Nilpotent orbits of a reductive group over a local field
(seminar talk at UMich)

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Overview

1. The landscape, quickly
2. Optimal $SL_2$-mappings
3. Nilpotent centralizers
4. Parahoric subgroups
Reductive groups: basic examples

$G$ connected, reductive group over the (arbitrary) field $K$.

Examples:

- $G = \text{GL}(V)$ for a $K$-vector space $V$
- $G = \text{Sp}(V)$ if $V$ has a non degenerate alternating form

I want to suppose $G$ is $D$-standard.

- if $G$ is semisimple, $D$-standard just means the char is “very good for $G$.
- Any $K$-form of $\text{GL}_n$ is also $D$-standard.
- $\text{Sp}(V)$ is $D$-standard just when $p \neq 2$.
- (the actual defn: let $H_1$ be semisimple in very good characteristic and $H = H_1 \times S$ for a $K$-torus $S$. Then $G$ is $D$-standard if it is separably isogenous to the centralizer in $H$ of a subgroup $M$ of multiplicative type for some $H_1, S$, and $M$).
Let $H$ be a linear algebraic group over $K$.

- Assume that $R_u H$ is defined over $K$.
- A Levi factor of $H$ (over $K$) is a $K$-subgroup $M \subset H$ for which $H$ is the semidirect product $R_u H \cdot M$.
- If $K$ has char. 0, then $H$ has a Levi factor (Mostow).
Levi factors: problems in positive char.

- In general, $R_u H$ need not be defined over $K$.
  - e.g. this fails for $H = R_{K_1/K}G_m$ when $K_1/K$ is purely inseparable.
- there may be reductive $M \subset H$ with $M(K_{alg}) \cdot R_u H(K_{alg}) = H(K_{alg})$ but for which $\text{Lie } M + \text{Lie } R_u H \neq \text{Lie } H$.
  - for an example, let $H$ a maximal parabolic of $GL_3$ in char. 2, and let $M$ be the image of the adjoint rep $SL_2 \to GL(\mathfrak{sl}_2) = GL_3$
- $H$ need not have a Levi factor.
  - $H = R_{K_1/K}G_m$ has no Levi factor over $K$
  - if $W_2 = \text{Witt vectors over alg. closed } k$, let $H = SL_2(W_2)$. $H$ is a 6 dimensional $k$-group, $R_u H$ is defined over $k$, and $H$ and has no Levi factor.
Let $G$ a $D$-standard reductive gp, and $X \in \mathfrak{g}(K)$ nilpotent.

- Let $S$ a maximal $K$-torus of $C = C_G(X)$.

- **Theorem (Premet, M)**
  
  There is a ($K$-)cochar. $\phi : G_m \to [C_G(S), C_G(S)]$ for which $X \in \mathfrak{g}(\phi; 2)$.
  
  If $S'$ is a second max torus of $C$, and if $\phi' : G_m \to [C_G(S'), C_G(S')]$ satisfies $X \in \mathfrak{g}(\phi'; 2)$, then $\phi$ and $\phi'$ are conjugate by a unique element of $R_u(C)(K)$.
  
  the parabolic subgroup $P = P(\phi)$ depends only on $X$.

- I’ll say that $\phi$ is **associated to** $X$.

- if $K$ has char. 0, the elements $X$ and $H = d\phi(1)$ may be completed to a unique $\mathfrak{sl}_2$-triple.
Now suppose the nilpotent $X \in \mathfrak{g}(K)$ satisfies $X^{[p]} = 0$.

- Fix a cocharacter $\lambda$ associated to $X$.

**Theorem (Seitz,M)**

*There is a unique homomorphism $\psi: \text{SL}_2 \to G$ for which*

$$\psi \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t-1} \end{pmatrix} = \phi(t) \quad \forall t \in K_{\text{alg}} \quad \text{and} \quad d\psi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X.$$ 

*Moreover, the representation $\text{Ad} \circ \psi$ of $\text{SL}_2$ on $\mathfrak{g}$ is a tilting module for which all weights $\mu$ satisfy $-2p + 2 \leq \mu \leq 2p - 2$.***
Good filtrations

Let $G$ be a reductive group, and suppose $K = K_{\text{alg}}$. Fix a maximal torus $T$ and a Borel subgroup $T \subset B$.

- The characters $\lambda \in X^*(T)$ parametrize the $G$-linearized line bundles $\mathcal{L}(\lambda)$ on $G/B$.
- $\lambda$ is dominant $\iff H^0(\lambda) = H^0(G/B, \mathcal{L}(\lambda)) \neq 0$.
- The simple $G$-modules are precisely the $L(\lambda) = \text{soc} \, H^0(\lambda)$ for dominant $\lambda$.
- A (finite dimensional) $G$-module $M$ is said to have a good filtration if there are submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ for which $M_i/M_{i-1} \simeq H^0(\lambda_i)$ for dominant weights $\lambda_i$. 
Tilting modules

- A $G$-module $M$ is said to be a \textit{tilting module} if both $M$ and $M^\vee$ have a good filtration.
- Any tilting mod $M$ is a $\bigoplus$ of \textit{indecomposable} tilting mods.
- The indec tilting modules $T(\lambda)$ are param by dominant $\lambda$.

\textbf{Theorem (Donkin, Mathieu)}

\textit{For all dominant weights $\lambda, \mu$, $H^0(\lambda) \otimes H^0(\mu)$ has a good filtration.}

- This essentially amounts to the assertion that there is a Frobenius splitting of $G/B \times G/B$ which “$B$-compatibly Frobenius splits” the diagonal.

\textbf{Corollary}

\textit{If $T_1$ and $T_2$ are tilting modules, so is $T_1 \otimes T_2$.}
Let $K$ have char. $p > 0$, Let $\lambda \in \mathbb{Z}$ with $0 \leq \lambda \leq 2p - 2$, and view $\lambda$ as a character of the standard max torus of $\text{SL}_2$.

- I’d like to describe the indec. tilting mod $T(\lambda)$.
- The standard module $H^0(\lambda)$ has dimension $\lambda + 1$ and coincides with $\text{Sym}^\lambda V$ where $V = K^2 = H^0(1)$ is natural rep.
- $H^0(\lambda) = L(\lambda)$ is simple $\iff \lambda < p$. In this case $T(\lambda) = L(\lambda)$. 
Now assume $\lambda = p + \mu$ for $0 \leq \mu \leq p - 2$.

Then $L(\lambda) \simeq V^{[1]} \otimes L(\mu) = L(1)^{[1]} \otimes L(\mu)$, where $[1]$ denotes a Frob. twist.

There is a non-split exact sequence of $SL_2$-modules

$$0 \to L(\lambda) \to H^0(\lambda) \to L(p - 2 - \mu) \to 0;$$

note that $L(p - 2 - \mu) = H^0(p - 2 - \mu)$.

There is a non-split exact sequence of $SL_2$-modules

$$0 \to H^0(p - 2 - \mu) \to T(\lambda) \to H^0(\lambda) \to 0.$$

In particular, $\dim T(\lambda) = (p - 2 - \mu + 1) + (\lambda + 1) = 2p$.

$T(\lambda)$ may be “constructed” as an indecomposable summand of $L(p - 1) \otimes L(\mu + 1)$. 
Tilting modules for SL$_2$, conclusion

Let $A$ a discrete valuation ring with fractions $K$ and residues $k = A/\pi A$.

- Let $L$ be a free $A$-mod of finite rank.
- Let $\rho : \text{SL}_2/A \to \text{GL}(L)$ be an $A$-representation – i.e. a morphism of $A$-group schemes.
- Write $X_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}_2(A)$, $X_0(k)$ the image in $\mathfrak{sl}_2(k)$ and $X_0(K)$ the image in $\mathfrak{sl}_2(K)$.
- Assume each weight $\lambda$ of the representation $L$ satisfies $-2p + 2 \leq \lambda \leq 2p - 2$.

**Proposition**

If $L/\pi L$ is a tilting module for $\text{SL}_2/k$, then $\dim_k \ker \rho(X_0(k))$ coincides with $\dim_K \ker \rho(X_0(K))$. 
Let $G$ be $D$-standard and $X \in \mathfrak{g}(K)$ nilpotent.

Recall $G$-orbits in nilp variety $\mathcal{N}$ are classified geometrically by Bala-Carter data $(L, Q)$: $L$ is the Levi subgroup of a parabolic of $G$, and $Q \subset L$ is a distinguished parabolic.

In particular, the geometric nilpotent orbits depend only on the root datum of $G$.

I want to comment on the structure of the centralizer $C = C_G(X)$.

If $\phi$ is a cocharacter associated with $X$, one knows that $M = C \cap C_G(\text{im } \phi)$ is a Levi factor of $C$ (over $K$).
The centralizer $C = C_G(X)$ has a Levi decomposition defined over $K$. Moreover, the following are independent of $p$ (under our standard hyp):

- the (geometric) root datum of a Levi factor of $C$,
- the (geometric) component group $C(K_{\text{alg}})/C^0(K_{\text{alg}})$

Method of proof: may suppose $K = K_{\text{alg}}$.

- let $\mathcal{A}$ be a DVR with residues $K$ and fractions of char 0.
- let $\mathcal{G}/\mathcal{A}$ be split reductive with root datum of $G$
- find a nilpotent section $X_1 \in \mathfrak{g}(\mathcal{A})$ specializing to $X$ for which the $\mathcal{A}$ group scheme $C_\mathcal{G}(X_1)$ is smooth over $\mathcal{A}$.
- Find the Levi factor “over $\mathcal{A}$”.
Motivating question(s)

Consider more general smooth group schemes over a DVR with reductive generic fiber, but which are not necessarily reductive. Can one:

- find nilpotent sections with smooth centralizer lifting classes on the special fiber?
- use such nilpotent sections to understand (something about) DeBacker’s parametrization of rational nilpotent orbits?
“Local” fields – notation etc.

- Let $\mathcal{A}$ be a discrete valuation ring with maximal ideal $\pi \mathcal{A}$; assume $\mathcal{A}$ is $\pi$-adically complete, assume $k = \mathcal{A}/\pi \mathcal{A}$ is \textit{perfect}, and let $K = \text{Frac}(\mathcal{A})$.

- Examples:
  - $K = k((\pi))$, $\mathcal{A} = k[[\pi]]$.
  - $K$ a finite extension of $\mathbb{Q}_p$, $\mathcal{A}$ int. closure of $\mathbb{Z}_p$ in $K$
Let $G$ be $D$-standard reductive over the “local” field $K$.

- Bruhat and Tits: parahoric “subgroups” – certain smooth $A$-group schemes $\mathcal{P}$ with $\mathcal{P}/K = G$
- determined by facets in the affine building of $G$.
- there is an $A$-split torus $S$ in $\mathcal{P}$ for which $S/K$ is a maximal $K$-split torus in $G$.

**Theorem (Bruhat-Tits, I think)**

*The special fiber $\mathcal{P}/k$ has a unique Levi factor containing $S/k$.*

- the case of $SU_3$ with $L/K$ totally ramified shows this not to be *too* trivial.
Let $\mathcal{P}$ a parahoric, write $\mathfrak{p} = \text{Lie}(\mathcal{P})$.

- Let $X \in \mathfrak{p}(A)$ nilpotent for which $X(k)$ lies in the Lie algebra of a Levi factor of $\mathcal{P}/k$.
- Up to conjugacy on the special fiber we may suppose that a cocharacter $\phi$ associated with $X(k) \in \mathfrak{p}$ (in a Levi factor) takes values in the fixed split maximal torus $S$ – thus $\phi$ may be viewed as an $A$-cocharacter of $\mathcal{P}$. 
Desired condition: Let $C \subset \mathfrak{p}$ an $\mathcal{A}$-submodule. Write $C/\mathbb{K} = C \otimes K$ and $C/\mathbb{k}$ for the image of $C$ in $\mathfrak{p}/\pi\mathfrak{p}$. Suppose that:

(C1) $C$ is stable under $\phi(\mathcal{A}^\times)$.
(C2) as an $\mathcal{A}$-module, $\mathfrak{p} = \mathfrak{p}(\mathcal{A})$ is the direct sum of $C$ and $[X, \mathfrak{p}(\mathcal{A})]$
(C3) $C/\mathbb{k} \cap \operatorname{Lie}(R_u \mathcal{P}/\mathbb{k})$ is a complement to $[X, \operatorname{Lie}(R_u \mathcal{P}/\mathbb{k})]$.

If there is $C$ satisfying (C1)–(C3), the centralizer $C_{\mathcal{P}}(X)$ is a smooth group scheme over $\mathcal{A}$.

If $\mathfrak{p} \gg 0$ one may use $C = \operatorname{Lie}(C_G(Y))$ determined by a suitable $\mathfrak{sl}_2$-triple $(X, H, Y)$ for which $H = d\phi(1)$. 

Write $p^+$ for the pre-image of $\text{Lie}(R_u P_{/k})$ under the mapping $p \rightarrow p_{/k} = \text{Lie}(P_{/k})$. Assume $C$ satisfies (C1)–(C3).

**Proposition (adaptation of DeBacker/Waldspurger)**

The $G(K)$-orbit of $X$ is the nilpotent orbit of minimal dimension having non-empty intersection with $X + p^+$.

The proof depends on viewing $P(\mathcal{A}/\pi^2 \mathcal{A})$ as the $k$-points of a linear group over $k = \mathcal{A}/\pi \mathcal{A}$ (à la Greenberg), and knowing that the centralizer of $X$ is *smooth*. 
A map in Galois cohomology, continued

**Corollary**

For $X$ as above, there is a natural mapping

$$H^1(k, C_{P/k}(X)) \to H^1(K, C_G(X)).$$

- Note the $H^1$ of $C_{P/k}(X)$ identifies with that of its reductive quotient.
- This natural mapping is *realized* by DeBacker’s mapping.
Let $\mathcal{P}$ be a parahoric of $G$, and let $M \subset \mathcal{P}/k$ be a Levi factor.

Let $X_0$ be a distinguished nilp element of $M$ s.t. $X_0^{[p]} = 0$.

Suppose that $M$ is $D$-standard (?!?), and let $\psi : \text{SL}_2/k \to M$ an optimal $\text{SL}_2$-mapping for $X_0$.

Hope

The representation $(\text{Ad} \circ \psi, \text{Lie}(\mathcal{P}/k))$ is a tilting module for $\text{SL}_2$ for which all weights $\mu$ satisfy $-2p + 2 \leq \mu \leq 2p - 2$.

Equivalently: the Lie algebra of the unipotent radical of $\mathcal{P}/k$ is a tilting module for $\text{SL}_2$ under the action determined by $\text{Ad} \circ \psi$ (with the indicated condition on the weights).
With notation as on the previous slide, assume that the hope holds for $X_0 \in \text{Lie}(M)(k)$.

**Proposition**

*Then there is an $A$-submodule $C \subset p$ for which (C1)–(C3) hold.*

An important point is the following: if $\phi$ is a cocharacter of $M$ associated with $X_0$, and if (as before) we arrange that $\phi$ is “defined over $A$”, one needs to know that $\phi$ is associated with $X$ for some nilpotent element $X \in p$ whose image in $p/k$ is $X_0$. 
The hope holds for $\text{GL}_n$.

It also holds for $G = \text{Sp}(V)$.

- Indeed, it is clear for the reductive parahoric.
- If $\mathcal{P}$ is a non-reductive parahoric, a Levi factor $M$ of the special fiber $\mathcal{P}_/k$ has the form $\text{Sp}(W_1) \times \text{Sp}(W_2)$.
- And as a module for $M$, $\text{Lie}(R_u \mathcal{P}_/k)$ is isomorphic to $(W_1 \otimes W_2) \oplus (W_1 \otimes W_2)$. 