Levi factors of a linear algebraic group
(special session; New Orleans meeting)

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Levi factors: definitions

$k$: field, $G$: linear algebraic group /$k$.

- I’ll always require (R): the unipotent radical $R$ of $G_{/k_{	ext{alg}}}$ is defined /$k$ (this condition always holds for perfect $k$).
- Write $\pi : G \to G/R$ for the quotient map.
- A Levi factor of $G$ is subgp $M \subset G$ s.t. $\pi|_M : M \to G/R$ is an isom. of alg. gps.
- Thus $G \simeq R \rtimes M$.
- Implicit in this terminology: $M$ is “defined over $k$”.
Main result, in brief

Goal of the talk is to explain the following result for $G$.

- “Condition (L)” will be discussed in a subsequent slide.

**Theorem (M; “Descent of Levi factors”)**

> Assume that (L) holds for the unipotent radical $R$ of $G$. If $G/\kappa_{\text{sep}}$ has a Levi factor (“defined over $\kappa_{\text{sep}}$”), then already $G$ has a Levi factor (“defined over $k$”).
Levi factors can fail to exist

Here is a general construction via cohomology

- Let $G$ a linear algebraic group and $V$ a linear $G$-representation.
- Fix $\alpha \in Z^2(G, V)$, i.e. a regular cocycle $\alpha : G \times G \rightarrow V$.
- $\alpha$ determines an extension

\[
(*) \quad 0 \rightarrow V \rightarrow E_\alpha \rightarrow G \rightarrow 1
\]

- $(*)$ is split $\iff 0 = [\alpha] \in H^2(G, V)$. 
Levi factors can fail to exist

- for any semisimple algebraic group $G$, we have
  $$H^2(G, \text{Lie}(G)^{[1]}) \neq 0.$$  

- This leads to a non-split seq of linear alg. gps
  $$0 \rightarrow \text{Lie}(G)^{[1]} \rightarrow H \rightarrow G \rightarrow 1;$$

- thus $H$ has no Levi factor.
To describe an application of the main theorem on descent of Levi factors, I need to discuss parahoric group schemes. Consider the following data:

- $A$: a complete DVR with fractions $K$ and perfect residues $k = A/\pi A$.
- $G$: a connected and reductive group over $K$.
- $\mathcal{P}$: a parahoric group scheme for $G$.

Thus $\mathcal{P}$ is a smooth affine group scheme over $A$ with $\mathcal{P}/K = G$.

and the special fiber $\mathcal{P}/k$ is a linear algebraic group over $k$ which is not reductive in general.
Levi factors for parahoric group schemes

**Theorem (M)**

If $G/L$ is split for some unramified extension $K \subset L$, then the special fiber $\mathcal{P}/k$ has a Levi factor, and any two $k$-Levi factors are conjugate by an element of $\mathcal{P}(k_{\text{alg}})$.

Here is a fairly simple example

- $G = \text{GL}(n+1)/K = \text{GL}(V_{K,n+1}), \mathcal{L} \subset V_{K,n+1}$ $A$-lattice
- parahoric $\mathcal{P}$: stab. of lattice flag $\pi\mathcal{L} \subset \mathcal{M} \subset \mathcal{L}$
- assume $\dim_k \mathcal{L}/\mathcal{M} = 1$
- $\mathcal{P}/k$ is the trivial extension

\[
0 \to (V_{k,n} \boxtimes k_{-1}) \oplus (V_{k,n}^\vee \boxtimes k_1) \to \mathcal{P}/k \to \text{GL}(V_{k,n}) \times \mathbb{G}_m \to 1
\]
Assume $G$ absolutely quasisimple and simply connected. If $G/L$ is split for a tamely ramified extension $K \subset L$, then $P/k$ has a Levi factor.

The proof of this theorem (i.e. the tamely ramified case) depends on the theorem on “descent of Levi factors”.

Indeed, using a theorem of Rousseau and of G. Prasad, I first proved this result when the residue field $k$ is alg closed, following a suggestion of Prasad.

in general, conjugacy of Levi factors is false if $G$ doesn’t split over an unramif. extension of $K$. 
Consider extension

\[(*) \quad 0 \to V \to E \to G \to 1 \]

for linear alg group \( G \), where \( V \) is a linear rep of \( G \).

- for each field ext \( k \subset \ell \), consider the group \( A(\ell) \) of all automorphisms \( \phi \) of \( E/\ell \) s.t.
  - \( \phi|_{V/\ell} = 1 \), and
  - \( \phi \) induces 1 on \( G/\ell \)

- \( A(\ell) \) may be ident. with the (in general, infinite dim’l) \( \ell \)-vector space \( Z^1(G, V/\ell) \)
Idea of proof of “descent of Levi factors” theorem

thus \( H^1(k, A) = H^1(\text{Gal}(k_{\text{sep}}/k), A) \) is trivial, by additive version of Hilbert 90.

- suppose given an isom. defined over \( k_{\text{sep}} \) between the extension
  \[
  (*) \quad 0 \rightarrow V \rightarrow E \rightarrow G \rightarrow 1
  \]
  and the trivial extension (**) of \( G \)

- since the set of isomorphisms between the extensions (*) and (**) forms a torsor for \( A \), the vanishing of \( H^1(k, A) \) shows there is such an isom. already over \( k \).
Let $k_{\text{sep}}$ be a separable closure of $k$, and let $G$ be a linear algebraic group over $k$ for which \((R)\) holds. Consider the condition

\[(L)\text{ } R \text{ has a filtration by closed } k\text{-subgroups which are normal in } G \text{ for which the successive quotients are vector groups each with a linear action of } G/R.\]

- when $G$ is absolutely quasisimple and simply connected, condition \((L)\) holds for the special fiber $\mathcal{P}_{/k}$ by a result of Prasad-Raghunathan.

- \((L)\) not true in general. When $p = 3$, there is a 2 dimensional vector group $V$ with non-linear action of $M = \text{SL}_2$ for which $\text{Lie}(V)$ is a simple $M$-module.
Thanks for listening.

- pre-prints pending...
- some results found in “Levi decompositions of a linear algebraic group” on arXiv (to appear: Transf. Groups)
- I’ll post these notes at www.tufts.edu/~gmcnin01