Problem Set 1 Answer Key

Tufts University
Fall 2025
Math 065
Prof. George McNinch

Due: September 8, 2025

Questions

1. **(Warm up; this question will not be graded.)**

   Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$. Determine the following sets:

   - $A \cup B = \{1, 2, 3, 4, 5\}$.
   - $A \cap B = \{3, 4\}$.
   - $A - B = \{1, 2\}$.
   - $B - A = \{5\}$.
   - $A \times B = \left\{ \begin{array}{ccc} (1,3) & (1,4) & (1,5) \\ (2,3) & (2,4) & (2,5) \\ (3,3) & (3,4) & (3,5) \\ (4,3) & (4,4) & (4,5) \end{array} \right\}$.

2. **For a natural number $n > 0$, consider the following intervals in the real line:**

$$I_n = \left(-\frac{1}{n}, \frac{1}{n}\right) = \{x \in \mathbb{R} \mid -\frac{1}{n} < x < \frac{1}{n}\}$$

   and

$$J_n = \left(n - \frac{1}{2}, n + \frac{1}{2}\right] = \{x \in \mathbb{R} \mid n - \frac{1}{2} < x \leq n + \frac{1}{2}\}.$$

   - **Find $\bigcap_{n=1}^{\infty} I_n$.**

     First note that $0 \in I_n$ for all $n \geq 1$, so $0 \in \bigcap\limits_{n=1}^{\infty} I_n$.

     For all $x \neq 0$, there exists n such that $|x| > \frac{1}{n}$, and so $x \notin I_n$. Thus, $x \notin \bigcap\limits_{n=1}^{\infty} I_n$.

     Therefore, $\bigcap\limits_{n=1}^{\infty} I_n = \{0\}$.

   - **Find $\bigcup_{n=1}^{\infty} J_n$.**

     First note that for all $n \geq 1$ and all $x \in J_n$, $x > \frac{1}{2}$. Thus, $\frac{1}{2}$ is a lower bound on $\bigcup\limits_{n=1}^{\infty} J_n$.

     Now consider any $x > \frac{1}{2}$. There exists $n \geq 1$ such that $n - \frac{1}{2} < x \leq n + \frac{1}{2}$. Thus, $x \in J_n$, and so $x \in \bigcup\limits_{n=1}^{\infty} J_n$.

     Therefore, $\bigcup\limits_{n=1}^{\infty} J_n = (\frac{1}{2}, \infty)$.

3. **Prove or disprove: if U is a set with subsets $A, B \subseteq U$, then $A \cup B = A \cap B$ if and only if $A = B$.**

   Note: To disprove the statement, you need to provide explicitly two subsets A, B such that $A \cup B = A \cap B$ and $A \neq B$.

   To prove the statement, you need to show two things: first, you must argue that if $A = B$, then $A \cup B = A \cap B$; second, you must argue that if $A \cup B = A \cap B$, then $A = B$.

   ($\Rightarrow$) Let $A \cup B = A \cap B$. Then for all $x \in A$, $x \in A \cup B = A \cap B$, which implies $x \in B$. Thus $A \subseteq B$. The same argument for all $x \in B$ shows $B \subseteq A$. Therefore, $A = B$.

   ($\Leftarrow$) Let $A = B$. Then
   $$A \cup B = A \cup A = A = A \cap A = A \cap B.$$
   Thus, $A \cup B = A \cap B$.

   $\blacksquare$

4. **Let**
   $$A = \{x \in \mathbb{R} \mid x^3 + 2x^2 - 3x \geq 0\} \quad \textbf{and} \quad B = \{x \in \mathbb{R} \mid 2 - |x| < 0\}.$$

   - **Describe A as a union of intervals on the real line.**

     Note $f(x) = x^3 + 2x^2 - 3x = x(x+3)(x-1)$ has roots at $x = -3, 0,$ and 1. Now we must test a value in each of these intervals to determine the sign of $f(x)$:

     $f(-4) = -64 + 32 + 12 < 0 \Rightarrow f(x) < 0$ for all $x < -3$.

     $f(-1) = -1 + 2 + 3 > 0 \Rightarrow f(x) \geq 0$ for all $x \in [-3, 0]$.

     $f(0.5) = 0.5^3 + 0.5 - 1.5 < 0 \Rightarrow f(x) < 0$ for all $x \in (0, 1)$.

     $f(2) = 8 + 8 - 6 > 0 \Rightarrow f(x) \geq 0$ for all $x \geq 1$.

     Thus, $f(x) \geq 0$ for all $x \in [-3, 0] \cup [1, \infty) = A$

   - **Describe B as a union of intervals on the real line.**

     $g(x) = 2 - |x|$ has roots at 2 and -2. $g(0) = 2 > 0$, but $g(\pm 3) = -1 < 0$.

     Thus, $g(x) < 0$ for all $x \in (-\infty, -2) \cup (2, \infty) = B$.

   - **Determine $A \cap B$. Write it as a union of disjoint intervals in the real line.**

     $$\begin{aligned} A \cap B &= \big([-3, 0] \cup [1, \infty)\big) \bigcap \big((-\infty, -2) \cup (2, \infty)\big) \\ &= \big([-3, 0] \cap (-\infty, -2)\big) \bigcup \big([1, \infty) \cap (2, \infty)\big) \\ &= [-3, -2) \cup (2, \infty) \end{aligned}$$

   - **Determine $A \cup B$. Write it as a union of disjoint intervals in the real line.**

     $$\begin{aligned} A \cup B &= \big([-3, 0] \cup [1, \infty)\big) \bigcup \big((-\infty, -2) \cup (2, \infty)\big) \\ &= [-3, 0] \cup (-\infty, -2) \bigcup [1, \infty) \cup (2, \infty) \\ &= (-\infty, 0] \cup [1, \infty) \end{aligned}$$