepsilon/2.$$

Using the triangle inequality, we find that

$$\begin{aligned} |a_{n_1} - a_{n_2}| &= |a_{n_1} - 1 + 1 - a_{n_2}| \\ &\leq |a_{n_1} - 1| + |1 - a_{n_2}| \\ &= |a_{n_1} - 1| + |a_{n_2} - 1| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

which confirms (*). Thus a_n is indeed Cauchy.

Problem 9. (8 points)

Let $n \in \mathbb{N}$. Prove by induction on n that $(1 + 7)^n \equiv 1 + 7n \pmod{49}$.

Solution for 9:

We treat the base-case first: let $n = 0$. We must argue that

$$(1 + 7)^0 \equiv 1 + 7 \cdot 0 \pmod{49}.$$

But $(1 + 7)^0 = 1$ and $1 + 7 \cdot 0 = 1$ so the required expressions are equal, hence in particular are congruent mod 49.

We now suppose that $n \in \mathbb{N}$ and that the induction hypothesis

$$(*) \quad (1 + 7)^n \equiv 1 + 7 \cdot n \pmod{49}$$

holds.

We must prove that

$$(1 + 7)^{n+1} \equiv 1 + 7 \cdot (n + 1) \pmod{49}.$$

Well, the induction hypothesis means that $(1 + 7)^n = 1 + 7 \cdot n + 49k$ for some $k \in \mathbb{Z}$. Thus

$$\begin{aligned} (1 + 7)^{n+1} - 1 - 7 \cdot (n + 1) &= (1 + 7) \cdot (1 + 7)^n - 1 - 7 \cdot n - 7 \\ &= (1 + 7) \cdot (1 + 7 \cdot n + 7^2 \cdot k) - 1 - 7 \cdot n - 7 \\ &= 1 + 7 \cdot n + 7^2 \cdot k + 7 + 7^2 \cdot n + 7^3 \cdot k - 1 - 7 \cdot n - 7 \\ &= 7^2 k + 7^2 n + 7^3 k \\ &= 7^2 \cdot (8k + n). \end{aligned}$$

This confirms that

$$(1 + 7)^{n+1} \equiv 1 + 7 \cdot (n + 1) \pmod{7^2}$$

as required. Now it follows by induction that

$$(1 + 7)^n \equiv 1 + 7 \cdot n \pmod{7^2}$$

for every $n \in \mathbb{N}$.