Nilpotent orbits of a reductive group over a local field

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Overview

Reductive groups: basic examples

G connected, reductive group over the (arbitrary) field K.
Examples:

- $G = \mathrm{GL}(V)$ for a K-vector space V
- $G = \mathrm{Sp}(V)$ if V has a non degenerate alternating form

I want to suppose G is D-standard.

- if G is semisimple, D-standard just means the char is "very good for G.
- Any K-form of GL_n is also D-standard.
- $\mathrm{Sp}(V)$ is D-standard just when $p \neq 2$.
- (the actual defn: let H_1 be semisimple in very good characteristic and $H = H_1 \times S$ for a K-torus S. Then G is D-standard if it is separably isogenous to the centralizer in H of a subgroup M of multiplicative type for some H_1, S, and M).

Levi factors of a linear group

Let H be a linear algebraic group over K.

- Assume that $R_u H$ is defined over K.
- A Levi factor of H (over K) is a K-subgroup $M \subset H$ for which H is the semidirect product $R_u H \cdot M$.
- If K has char. 0, then H has a Levi factor (Mostow).

Levi factors: problems in positive char.

- In general, $R_u H$ need not be defined over K.
  - e.g. this fails for $H = R_{K_1/K}\mathbf{G}_m$ when K_1/K is purely inseparable.
- there may be reductive $M \subset H$ with $M(K_{\mathrm{alg}}) \cdot R_u H(K_{\mathrm{alg}}) = H(K_{\mathrm{alg}})$ but for which $\operatorname{Lie} M + \operatorname{Lie} R_u H \neq \operatorname{Lie} H$.
  - for an example, let H a maximal parabolic of GL_3 in char. 2, and let M be the image of the adjoint rep $\mathrm{SL}_2 \to \mathrm{GL}(\mathfrak{sl}_2) = \mathrm{GL}_3$
- H need not have a Levi factor.
  - $H = R_{K_1/K}\mathbf{G}_m$ has no Levi factor over K
  - if W_2 = Witt vectors over alg. closed k, let $H = \mathrm{SL}_2(W_2)$. H is a 6 dimensional k-group, $R_u H$ is defined over k, and H and has no Levi factor.

Instability cocharacters

Let G a D-standard reductive gp, and $X \in \mathfrak{g}(K)$ nilpotent.

- Let S a maximal K-torus of $C = C_G(X)$.

- **Theorem (Premet, M)**

  - *There is a (K-)cochar.* $\phi : \mathbf{G}_m \to [C_G(S), C_G(S)]$ *for which* $X \in \mathfrak{g}(\phi; 2)$.
  - *If* S' *is a second max torus of* C, *and if* $\phi' : \mathbf{G}_m \to [C_G(S'), C_G(S')]$ *satisfies* $X \in \mathfrak{g}(\phi'; 2)$, *then* ϕ *and* ϕ' *are conjugate by a unique element of* $R_u(C)(K)$.
  - *the parabolic subgroup* $P = P(\phi)$ *depends only on* X.

- I'll say that ϕ is *associated to* X.
- if K has char. 0, the elements X and $H = d\phi(1)$ may be completed to a unique $\mathfrak{sl}_2$-triple.

Optimal SL_2-mappings

Now suppose the nilpotent $X \in \mathfrak{g}(K)$ satisfies $X^{[p]} = 0$.

- Fix a cocharacter λ associated to X.

- **Theorem (Seitz,M)**

  There is a unique homomorphism $\psi : \mathrm{SL}_2 \to G$ for which

  $$\psi \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \phi(t) \; \forall t \in K_{\mathrm{alg}} \quad \textit{and} \quad d\psi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X.$$

  Moreover, the representation $\mathrm{Ad} \circ \psi$ of SL_2 on $\mathfrak{g}$ is a tilting module for which all weights μ satisfy $-2p + 2 \leq \mu \leq 2p - 2$.

Good filtrations

Let G be a reductive group, and suppose $K = K_{\mathrm{alg}}$. Fix a maximal torus T and a Borel subgroup $T \subset B$.

- The characters $\lambda \in X^*(T)$ parametrize the G-linearized line bundles $\mathcal{L}(\lambda)$ on G/B.
- λ is dominant $\iff H^0(\lambda) = H^0(G/B, \mathcal{L}(\lambda)) \neq 0$.
- the simple G-modules are precisely the $L(\lambda) = \operatorname{soc} H^0(\lambda)$ for dominant λ.
- a (finite dimensional) G-module M is said to have a *good filtration* if there are submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ for which $M_i/M_{i-1} \simeq H^0(\lambda_i)$ for dominant weights λ_i.

Tilting modules

- a G-module M is said to be a *tilting module* if both M and $M^\vee$ have a good filtration.
- any tilting mod M is a $\bigoplus$ of *indecomposable* tilting mods
- the indec tilting modules $T(\lambda)$ are param by dominant λ.

Theorem (Donkin, Mathieu)

For all dominant weights λ, μ, $H^0(\lambda) \otimes H^0(\mu)$ has a good filtration.

- this essentially amounts to the assertion that there is a Frobenius splitting of $G/B \times G/B$ which "B-compatibly Frobenius splits" the diagonal.

Corollary

If T_1 and T_2 are tilting modules, so is $T_1 \otimes T_2$.

Tilting modules for SL_2

Let K have char. $p > 0$, Let $\lambda \in \mathbf{Z}$ with $0 \leq \lambda \leq 2p - 2$, and view λ as a character of the stndrd max torus of SL_2.

- I'd like to describe the indec. tilting mod $T(\lambda)$.
- The standard module $H^0(\lambda)$ has dimension $\lambda + 1$ and coincides with $\mathrm{Sym}^\lambda V$ where $V = K^2 = H^0(1)$ is natural rep.
- $H^0(\lambda) = L(\lambda)$ is simple $\iff \lambda < p$. In this case $T(\lambda) = L(\lambda)$.

Tilting modules for SL_2, continued

- Now assume $\lambda = p + \mu$ for $0 \leq \mu \leq p - 2$.
- Then $L(\lambda) \simeq V^{[1]} \otimes L(\mu) = L(1)^{[1]} \otimes L(\mu)$, where $[1]$ denotes a Frob. twist.
- There is a non-split exact sequence of SL_2-modules

  $$0 \to L(\lambda) \to H^0(\lambda) \to L(p - 2 - \mu) \to 0;$$

  note that $L(p - 2 - \mu) = H^0(p - 2 - \mu)$.
- There is a non-split exact sequence of SL_2-modules

  $$0 \to H^0(p - 2 - \mu) \to T(\lambda) \to H^0(\lambda) \to 0.$$

- In particular, $\dim T(\lambda) = (p - 2 - \mu + 1) + (\lambda + 1) = 2p$.
- $T(\lambda)$ may be "constructed" as an indecomposable summand of $L(p - 1) \otimes L(\mu + 1)$.

Tilting modules for SL_2, conclusion

Let $\mathcal{A}$ a discrete valuation ring with fractions K and residues $k = \mathcal{A}/\pi\mathcal{A}$.

- Let $\mathcal{L}$ be a free $\mathcal{A}$-mod of finite rank.
- Let $\rho : \mathrm{SL}_{2/\mathcal{A}} \to \mathrm{GL}(\mathcal{L})$ be an $\mathcal{A}$-representation – i.e. a morphism of $\mathcal{A}$-group schemes.
- Write $X_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathcal{A})$, $X_0(k)$ the image in $\mathfrak{sl}_2(k)$ and $X_0(K)$ the image in $\mathfrak{sl}_2(K)$.
- Assume each weight λ of the representation $\mathcal{L}$ satisfies $-2p + 2 \leq \lambda \leq 2p - 2$.

Proposition

If $\mathcal{L}/\pi\mathcal{L}$ is a tilting module for $\mathrm{SL}_{2/k}$, then $\dim_k \ker \rho(X_0(k))$ *coincides with* $\dim_K \ker \rho(X_0(K))$.

Levi factors of nilpotent centralizers

- Let G be D-standard and $X \in \mathfrak{g}(K)$ nilpotent.
- Recall G-orbits in nilp variety $\mathcal{N}$ are classified *geometrically* by Bala-Carter data (L, Q): L is the Levi subgroup of a parabolic of G, and $Q \subset L$ is a distinguished parabolic.
- In particular, the *geometric* nilpotent orbits depend only on the root datum of G.
- I want to comment on the structure of the centralizer $C = C_G(X)$.
- If ϕ is a cocharacter associated with X, one knows that $M = C \cap C_G(\operatorname{im} \phi)$ is a Levi factor of C (over K).

Structure of a nilpotent centralizer

Theorem (M)

The centralizer $C = C_G(X)$ has a Levi decomposition defined over K. Moreover, the following are independent of p (under our standard hyp):

- *the (geometric) root datum of a Levi factor of C,*
- *the (geometric) component group $C(K_{\mathrm{alg}})/C^0(K_{\mathrm{alg}})$*

- Method of proof: may suppose $K = K_{\mathrm{alg}}$.
  - let $\mathcal{A}$ be a DVR with residues K and fractions of char 0.
  - let $\mathcal{G}_{/\mathcal{A}}$ be split reductive with root datum of G
  - find a nilpotent section $X_1 \in \mathfrak{g}(\mathcal{A})$ specializing to X for which the $\mathcal{A}$ group scheme $C_{\mathcal{G}}(X_1)$ is *smooth* over $\mathcal{A}$.
  - Find the Levi factor "over $\mathcal{A}$".

Motivating question(s)

Consider more general smooth group schemes over a DVR with reductive generic fiber, but which are not necessarily reductive. Can one:

- find nilpotent sections with smooth centralizer lifting classes on the special fiber?
- use such nilpotent sections to understand (something about) DeBacker's parametrization of rational nilpotent orbits?

“Local” fields – notation etc.

- Let $\mathcal{A}$ be a discrete valuation ring with maximal ideal $\pi\mathcal{A}$; assume $\mathcal{A}$ is π-adically complete, assume $k = \mathcal{A}/\pi\mathcal{A}$ is *perfect*, and let $K = \mathrm{Frac}(\mathcal{A})$.
- Examples:
  - $K = k((\pi))$, $\mathcal{A} = k[[\pi]]$.
  - K a finite extension of $\mathbf{Q}_p$, $\mathcal{A}$ int. closure of $\mathbf{Z}_p$ in K

Parahoric subgroups

Let G be D-standard reductive over the "local" field K.

- Bruhat and Tits: parahoric "subgroups" – certain smooth $\mathcal{A}$-group schemes $\mathcal{P}$ with $\mathcal{P}_{/K} = G$
- determined by *facets* in the affine *building* of G.
- there is an $\mathcal{A}$-split torus S in $\mathcal{P}$ for which $S_{/K}$ is a maximal K-split torus in G.

Theorem (Bruhat-Tits, I think)

The special fiber $\mathcal{P}_{/k}$ has a unique Levi factor containing $S_{/k}$.

- the case of SU_3 with L/K totally ramified shows this not to be *too* trivial.

A map in Galois cohomology

Let $\mathcal{P}$ a parahoric, write $\mathfrak{p} = \mathrm{Lie}(\mathcal{P})$.

- Let $X \in \mathfrak{p}(\mathcal{A})$ nilpotent for which $X(k)$ lies in the Lie algebra of a Levi factor of $\mathcal{P}_{/k}$.
- Up to conjugacy on the special fiber we may suppose that a cocharacter ϕ associated with $X(k) \in \mathfrak{p}$ (in a Levi factor) takes values in the fixed split maximal torus S – thus ϕ may be viewed as an $\mathcal{A}$-cocharacter of $\mathcal{P}$.

A map in Galois cohomology, continued

- **Desired condition:** Let $C \subset \mathfrak{p}$ an $\mathcal{A}$-submodule. Write $C_{/K} = C \otimes K$ and $C_{/k}$ for the image of C in $\mathfrak{p}/\pi\mathfrak{p}$. Suppose that:
  (C1) C is stable under $\phi(\mathcal{A}^\times)$.
  (C2) as an $\mathcal{A}$-module, $\mathfrak{p} = \mathfrak{p}(\mathcal{A})$ is the direct sum of C and $[X, \mathfrak{p}(\mathcal{A})]$
  (C3) $C_{/k} \cap \mathrm{Lie}(R_u\mathcal{P}_{/k})$ is a complement to $[X, \mathrm{Lie}(R_u\mathcal{P}_{/k})]$.
- If there is C satisfying (C1)–(C3), the centralizer $C_{\mathcal{P}}(X)$ is a smooth group scheme over $\mathcal{A}$.
- If $p >> 0$ one may use $C = \mathrm{Lie}(C_G(Y))$ determined by a suitable $\mathfrak{sl}_2$-triple (X, H, Y) for which $H = d\phi(1)$.

A map in Galois cohomology, continued

Write $\mathfrak{p}^+$ for the pre-image of $\mathrm{Lie}(R_u \mathcal{P}_{/k})$ under the mapping $\mathfrak{p} \to \mathfrak{p}_{/k} = \mathrm{Lie}(\mathcal{P}_{/k})$. Assume C satisfies (C1)–(C3).

Proposition (adaptation of DeBacker/Waldspurger)

The $G(K)$-orbit of X is the nilpotent orbit of minimal dimension having non-empty intersection with $X + \mathfrak{p}^+$.

The proof depends on viewing $\mathcal{P}(\mathcal{A}/\pi^2\mathcal{A})$ as the k-points of a linear group over $k = \mathcal{A}/\pi\mathcal{A}$ (à la Greenberg), and knowing that the centralizer of X is *smooth*.

A map in Galois cohomology, continued

Corollary

For X as above, there is a natural mapping

$$H^1(k, C_{\mathcal{P}_{/k}}(X)) \to H^1(K, C_G(X)),$$

- Note the H^1 of $C_{\mathcal{P}_{/k}}(X)$ identifies with that of its reductive quotient.
- This natural mapping is *realized* by DeBacker's mapping.

The hope!!

- Let $\mathcal{P}$ be a parahoric of G, and let $M \subset \mathcal{P}_{/k}$ be a Levi factor.
- Let X_0 be a distinguished nilp element of M s.t. $X_0^{[p]} = 0$.
- Suppose that M is D-standard (!?!), and let $\psi : \mathrm{SL}_{2/k} \to M$ an optimal SL_2-mapping for X_0.

Hope

The representation $(\mathrm{Ad} \circ \psi, \mathrm{Lie}(\mathcal{P}_{/k}))$ is a tilting module for SL_2 for which all weights μ satisfy $-2p + 2 \leq \mu \leq 2p - 2$.

- Equivalently: the Lie algebra of the unipotent radical of $\mathcal{P}_{/k}$ is a tilting module for SL_2 under the action determined by $\mathrm{Ad} \circ \psi$ (with the indicated condition on the weights).

The utility of hope...

- With notation as on the previous slide, assume that *the hope* holds for $X_0 \in \mathrm{Lie}(M)(k)$.

Proposition

Then there is an $\mathcal{A}$-submodule $C \subset \mathfrak{p}$ for which (C1)–(C3) hold.

- An important point is the following: if ϕ is a cocharacter of M associated with X_0, and if (as before) we arrange that ϕ is "defined over $\mathcal{A}$", one needs to know that ϕ is associated with X for some nipotent element $X \in \mathfrak{p}$ whose image in $\mathfrak{p}_{/k}$ is X_0.

Verifying hope...

- The hope holds for GL_n.
- It also holds for $G = \mathrm{Sp}(V)$.
  - Indeed, it is clear for the reductive parahoric.
  - If $\mathcal{P}$ is a non-reductive parahoric, a Levi factor M of the special fiber $\mathcal{P}_{/k}$ has the form $\mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2)$.
  - And as a module for M, $\mathrm{Lie}(R_u\mathcal{P}_{/k})$ is isomorphic to $(W_1 \otimes W_2) \oplus (W_1 \otimes W_2)$.