Problem Set week 11

Stars and Bars is a method for deriving various combinatorial formulas. For example, it can be used to answer the question: if $k \leq m$, given m (indistinguishable) objects, how many ways can they be placed in k (distinguishable) bins?

Let's suppose for example that $m = 4$ and $k = 3$. Then we can separate the 4 objects into 3 bins by placing boundaries between them. Thus, we count all arrangements of $m = 4$ (indistinguishable) stars, and $k - 1 = 2$ bars. For example here are 4 such arrangements:

$$\star \star \star \star \mid \mid \quad , \quad \star \star \star \mid \star \mid \quad , \quad \star \mid \star \star \mid \star \quad , \quad \mid \star \star \star \mid \star$$

which correspond respectively to:

$$\text{all in 1st bin} \;\; ; \;\; \text{3 in 1st bin, 1 in 2nd} \;\; ; \;\; \text{1 in 1st, 2 in 2nd, 1 in 3rd} \;\; ; \;\; \text{3 in 2nd, 1 in 3rd}$$

The number of such lists is given by the formula $\binom{n+k-1}{k-1}$. To see why, note that any such list is a string of $n + k - 1$ characters, and is completely determined by the positions of the bars; these positions are determined by choosing $k - 1$ numbers from the list $\{1, 2, ..., n + k - 1\}$. For example, the 4 lists above are determined by the following subsets of $\{1, 2, 3, 4, 5, 6\}$:

$$\{5, 6\} \quad , \quad \{4, 6\} \quad , \quad \{2, 5\} \quad , \quad \{1, 5\}$$

Thus there are $\binom{4+3-1}{3-1} = \binom{6}{2} = 15$ ways of placing 4 objects into 3 bins.

Problem 1: Suppose that there are four donuts available at a certain bakery: Boston cream, pumpkin spice, glazed, and chocolate.

Boxes containing 6 donuts are sold.

An industrious group of bakers has packed boxes will all possible assortments and created a stack of boxes for each possible assortment. Thus there is stack for "6 Boston creams", another stack for "4 Boston cream, 1 glazed, 1 chocolate", etc.

a. Imagine choosing a box of donuts from a random stack.

   Do you expect the events "the box contains at least 2 chocolate donuts" and "the box contains at most 1 pumpkin spice donut" are independent?

   Don't make any calculations yet, just consider the problem (or discuss with colleagues in the class!) We are going to calculate in the remainder of the problem!

b. Use the method of stars-and-bars to determine the number of stacks of donut boxes. (Think of donut style – Boston cream, glazed, etc. – as the bins.)

c. Use the method of stars-and-bars to determine the number of stacks for which each box contains at least two chocolate donuts.

d. What is the probability that a box from a randomly chosen stack contains at least two chocolate donuts?

e. What is the probability that a box from a randomly chosen stack contains at most one pumpkin spice donut?

f. What is the probability that a box from a randomly chosen stack contains at least two chocolate donuts and at most one pumkin spice donut?

g. If a box from a randomly chosen stack has at least two chocolate donuts, what is the probability that it has at most 1 pumpkin spice donut. (This is the *conditional probability*).

h. After these calculations, answer the question: are the events "at least 2 are chocolate" and "at most 1 is pumpkin spice" independent?

Problem 2: Two events in a finite probability space are said to be *independent* if $P(A|B) = P(A)$. Prove or disprove:

a. If A and B are independent, then A and $B^c = S \setminus B$ are independent.

b. If $P(A \mid B) = P(B \mid A)$, then A and B are independent.

c. If A and B are independent, then $P(A \mid B) = P(B \mid A)$.

Problem 3: In the contract bridge card game, a player is dealt a 13-card hand from a 52-card deck. There are 13 kinds of cards: Ace, King, Queen, Jack, 10, 9, … 2, and 4 suits: clubs (♣), diamons (♦), hearts (♥), spades (♥). Each suit contains exactly one card of each kind.

a. What is the probability of being dealt a hand with no Jack, Queen, or King?
b. What is the probability of being dealt a hand with all 4 kings and exactly 3 queens.

Problem 4: You hold a bag of ten coins. Nine of them are fair, but one is loaded - it shows heads with probability $9/10$. You draw out a coin at random and begin flipping it. The first five tosses are $HHHTH$.

Let's write A for the event: "the chosen coin is fair" and B for the event "the outcomes of five consecutive coin tosses are $HHHTH$."

a. What is the probability $P(A)$? (This probability is only concerned with selecting a coin from the bag; it is unrelated to the outcome of the coin tosses.)
b. What is the probability $P(B \mid A)$?

   (Notice for this computation that you don't have to view this as a conditional probability; this is just the probability of getting "$HHHTH$" from a fair coin!)
c. Find $P(A \mid B)$. What is the probability that you are flipping a fair coin?